A New a Posteriori Error Estimate for Adaptive Finite Element Methods

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1 Introduction

In many scientific problems, adaptive finite element methods has been widely used to improve the accuracy of numerical solutions. The general idea is to refine or adjust the mesh such that the errors are "equally" distributed over the computational mesh, with the aim of achieving a better accurate solution using an optimal number of degrees of freedom. By using the information from the approximated solution and the known data, the a posteriori error estimator provides the information about the size and the distribution of the error of the finite element approximation. There is a large numerical analysis literature on adaptive finite element methods, and various types of a posteriori estimates have been proposed for different problems, see e.g. [1]. The a posterior error estimate and adaptive finite element method were first introduced by [2]. Since the later 1980s, much research work on a posteriori error estimate has been developed including the residual type a posteriori error estimate [8], recovery type a posteriori error estimate [16], a posteriori error estimate based on hierarchic basis [4, 5], and so on. For the literature, the readers are referred to the books [1, 3, 12, 14], the papers [6, 13, 15], and the references cited therein.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We assume that $\mathcal{T}_h$ is a shape regular triangulation of Ω. Let $V_h \subset H^1(\Omega)$ be the corresponding continuous piecewise linear finite element space associated with $\mathcal{T}_h$, and $u_h \in V_h$ be a finite element approximation to a second order elliptic boundary value problem.

In this paper, we consider the adaptive finite element methods for a second order elliptic boundary value problem. We propose a new a posteriori error estimate which is motivated from the smoothing iteration of the multilevel iterative methods. In particular, on current mesh $\mathcal{T}_h$, we solve the equation to obtain the finite element solution u_h, then global refine the mesh $\mathcal{T}_h$ to obtain the auxiliary mesh $\mathcal{T}_{h/2}$. On the fine mesh, we use a simple smoother such as Gauss–Seidel iteration with u_h as the initial value. After m iterations, we obtain an approximation solution $u_{h/2,m}$ of finite element solution $u_{h/2}$ on fine mesh $\mathcal{T}_{h/2}$. Then take $\|\nabla(u_h - u_{h/2,m})\|$ as the a posteriori estimate to guide the mesh refinement on $\mathcal{T}_h$. In practice, it only need small number of smoothing steps to obtain an efficient a posteriori error estimator $\|\nabla(u_h - u_{h/2,m})\|$, the computational cost is relatively small.

The rest of the paper is organized as follows: In Sect. 2 we propose the new a posteriori error estimate and investigate its properties. And we describe adaptive finite element algorithm with our new a posteriori error estimator for a second order elliptic boundary value problem. We present some numerical investigations in the efficiency of the new a posteriori error estimate and the performance of the corresponding adaptive finite element algorithm in Sect. 3.

2 A Posteriori Error Estimate

We consider the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \in \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$, for simplicity, Ω is assumed to be a polygonal domain.

In weak form, this problem reads: Find $u \in V = \{v \in H^1(\Omega) : v|_{\partial\Omega} = g\}$ such that

$$a(u, v) = f(v) \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where

$$a(u, v) = \int_\Omega \nabla u \nabla v dx,$$

and

$$f(v) = \int_\Omega f v dx.$$

Let $\mathcal{T}_h$ be a shape regular triangulation of Ω. Consider the C^0 linear finite element space V_h associated with $\mathcal{T}_h$ and defined by

$$V_h = \{v \in H^1(\Omega) : v \in P_1(\tau), \forall \tau \in \mathcal{T}_h\},$$

where $P_l(D)$ denotes the set of all polynomials defined of $D \subseteq \mathbb{R}^2$ of total degree $\leq l$. The discrete approximation to (1) is obtained in the standard way: Find $u_h \in V_h \cap V$ such that

$$a(u_h, v) = f(v) \quad \forall v \in V_h \cap H_0^1(\Omega). \tag{3}$$

Suppose that $\{\psi_i : i = 1, 2, \cdots, N\}$ are the basis for V_h, and define the matrix A^h, and a vector, F^h, via

$$A_{ij}^h := a(\psi_i, \psi_j) \qquad \text{and} \qquad F_i^h := f(\psi_i) \qquad \forall i, j = 1, 2, \cdots, N.$$

Then (3) is equivalent to solving $A^h U = F^h$ with $u_h = \sum_{i=1}^{N} u_i \psi_i$ and $U = (u_i)$. Clearly, the matrix A^h is a symmetric positive definite (SPD) matrix as $a(\cdot, \cdot)$ is SPD.

Let $\mathcal{T}_{h/2}$ be a global refinement of the triangulation $\mathcal{T}_h$ and $V_h \subset V_{h/2}$, suppose $u_h, u_{h/2}$ are then the discrete finite element solutions over $\mathcal{T}_h$ and $\mathcal{T}_{h/2}$, respectively. We have the following orthogonality relation between $u - u_{h/2}$ and $u_h - u_{h/2}$, which follows immediately from the Galerkin orthogonality.

$$\|\nabla(u - u_{h/2})\|_{0,\Omega}^2 = \|\nabla(u - u_h)\|_{0,\Omega}^2 - \|\nabla(u_h - u_{h/2})\|_{0,\Omega}^2. \tag{4}$$

Using the orthogonality (4), we have

$$\begin{aligned}\frac{\|\nabla u_{h/2} - \nabla u_h\|_{0,\Omega}^2}{\|\nabla u - \nabla u_h\|_{0,\Omega}^2} &= \frac{\|\nabla u - \nabla u_h\|_{0,\Omega}^2 - \|\nabla u - \nabla u_{h/2}\|_{0,\Omega}^2}{\|\nabla u - \nabla u_h\|_{0,\Omega}^2} \\ &= 1 - \frac{\|\nabla u - \nabla u_{h/2}\|_{0,\Omega}^2}{\|\nabla u - \nabla u_h\|_{0,\Omega}^2}.\end{aligned}$$

With the saturation assumption:

$$\|\nabla u - \nabla u_{h/2}\|_{0,\Omega} \leq \beta \|\nabla u - \nabla u_h\|_{0,\Omega}, \qquad \beta \in [0, 1),$$

we have

$$\sqrt{1 - \beta^2} \leq \frac{\|\nabla u_{h/2} - \nabla u_h\|_{0,\Omega}}{\|\nabla u - \nabla u_h\|_{0,\Omega}} \leq 1. \tag{5}$$

Numerical examples show that

$$\frac{\|\nabla u_{h/2} - \nabla u_h\|_{0,\Omega}}{\|\nabla u - \nabla u_h\|_{0,\Omega}} \to \frac{\sqrt{3}}{2}. \tag{6}$$

So $\|\nabla(u_{h/2} - u_h)\|_{0,\Omega}$ can be used as a posteriori error estimate if $u_{h/2}$ is at hand. Notice that $u_{h/2} - u_h$ is of high frequency which can be easily obtained by a few smoothing iterations. So we can use the $\|\nabla(u_{h/2,m} - u_h)\|_{0,\Omega}$ instead of $\|\nabla(u_{h/2} - u_h)\|_{0,\Omega}$ after m steps of the a posteriori error estimate, where $u_{h/2,m}$ is an approximation of $u_{h/2}$ by the smoothing iterations, and the computational cost is much cheaper. From (6), it is possible that

$$\frac{\|\nabla u_{h/2,m} - \nabla u_h\|_{0,\Omega}}{\|\nabla u - \nabla u_h\|_{0,\Omega}} \to \frac{\sqrt{3}}{2}. \tag{7}$$

Note that if we have the approximation $u_{h/2,m}$ on $\mathcal{T}_{h/2}$, we then could obtain $I_2 u_{h/2,m}$ by interpolating $u_{h/2,m}$ into the piecewise quadratic finite element spaces on $\mathcal{T}_h$. In Sect. 3, the numerical examples show

$$\frac{\|\nabla I_2 u_{h/2,m} - \nabla u_h\|_{0,\Omega}}{\|\nabla u - \nabla u_h\|_{0,\Omega}} \to 1, \tag{8}$$

it means that the error estimate $\|\nabla I_2 u_{h/2,m} - \nabla u_h\|_{0,\Omega}$ is an asymptotically exact a posteriori error estimate for adaptive finite element methods.

For our error estimator, we find a better approximation $u_{h/2,m}$ in a bigger space, which shares the same principle as the hierarchical basis error estimator of [4]. Comparing with the hierarchal basis error estimator, we obtain the error estimator by solving the problem on the finer mesh, and Bank and Smith solve an approximation problem on the enriched subspace to estimate the error.

We now describe an algorithm to obtain our new a posteriori error estimate for mesh $\mathcal{T}_h$ in detail. Given the finite element solution u_h, the number of smoothing iterations m, we carry out the following steps to obtain the new a posteriori error estimate.

1. Global refine $\mathcal{T}_h$ to obtain an auxiliary fine mesh $\mathcal{T}_{h/2}$.
2. Build the finite element space $V_{h/2}$ on the fine mesh $\mathcal{T}_{h/2}$, and the corresponding stiffness matrix $A^{h/2}$ and load vector $F^{h/2}$.
3. Obtain $I_h^{h/2} u_h$ by interpolating u_h from V_h to $V_{h/2}$, taking $I_h^{h/2} u_h$ as the initial value $u_{h/2,0}$ and solving the linear equations
$$A^{h/2} U = F^{h/2} \tag{9}$$
in m smoothing iterations to obtain $U^m = (u_i^m)$. We then obtain an approximation of $u_{h/2}$
$$u_{h/2,m} = \sum_{i=1}^{N_{h/2}} u_i^m \psi_i,$$
where $N_{h/2}$ is the number of basis function of $V_{h/2}$.
4. For each $\tau \in \mathcal{T}_h$, we calculate
$$\eta_{\tau,m} = \|\nabla(u_h - u_{h/2,m})\|_{0,\tau}$$
as the error estimator on τ, and take
$$\eta_{h,m}^2 = \sum_{\tau \in \mathcal{T}_h} \eta_{\tau,m}^2$$
as the a posteriori error estimate.

For the condition number of the finite element equations on adaptively refined meshes $\{\mathcal{T}_l : l \in \mathcal{N}\}$, a mesh family $\{\mathcal{T}_l : l \in \mathcal{N}\}$ is said to be nondegenerate if there exists a constant $\rho > 0$ such that for all $l \in \mathcal{N}$ and for all $\tau \in \mathcal{T}_l$ there is a ball of radius $\rho \cdot diam(\tau)$ contained in τ, where $diam(\tau)$ denotes the diameter of τ.

Following [7], we assume that the basis $\{\psi_i : i = 1, 2, \cdots, N\}$ of V_h is a local basis:

$$\max_{1 \le i \le N} cardinality\{\tau \in \mathcal{T}_h, supp(\psi_i) \cap \tau \neq \emptyset\} \le C. \tag{10}$$

We have the following estimates:

Lemma 1. *Suppose that the mesh $\mathcal{T}_h$ is nondegenerate. Let A^h denote the matrix corresponding to the inner product $a(\cdot,\cdot)$, i.e., $A^h_{ij} = a(\psi_i, \psi_j)$ where $\{\psi_i : i = 1, 2, \cdots, N\}$ are the standard linear Lagrange basis. Then the maximum eigenvalue λ_{max} of A^h is bounded by*

$$\lambda_{max} \leq C. \tag{11}$$

Proof. First note that if we set $v = \sum_{i=1}^{N} v_i \psi_i$ then

$$a(v, v) = V^t A^h V,$$

where $V = (v_i)$, because $a(\cdot,\cdot)$ is bilinear. From the inverse estimate and (10), we have

$$\begin{aligned} a(v, v) \leq C\|v\|_1^2 = C \sum_{\tau \in \mathcal{T}_h} \|v\|_{1,\tau}^2 \leq C \sum_{\tau \in \mathcal{T}_h} \|v\|_{0,\infty,\tau}^2 \\ \leq C \sum_{\tau \in \mathcal{T}_h} \sum_{supp(\psi_i) \cap \tau \neq \emptyset} v_i^2 \leq C V^t V. \end{aligned}$$

Then we obtain (11).

For solving the linear equations $AU = F$, a basic linear iterative method can be written in the following form:

$$U^{k+1} = U^k + B(F - AU^k), \quad k = 0, 1, 2, \cdots, \tag{12}$$

starting from an initial guess $U^0 \in R^n$.

The Richardson iterative scheme corresponds to (12) with $B = \frac{\omega}{\rho(A)} I$. Namely,

$$U^{k+1} = U^k + \frac{\omega}{\rho(A)}(F - AU^k), \quad k = 0, 1, 2, \cdots. \tag{13}$$

We first discuss its “smoothing property”. Set $\omega = 1$ in (13) and define

$$S = I - \frac{1}{\rho(A)} A.$$

Theorem 1. *For the smoother S, we have*

$$\|S^m V\|_A \leq C m^{-1/2} \|V\|_0, \qquad \forall V \in R^n, \tag{14}$$

where $\|V\|_0 = (V, V)^{1/2}$ is the l^2-norm in R^n and

$$\|V\|_A = (AV, V)^{1/2}, \tag{15}$$

is the A-norm corresponding to the linear system we wish to solve.

Proof. Since A is an symmetric positive define matrix, then we have $A\phi_i = \lambda_i\phi_i$ with $\lambda_{min} = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n = \lambda_{max}$, $(\phi_i, \phi_j) = \delta_{ij}$, and $\forall v \in \mathbb{R}^n$,

$$V = \sum_{i=1}^{n} v_i\phi_i.$$

Then

$$S^m V = \left(I - \frac{1}{\rho(A)}A\right)^m V = \sum_{i=1}^{n}\left(1 - \frac{\lambda_i}{\lambda_{max}}\right)^m v_i\phi_i.$$

And

$$\begin{aligned}\|S^m V\|_A^2 &= \sum_{i=1}^{n}\left(1 - \frac{\lambda_i}{\lambda_{max}}\right)^{2m} v_i^2\lambda_i \\ &= \lambda_{max}\left(\sum_{i=1}^{n}\left(1 - \frac{\lambda_i}{\lambda_{max}}\right)^{2m}\frac{\lambda_i}{\lambda_{max}}v_i^2\right) \\ &\le \lambda_{max}\left\{\sup_{0\le x\le 1}(1-x)^{2m}x\right\}\sum_{i=1}^{n} v_i^2.\end{aligned}$$

Clearly,

$$\sup_{0\le x\le 1}(1-x)^{2m}x \le \frac{1}{2m+1}.$$

From (11), we have

$$\lambda_{max} \le C.$$

Then, from the above inequalities, we obtain

$$\|S^m V\|_A^2 \le Cm^{-1}\|V\|_0^2.$$

On the quasi-uniformly meshes, the smoother operator S have the well known smoothing property

$$\|S^m v_h\|_A \le C\frac{h^{-1}}{m^{1/2}}\|v_h\|_{0,\Omega}, \qquad \forall v_h \in V_h.$$

In the following, from a numerical example, we investigate the smoothing property of Gauss–Seidel smoother on locally refined meshes. We solve the Laplace equation with the exact solution $u = r^{\frac{2}{3}}\sin(\frac{2}{3}\theta), r = \sqrt{x^2+y^2}$ on a L-Shape domain by the adaptive algorithm. We consider one of the adaptive level, we obtain the finite element solution u_h on $\mathcal{T}_h$, then we get $\mathcal{T}_{h/2}$ (see Fig. 1 (Left)) by globally refining $\mathcal{T}_h$. Set u_h as the initial value, and solve the Eq. (16) by executing m smoothing steps on the $\mathcal{T}_{h/2}$, the results are plotted in Fig. 1 (Right), we see that the smoother operator S admits the similar property on the locally refined meshes.

It is obviously that we can obtain an approximation $u_{h/2,m}$ for $u_{h/2}$ at any accuracy with a larger m. And we known that the error between $u_{h/2}$ and $u_{h/2,m}$is reduced quickly at the beginning of several iterative steps, then we need to do only

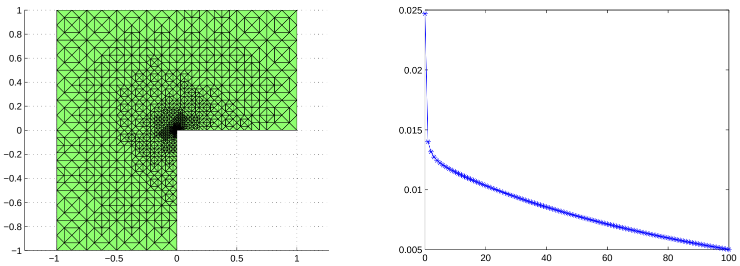


Fig. 1. *Left*: Refined mesh. *Right*: Gauss–Seidel convergence history.

a few smoothing steps to obtain an approximation $u_{h/2,m}$ for our a posteriori error estimator. From our numerical examples in Sect. 3, $m = 3$ performs well.

The standard adaptive finite element methods through local refinement can be written in the following loop

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}.$$

Using the above new a posteriori error estimator, the adaptive algorithm has the following general steps:

1. Construct an initial coarse mesh $\mathcal{T}_0$ representing sufficiently well the geometry of the problem. Put $k := 0$.
2. Solve the discrete problem on $\mathcal{T}_k$ to obtain the solution u_k.
3. For each element $\tau \in \mathcal{T}_k$ compute the a posteriori error estimate. In detail, first globally refine $\mathcal{T}_k$ to obtain the fine mesh $\mathcal{T}_k^{'}$, then take u_k as the initial value, use the Gauss–Seidel iteration in m steps, solve the discrete problem on $\mathcal{T}_k^{'}$ to obtain the approximation $u_{k,m}$. Then we get the error estimator $\|\nabla u_k - \nabla u_{k,m}\|_{0,\tau}$ on each $\tau \in \mathcal{T}_k$.
4. If the estimated global error $\|\nabla u_k - \nabla u_{k,m}\|_{o,\Omega}$ is sufficiently small then **stop**. Otherwise, using a suitable marking strategy, decide which elements have to be refined and construct the next mesh $\mathcal{T}_{k+1}$ through local refinement. Replace k by $k+1$ and return to step 2.

One drawback of hierarchical type error estimators is the computational cost to refine the mesh and assemble the matrix equation on the finer mesh. For our error estimator, in step 3, we can assemble the matrix equation in the finer mesh $\mathcal{T}_{h/2}$ by using the element stiffness matrix in $\mathcal{T}_h$, as the finer mesh $\mathcal{T}_{h/2}$ is the global refinement of $\mathcal{T}_h$, each element are refined into four children elements, the children's element stiffness matrix is the same as its farther's element stiffness matrix for constant coefficients. For smoothing coefficient we can also use the element stiffness matrix on $\mathcal{T}_h$ to assemble the stiffness matrix on $\mathcal{T}_{h/2}$. Then we obtain the a posteriori error estimator at a relatively small computational cost. Thus the adaptive algorithm with our new a posteriori error estimate is efficient and simple in practice. We present some numerical examples in the following section to investigate in the performance of the adaptive finite element algorithm.

3 Numerical Validation and Applications

In this section, we present some numerical examples to verify the results in Sect. 2 with the model problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{16}$$

where $\Omega \in \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$.

For a $\tau \in \mathcal{T}_h$,

$$\eta_\tau = \|\nabla u_{h/2} - \nabla u_h\|_{0,\tau}, \quad \text{and} \quad \eta_h = \|\nabla u_{h/2} - \nabla u_h\|_{0,\Omega},$$

the new a posteriori error estimator in τ is

$$\eta_{\tau,m} = \|\nabla u_{h/2,m} - \nabla u_h\|_{0,\tau}, \quad \text{and} \quad \eta_{h,m} = \|\nabla u_{h/2,m} - \nabla u_h\|_{0,\Omega}.$$

To measure the accuracy of η_m, we use the index θ_τ, θ_h defined by

$$\theta_\tau = \frac{\eta_{\tau,m}}{\|\nabla u - \nabla u_h\|_{0,\tau}}, \quad \text{and} \quad \theta_h = \frac{\eta_{h,m}}{\|\nabla u - \nabla u_h\|_{0,\Omega}}.$$

Accordingly, for the error estimator $\eta'_{\tau,m} = \|\nabla I_2 u_{h/2,m} - \nabla u_h\|_{0,\tau}$ and $\eta'_{h,m} = \|\nabla I_2 u_{h/2,m} - \nabla u_h\|_{0,\Omega}$, where $I_2 u_{h/2,m}$ is a piecewise quadratic polynomial which obtained by the interpolation postprocessing. We define

$$\theta'_\tau = \frac{\eta'_{\tau,m}}{\|\nabla u - \nabla u_h\|_{0,\tau}}, \qquad \theta'_h = \frac{\eta'_{h,m}}{\|\nabla u - \nabla u_h\|_{0,\Omega}}.$$

In the following examples, we investigate the performance of our new a posteriori error estimator. In detail, we consider two types of methods for local mesh refinement, one based on Centroidal Voronoi Delaunay Triangulation(CVDT) [10, 11], the other on bisection, 3 Gauss–Seidel iterations are used to obtain the approximation $u_{h/2,m}$, and then $\eta_{h,\tau}$ is used as the error estimator. We implement our numerical tests with the Matlab package iFEM [9].

Example 1 In this example, we solve (16) with $f = 0$ and the exact solution $u = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta), r = \sqrt{x^2 + y^2}$ on the L-Shape domain $\Omega = \{-1 \le x, y \le 1\} \setminus \{0 \le x \le 1, -1 \le y \le 0\}$. The mesh refinement is based on CVDT. The results are shown in Fig. 2. We see that

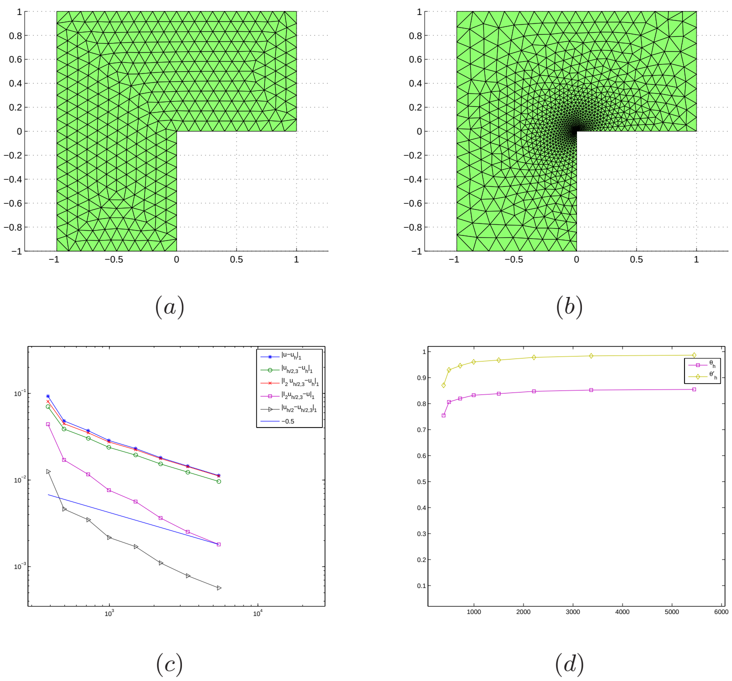
$$\|\nabla u - \nabla u_h\|_0 = O(N^{-1/2}), ,$$

$$\|\nabla u - \nabla I_2 u_{h/2,3}\|_0 = O(N^{-0.7}), \qquad \|\nabla u_{h/2} - \nabla u_{h/2,3}\|_0 = O(N^{-0.67}).$$

For the efficient index, it shows that

$$\theta_h \to \frac{\sqrt{3}}{2}, \qquad \theta'_h \to 1.$$

Notice that the decay of $\|\nabla u - \nabla u_h\|_0$ is quasi-optimal.



(a) (b) (c) (d)

Fig. 2. Results of example 1. (**a**): initial mesh; (**b**): refined mesh after 4 refinements; (**c**): errors; (**d**): effectivity index.

Example 2 In this example, as in Example 1, we solve (16) with the exact solution $u = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$ on the L-Shape domain. But, we use the bisection for local mesh refinement. We obtain similar results; Fig. 3 plots the initial mesh and the adaptively refined mesh after 8 adaptive iterations. From Fig. 3, we see that

$$\|\nabla u - \nabla u_h\|_0 = O(N^{-1/2}),$$

$$\|\nabla u - \nabla I_2 u_{h/2,3}\|_0 = O(N^{-0.85}), \qquad \|\nabla u_{h/2} - \nabla u_{h/2,3}\|_0 = O(N^{-3/4}).$$

For the efficient index, it shows that

$$\theta_h \to \frac{\sqrt{3}}{2}, \qquad \theta'_h \to 1.$$

Notice that the decay of $\|\nabla u - \nabla u_h\|_0$ is also quasi-optimal.

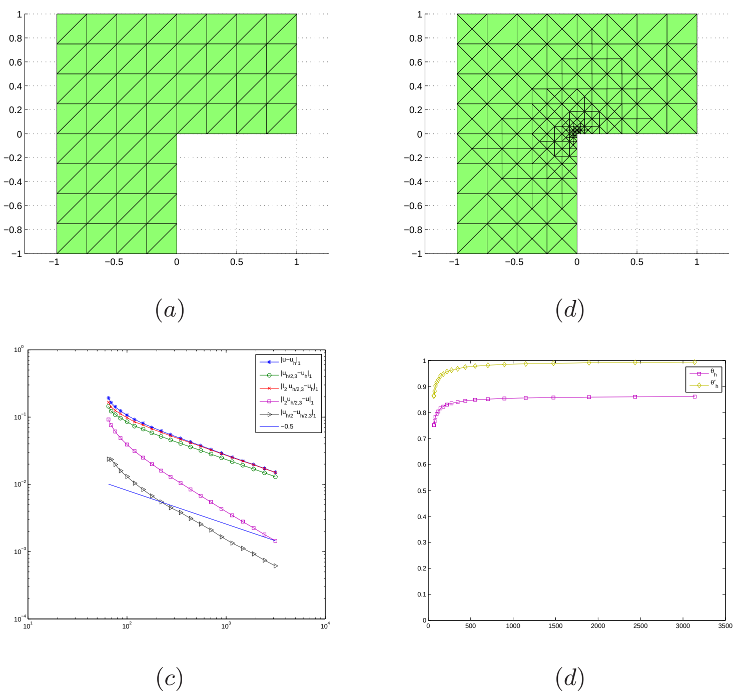


Fig. 3. Results of example 2. (**a**): initial mesh; (**b**): refined mesh after 8 refinements; (**c**): errors; (**d**): effectivity index.

Example 3 In this example, we solve (16) with $f = 1$ and the exact solution $u = \sqrt{\frac{1}{2}(r-x)} - \frac{1}{4}r^2, r = \sqrt{x^2+y^2}$ on a crack domain $\Omega = \{|x|+|y| < 1\} \setminus \{0 \le x \le 1, y = 0\}$. Figure 4 plots the initial mesh and the adaptively refined mesh after 8 adaptive iterations, and shows the performance of the error estimator. We see that

$$\|\nabla u - \nabla u_h\|_0 = O(N^{-1/2}),$$
$$\|\nabla u - \nabla I_2 u_{h/2,3}\|_0 = O(N^{-0.65}), \qquad \|\nabla u_{h/2} - \nabla u_{h/2,3}\|_0 = O(N^{-0.65}).$$

For the efficient index, it shows that

$$\theta_h \to \frac{\sqrt{3}}{2}, \qquad \theta'_h \to 1.$$

The decay of $\|\nabla u - \nabla u_h\|_0$ is also quasi-optimal.

Finally, based on the numerical observation and rough analysis, we may propose a conjecture on the convergence property of the finite element method.

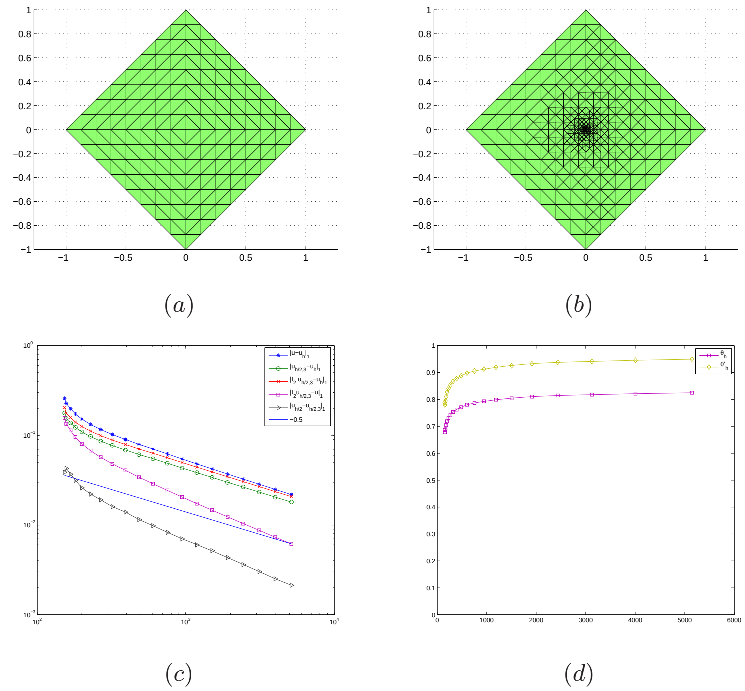


Fig. 4. Results of example 3. (**a**): initial mesh; (**b**): refined mesh after 8 refinements; (**c**): errors; (**d**): efficient index.

Conjecture For linear triangular element approximation on a sequence of triangulations $\mathcal{T}_h$, if the convergence rate is optimal in the sense of

$$\|u - u_h\|_1 \leq CN^{-1/2},$$

where N is the total number of unknowns. Then there holds

$$\frac{\|u_h - u_{h/2}\|_1}{\|u - u_h\|_1} \to \frac{\sqrt{3}}{2}\ (N \to \infty) \quad \text{and} \quad \frac{\|u_h - I_2 u_{h/2}\|_1}{\|u - u_h\|_1} \to 1\ (N \to \infty).$$

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