Equidistribution and Optimal Approximation Class*

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1 Introduction

Local adaptive grid refinement is an important technique in finite element methods. Its study can be traced back to the pioneering work [2] in one dimension. In recent years, mathematicians start to prove the convergence and optimal complexity of the adaptive procedure in multi-dimensions. Dörfler [11] first proved an error reduction in the energy norm for the Poisson equation provided the initial mesh is fine enough. Morin et al. [15, 16] extended the convergence result without the constrain of the initial mesh and they also reveal the importance of data oscillation. But results in [11, 15, 16] only establish the qualitative convergence estimate by a proof of an error reduction property. The number of elements generated by the adaptive algorithm is not under control. A natural theoretical question is if a standard adaptive finite element scheme would give an optimal asymptotic convergence rate in terms of the number of elements. For linear finite element approximation to second order elliptic boundary value problems in two dimensions, for example, an optimal asymptotic error estimate would be something like

$$|u-u_N|_{1,\Omega} \le C(u)N^{-1/2}, \tag{1}$$

where u_N is a finite element approximation of the Poisson equation with homogenous Dirichlet boundary condition based on an adaptive grid with at most N elements.

An important progress has been made by Binev et al. [7] concerning the asymptotic estimate (1). In their algorithm, an additional coarsening step is required to achieve optimal complexity. However in practice the nearly optimal complexity

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is obtained without the coarsening step. Such theoretical gap is filled by Stevenson [18] which shows that the practical refinement without a recurrent coarsening will also generate finite element solution with quasi-optimal computational complexity. But marking for oscillation and refinement with interior nodes assumptions are still needed. Recently, [8] presented the most standard AFEM and proved a contraction property and quasi-optimal cardinality without any additional assumptions. Their results show that if the solution $u \in \mathscr{A}_s$, where $\mathscr{A}_s$ is the approximation class space of rate s, then $|u-u_N|_{1,\Omega} \le |u|_{\mathscr{A}_s} N^{-s}$.

Another important theoretical and practical issue is to characterize the approximation class $\mathscr{A}_{1/2}$ using the smoothness of u. A near characterization of $\mathscr{A}_{1/2}$ in terms of Besov spaces $B^k_{p,q}(\Omega)$ in two dimensions can be found in [6, 7] which shows that $u \in \mathscr{A}_{1/2}$ implies that $u \in B^2_{1,1}(\Omega)$ and $u \in B^2_{p,p}(\Omega)$ for $p > 1$ implies that $u \in \mathscr{A}_{1/2}$.

In this paper, we shall provide a sharper result: We prove that if $u \in W^{2,L\log L}(\Omega)$, i.e.,

$$\int_\Omega |D^2 u \log |D^2 u|| \, dx < \infty,$$

then $u \in \mathscr{A}_{1/2}$. This is an improved result since, when $p > 1$, $B^2_{p,p}(\Omega) \subset W^{2,L\log L}(\Omega)$ from the Hölder inequality. With the regularity theory of elliptic equations, which ensures $u \in W^{2,L\log L}(\Omega)$, we are led to conclude the following practical statement: linear adaptive finite element approximation of second order elliptic equations in two dimensions will achieve optimal rate of convergence.

Our contribution in this paper is further related with recent work on equidistribution and refinement strategies as follows:

1. The role of the equidistribution. In Sect. 2 we reveal that the equidistribution principle can be severely violated but asymptoticly optimal error estimates can still be maintained. The result (Theorem 1) is firstly presented in [9] and similar idea can be also found in [8] around the same time.
2. The proof of the bound of the pollution of the local mesh refinement in the completion is of its own interest. The estimate (Theorem 2) is a much sharper constant comparing with existing results in [7]. The idea of the proof is borrowed from [1] and the result is generalized from the uniform grids in [1] to compatible divisible unstructured grids.

The rest of the paper is organized as follows. In Sect. 2 we explain the equidistribution principle for the case when the function to be approximated belongs to $W^{2,1}(\Omega)$. The advantage of our approach is that only standard approximation for the interpolation operator are used, and approximation theory for Besov spaces is not needed. In Sect. 3, we review the newest vertex bisection refinement strategy and provide a sharp estimate for the number of triangle needed for the completion of the mesh after an arbitrary marking and bisection refinement is performed. In Sect. 4, we present a new approach for the local grid refinement based on the error estimate and the equidistribution principle.

2 Error Estimate and Equidistribution Principle

We shall consider a simple elliptic boundary value problem

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{2}$$

where, for simplicity, we assume Ω is a polygon and is partitioned by a shape regular conforming triangulation $\mathscr{T}_N$ with N number of triangles. Let $\mathscr{V}_N \subset H_0^1(\Omega)$ be the corresponding continuous piecewise linear finite element space associated with this triangulation $\mathscr{T}_N$.

A finite element approximation of the above problem is to find $u_N \in \mathscr{V}_N$ such that

$$a(u_N, v_N) = (f, v_N) \quad \forall v_N \in \mathscr{V}_N, \tag{3}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx, \text{ and } (f,v) = \int_\Omega f v\, dx.$$

For this problem, it is well known that for a fixed finite element space $\mathscr{V}_N$

$$|u - u_N|_{1,\Omega} = \inf_{v_N \in \mathscr{V}_N} |u - v_N|_{1,\Omega}. \tag{4}$$

We then present a H^1 error estimate for linear triangular element interpolation in two dimensions. We note that in two dimensions, the following two embeddings are both valid:

$$W^{2,1}(\Omega) \subset W^{1,2}(\Omega) \equiv H^1(\Omega) \text{ and } W^{2,1}(\Omega) \subset C(\bar{\Omega}). \tag{5}$$

Given $u \in W^{2,1}(\Omega)$, let u_I be the linear nodal value interpolant of u on $\mathscr{T}_N$. For any triangle $\tau \in \mathscr{T}_N$, thanks to (5) and the assumption that τ is shape-regular, we have

$$|u - u_I|_{1,\tau} \lesssim |u|_{2,1,\tau}.$$

As a result,

$$|u - u_I|_{1,\Omega}^2 \lesssim \sum_{\tau \in \mathscr{T}_N} |u|_{2,1,\tau}^2.$$

To minimize the error, we can try to minimize the right hand side. By Cauchy-Schwarz inequality,

$$|u|_{2,1,\Omega} = \sum_{\tau \in \mathscr{T}_N} |u|_{2,1,\tau} \le (\sum_{\tau \in \mathscr{T}_N} 1)^{1/2} (\sum_{\tau \in \mathscr{T}_N} |u|_{2,1,\tau}^2)^{1/2} = N^{1/2} (\sum_{\tau \in \mathscr{T}_N} |u|_{2,1,\tau}^2)^{1/2}.$$

Thus, we have the following lower bound:

$$(\sum_{\tau \in \mathscr{T}_N} |u|_{2,1,\tau}^2)^{1/2} \ge N^{-1/2} |u|_{2,1,\Omega}. \tag{6}$$

The equality holds if and only if

$$|u|_{2,1,\tau} = \frac{1}{N}|u|_{2,1,\Omega}. \tag{7}$$

The condition (7) is hard to be satisfied in general. But we can considerably relax this condition to ensure the lower bound estimate (6) is still achieved asymptotically. The relaxed condition is as follows:

$$|u|_{2,1,\tau} \leq \kappa_{\tau,N}|u|_{2,1,\Omega} \tag{8}$$

and

$$\sum_{\tau\in\mathscr{T}_N} \kappa_{\tau,N}^2 \leq c_1 N^{-1}. \tag{9}$$

When the above two inequalities hold, we have

$$|u-u_I|_{1,\Omega} \lesssim N^{-1/2}|u|_{2,1,\Omega}.$$

In summary, we have the following theorem.

Theorem 1. *If $\mathscr{T}_N$ is a triangulation with at most N triangles and satisfying (8) and (9), then*

$$|u-u_N|_1 \leq |u-u_I|_{1,\Omega} \lesssim N^{-1/2}|u|_{2,1,\Omega}. \tag{10}$$

In the above analysis, we see how equidistribution principle plays an important role in achieving asymptotically optimal accuracy for adaptive grids. We would like to further elaborate that, in the current setting, equidistribution is indeed a sufficient condition for optimal error, but by no means this has to be a necessary condition. Namely the equidistribution principle can be severely violated but asymptoticly optimal error estimates can still be maintained. For example, the following mild violation of this principle is certainly acceptable:

$$|u|_{2,1,\tau} \leq \frac{c}{N}|u|_{2,1,\Omega}. \tag{11}$$

In fact, this condition can be more significantly violated on a finitely many elements $\{\tau\}$

$$|u|_{2,1,\tau} \leq \frac{c}{\sqrt{N}}|u|_{2,1,\Omega}. \tag{12}$$

It is easy to see if a bounded number of elements satisfy (12) and the rest satisfy (11), the estimate (9) is satisfied and hence the optimal error estimate (10) is still valid.

As we can see that the condition (12) is a very serious violation of equidistribution principle, nevertheless, as long as such violations do not occur on too many elements, asymptotically optimal error estimates are still valid. This simple observation is important from both theoretical and practical points of view. The marking strategy proposed by Dörfler [11] may also be interpreted in this way in its relationship with equidistribution principle. In [5], they propose to use certain penalty in using equidistribution principle. Such a modification certainly has similar spirit.

We shall discuss how to generate a mesh $\mathscr{T}_N$ to satisfy (8) and (9) in the next two sections. To this end, we shall introduce the local refinement method: newest vertex bisection, in the next section.

3 Newest Vertex Bisection

In this section we shall give a brief introduction of the newest vertex bisection and mainly concern the number of elements added by the completion process. We refer to [14, 19] and [7] for detailed description of the newest vertex bisection refinement procedure.

Given an initial shape regular triangulation $\mathscr{T}_0$ of Ω, it is possible to assign to each $\tau \in \mathscr{T}_0$ exactly one vertex called *the newest vertex*. The opposite edge of the newest vertex is called *refinement edge*. The rule of the newest vertex bisection includes:

1. A triangle is divided to two new children triangles by connecting the newest vertex to the midpoint of the refinement edge;
2. The new vertex created at a midpoint of a refinement edge is assigned to be the newest vertex of the children.

It is easy to verify that all the descendants of an original triangle fall into four similarity classes (see Fig. 1) and hence the angles are bounded away from 0 and π and all triangulations refined from $\mathscr{T}_0$ using newest vertex bisection forms a shape regular class of triangulations.

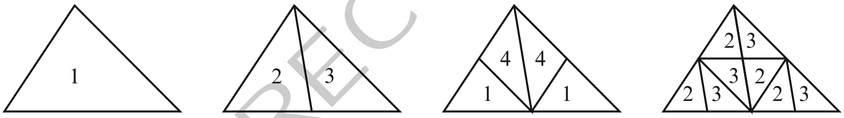

Fig. 1. Four similarity classes of triangles generated by the newest vertex bisection

The triangulation obtained by the newest vertex might have hanging nodes. We have to make additional subdivisions to eliminate the hanging nodes, i.e., complete the new partition. The completion should also follow the bisection rules. We shall consider more combinatory properties of the completion.

Let the triangles of the initial triangulation be assigned generation 0. We refer to the two triangles obtained by splitting a triangle τ in two sub-triangles by the newest vertex procedure as being the children of τ. For $i = 1, 2, \ldots$, we define the generation of the children of τ to be i if the parent τ has the generation $i-1$. It can be shown that the completion will terminate in finite steps, due to the fact that the completion process will not create new generations of triangles (see [3, 13]).

We ask more than the termination of the completion process. That is we want to control the number of elements refined due to the completion. To this end, we have to carefully assign the newest vertexs for the initial partition $\mathscr{T}_0$. A triangle is called *compatible divisible* if its refinement edge is either the refinement edge of the triangle that shares that edge or an edge on the boundary. A triangulation $\mathscr{T}$ is called *compatible divisible* or *compatible labled* if every triangle is compatible divisible. See Fig. 2 for an example of such compatible initial labeling.

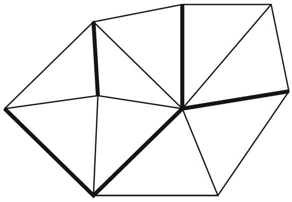

Fig. 2. A conforming divisible labeling of the initial triangulation where edges in *bold case* are refinement edges

It is obvious that the completion for a compatible triangulation is terminated in one step. Mitchell [13] proves that for any conforming triangulation $\mathscr{T}$, there exist a compatible label scheme. Biedl et al. [4] present an $O(N)$ algorithm to find a compatible labeling for a triangulation $\mathscr{T}$ with N elements.

Let $\mathscr{T}_0$ be a compatible triangulation and let $\mathscr{T}_{\frac{1}{2}}$ be a triangulation obtained by the newest vertex bisection by performing m_0 bisections starting from $\mathscr{T}_0$. Denote by $\mathscr{M}_0$ the set of all m_0 marked and split triangles. Note that not all the triangles of $\mathscr{M}_0$ have to be in $\mathscr{T}_0$. Let $\mathscr{T}_1$ be the (minimal) conforming refinement of $\mathscr{T}_{\frac{1}{2}}$ and denote by n_k the number of triangles of $\mathscr{T}_k$, $k = 0, 1$ (Fig. 3).

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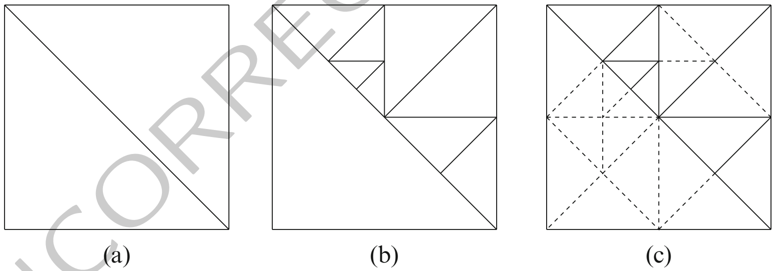


Fig. 3. Marking, splitting, and completing. **(a)** $\mathscr{T}_0$. **(b)** $\mathscr{T}_{\frac{1}{2}}$. **(c)** $\mathscr{T}_1$

Theorem 2. *Let $\mathscr{T}_0$ be a compatible triangulation and $\mathscr{T}_1$ be obtained as above. Then there exists a constant C only depending on the minimal angle of $\mathscr{T}_0$ such that*

$$n_1 \le n_0 + (C+1)\, m_0. \tag{13}$$

Remark 1. It is a temptation to repeat the Theorem 2 to conclude: for $j = 1, 2, \ldots, p-1$, we have that $\mathscr{T}_{j+1}$ is obtained from $\mathscr{T}_j$, by m_j markings and then minimal completion, then

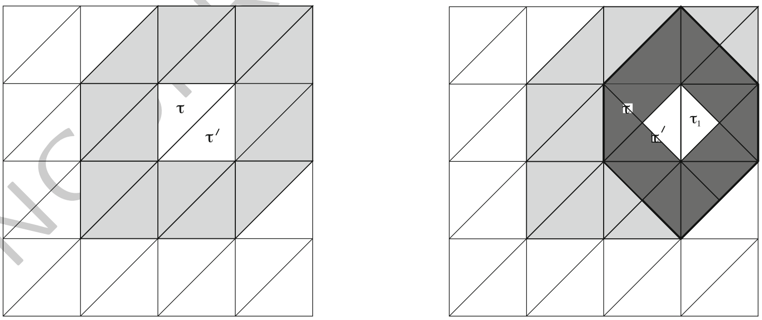
$$n_p \le n_0 + (C+1)\,(m_0 + m_1 + \cdots + m_{p-1}). \tag{14}$$

Unfortunately this argument does not work since $\mathscr{T}_1$ may not be compatible divisible anymore. The inequality (14) still holds but the proof is much involved; See Theorem 2.4 in [7]. The bound (13) can be derived from that theorem; See Lemma 2.5 in [7]. However, careful tracing the argument in [7] would give a huge constant in (14) in the magnitude of 10,000. We shall give another more direct and simpler proof based on an improved technique in [1]. The constant in our proof is much smaller and usually below 100. Note that numerically in the average case of the constant is around 4 and in the worst case is around 14; see [1].

Let us introduce notation for uniform bisection by setting $\overline{\mathscr{T}}_k$ as the triangulation obtained by bisecting each triangle in $\mathscr{T}_0$ completely up to the k-th generation. The assumption: $\mathscr{T}_0$ is compatible divisible implies that $\overline{\mathscr{T}}_k$ is conforming and compatible divisible for all $k \geq 1$. Note that this may not hold if the initial labeling is not compatible divisible.

For a triangle τ, we define a neighbor of τ as another triangle sharing a common edges of τ. By the definition, a triangle has at most three neighbors. Among them, for $\tau \in \overline{\mathscr{T}}_k$, we define the *refinement neighbor* of τ as the triangle $\tau' \in \overline{\mathscr{T}}_k$ such that τ and τ' use the same edge as their refinement edges. We allow $\tau' = \varnothing$ for τ touching the boundary. We define the *barrier* of τ as all triangles in $\overline{\mathscr{T}}_{g(\tau)}$ which intersect $\tau \cup \tau'$ and denoted by $B(\tau)$, i.e.,

$$B(\tau) = \{\hat{\tau} \in \overline{\mathscr{T}}_{g(\tau)}, \hat{\tau} \cap (\tau \cup \tau') \neq \varnothing\}.$$



(a) (b)

Fig. 4. Barrier of a safe triangle. **(a)** Barrier 1. **(b)** Barrier 2

Definition 1. *We say that τ is a* safe *triangle if none of the barrier elements of τ is marked in going from $\mathscr{T}_0$ to $\mathscr{T}_1$, namely $\hat{\tau} \notin \mathscr{M}_0$ for any $\hat{\tau} \in B(\tau)$.*

The following lemma will justify the name of safe triangles. They are triangles that not touched going from $\mathscr{T}_0$ to $\mathscr{T}_1$.

Lemma 1. *Any safe triangle τ in $\mathscr{T}_0$ or born in the marking and completion process of going from $\mathscr{T}_0$ to $\mathscr{T}_1$ will never be bisected during the completion process.*

Proof. We shall prove it by the induction over the generation of τ. Suppose $g(\tau) = \max_{\tilde{\tau}\in\mathscr{T}_{\frac{1}{2}}} g(\tilde{\tau})$ and τ is safe. Then τ will not be bisected during the completion since the completion will not increase the maximal generation.

Assume that our statement holds for all safe triangles of generation $p+1$. We will show that the statement also holds for a safe triangle with generation p. Note that to trigger the bisection of τ, one has to refine one of the two neighbors of τ (which do not share the refinement edge with τ) twice or two such neighbors of τ' twice (since τ and τ' share the refinement edge). Without loss of generality, let us say that one of the neighbor τ' is bisected once in the completion process. Then it produces a children triangle τ_1 of generation $p+1$ which has a common edge with τ'. It is important to note that $B(\tau_1) \subset B(\tau)$ and thus τ_1 is safe; See Fig. 4 for an illustration. By the inductive hypothesis τ_1 will never be bisected anymore during the completion process. Consequently, τ will never be bisected during the completion process.

Now we are in the position to prove Theorem 2.

Proof. (of Theorem 2) We denote by $\mathscr{M}_{\frac{1}{2}}$ as the set of all triangles τ which are split in the completion process of going from $\mathscr{T}_{\frac{1}{2}}$ to $\mathscr{T}_1$. Let us choose a triangle $\tau \in \mathscr{M}_{\frac{1}{2}}$. Since τ is split in the completion process, by the above Lemma, τ is not safe. It implies that there should exist a same-generation triangle $F(\tau)$ in $B(\tau)$ such that $F(\tau) \in \mathscr{M}_0$. In this way, we defined a map from $F : \mathscr{M}_{\frac{1}{2}} \to \mathscr{M}_0$.

Note that F is not necessary a one-to-one map, but a triangle $\tau \in \mathscr{M}_0$ could be in only finite number of barriers, due to the space limitation of the same-generation assumption. Given a triangle τ, we define the first ring of τ as all triangles intersect τ and the second ring of τ as the union of first rings of triangles in the first ring of τ. Then τ can be only in the barrier of triangles in its second ring and thus the number is bounded by the maximum number of triangles in the second ring of a triangle, say C, which is usually below 100. Thus any triangle in $\mathscr{M}_0$ is the image of at most C triangles from $\mathscr{M}_{\frac{1}{2}}$. This leads to the fact that the number of splittings needed for completion can be bounded by Cm_0. Since any splitting in the completion process adds one more triangle towards the completed mesh $\mathscr{T}_1$, we have proved (13).

4 Local Grid Refinement Algorithm

In this section we shall propose a new approach for the local grid refinement based on the error estimate and the equidistribution principle. We will use newest vertex bisection to refine the grid and use $|u|_{2,1,\tau}$ as an error indicator. With a little bit higher regularity requirement of u, we are able to prove the effectiveness of our algorithm. Namely, it will end with an optimal asymptotic error estimate similar to (1).

4.1 Local Refinement Strategy

We will illustrate a way to find a nearly optimal grid for the solution of (2). We will use the newest vertex bisection refinement procedure with the marking strategy given by (11). For the later analysis, we will have to assume that the solution u is in $W^{2,1}$ and that the Hardy-Littlewood maximal function of D^2u is in $L^1(\Omega)$. Due to a result of [17], this is equivalently $D^2u \in L\log L(\Omega)$. Such further assumption holds if for example $u \in W^{2,p}$ for some $p > 1$.

The maximal function of an integrable function f on Ω is defined by

$$\widetilde{M}f(x) = \sup \frac{1}{|Q|}\int_Q |f(y)|\, dy,$$

where the supremum is taken over all square domains contained in Ω and containing x.

For a triangulation obtained by the newest vertices bisection from $\mathscr{T}_0$. The similarity classes are in fact completely represented by the children and grandchildren of all triangles from $\mathscr{T}_0$. Let us denote by $\mathscr{C}_0$ the following family of triangles:

$$\begin{aligned}\mathscr{C}_0 = \{\tau|\ \tau \text{ is a triangle contained in } \Omega \text{ and is similar with}\\ \text{a child or grandchild of a triangle from } \mathscr{T}_0\}\end{aligned}$$

We define another maximal function

$$Mf(x) = \sup \frac{1}{|\tau|}\int_\tau |f(y)|\, dy,$$

where the supremum is taken over all triangles $\tau \in \mathscr{C}_0$ containing x. Then it is easy to show that $\widetilde{M}$ and M are equivalent in the sense that

$$c_1\widetilde{M}f(x) \le Mf(x) \le c_2\widetilde{M}f(x), \quad \forall x \in \Omega$$

with c_1 and c_2 independent of x. Thus, for theoretical purposes, the two operators M and $\widetilde{M}$ are interchangeable.

The following result concerns the number of the new triangles added in the refinement procedure. The main idea of the proof for the 1-D case was showed to the authors by DeVore and can be found in [10].

Theorem 3. *Let f be an integrable function on Ω such that $Mf \in L^1(\Omega)$, and let $\varepsilon > 0$ be given. Assume that the newest vertex bisection refinement procedure is applied to an compatible initial triangulation $\mathscr{T}_0$ with n_0 triangles. Let the marking strategy be given by: a triangle τ is marked if*

$$\int_\tau |f(x)|\, dx > \varepsilon.$$

Denote by $\mathscr{M}_0$ the set of all marked and split triangles. Then, the marking and refinement procedure will terminate in finite steps and we have

$$n_0 + m_0 < \frac{2}{\varepsilon} \int_\Omega Mf(x)\, dx, \tag{15}$$

where m_0 is the number of elements of $\mathscr{M}_0$. Assume that $\mathscr{T}_{\frac{1}{2}}$ is the triangulation obtained from $\mathscr{T}_0$ after the m_0 bisections. Let $\mathscr{T}_1$ be the (minimal) conforming refinement of $\mathscr{T}_{\frac{1}{2}}$ and denote by n_1 the number of triangles of $\mathscr{T}_1$. Then,

$$n_1 \le \frac{C_1}{\varepsilon} \int_\Omega |Mf(x)|\, dx, \tag{16}$$

with a constant C_1 independent of the function f and the number ε. More precisely, $C_1 = 2(C+1)$, with C the constant of Theorem 2.

Proof. Since $\lim_{|\tau|\to 0} \int_\tau |f(x)|\, dx = 0$ and the areas of new triangles are exponentially decreased, the refinement procedure will terminate in finite steps.

We can assume without loss of generality that each triangle in $\mathscr{T}_{\frac{1}{2}}$ is not a triangle in $\mathscr{T}_0$. Now, let $\tau \in \mathscr{T}_{\frac{1}{2}}$ and let $\tilde{\tau}$ be its parent. Then $\tilde{\tau} \in \mathscr{M}_0$. (Recall that $\mathscr{M}_0$ is the collection of marked triangles in the refinement procedure.) By our refinement strategy

$$\int_{\tilde{\tau}} |f(x)|\, dx > \varepsilon,$$

Thus,

$$Mf(x) > \frac{1}{|\tilde{\tau}|} \int_{\tilde{\tau}} |f(y)|\, dy > \frac{\varepsilon}{|\tilde{\tau}|}, \quad \forall x \in \tau.$$

Integrating the above inequality on τ we have,

$$\int_\tau Mf(x)\, dx > \frac{\varepsilon}{2}. \tag{17}$$

Here we use fact $|\tilde{\tau}| = 2|\tau|$. If we sum up (17) over all $n_0 + m_0$ triangles $\tau \in \mathscr{T}_{\frac{1}{2}}$ we obtain (15).

By using Theorem 2 we have that

$$n_1 \le n_0 + m_0 + C\, m_0 \le (C+1)\, (n_0 + m_0).$$

The estimate (16) follows now as a direct consequence of (15) and the above inequality.

An application of Theorem 1 and the estimate (16) for $f = D^2 u$ and $\varepsilon = 1/N$, leads to the proof of the existence of a nearly optimal grid. Starting from a coarse grid $\mathscr{T}_0$, we define the approximation class $\mathscr{A}_{1/2}$ as

$$\mathscr{A}_{1/2} = \{u \in H_0^1(\Omega) : |u|_{\mathscr{A}_{1/2}} := \sup_{N \ge \#\mathscr{T}_0} N^{-1/2} \inf_{\#\mathscr{T} \le N} \inf_{v_h \in V(\mathscr{T})} |u - v_h|_1 < \infty\}.$$

Corollary 1. *If $u \in W^{2, L\log L}(\Omega)$, then $u \in \mathscr{A}_{1/2}$.*

Remark 2. The $(L\log L)$ norm is needed only for proving the success of the algorithm but is not effectively needed for the implementation of the algorithm. If we can find good approximations or upper bound for $\int_\tau D^2 u\,dx$ on triangles using e.g., gradient and Hessian recovery methods (from the discrete Galerkin approximation of u) or using regularity result in [12], then the ideas presented in this paper can lead to new and optimal adaptive methods.

Bibliography

[1] F. B. Atalay and D. M. Mount. The cost of compatible refinement of simplex decomposition trees. In *Proceedings of the 15th International Meshing Roundtable, Birmingham, AL, September 2006*, pages 57–70, 2006.

[2] I. Babuška and W. C. Rheinboldt. A posteriori error estimates for the finite element method. *Internat. J. Numer. Methods Engrg.*, 12:1597–1615, 1978.

[3] E. Bänsch. Local mesh refinement in 2 and 3 dimensions. *Impact Comput. Sci. Engrg.*, 3:181–191, 1991.

[4] T. C. Biedl, P. Bose, E. D. Demaine, and A. Lubiw. Efficient algorithms for Petersen's matching theorem. *J. Algorithms*, 38(1):110–134, 2001. ISSN 0196-6774. doi: http://dx.doi.org/10.1006/jagm.2000.1132.

[5] P. Binev and R. DeVore. Fast computation in adaptive tree approximation. *Numer. Math.*, 97:193–217, 2004.

[6] P. Binev, W. Dahmen, R. DeVore, and P. Petrushev. Approximation classes for adaptive methods. *Serdica Math. J*, 28:391–416, 2002.

[7] P. Binev, W. Dahmen, and R. DeVore. Adaptive finite element methods with convergence rates. *Numer. Math.*, 97(2):219–268, 2004.

[8] J. M. Cascón, C. Kreuzer, R. H. Nochetto, and K. G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method. *SIAM J. Numer. Anal.*, 46(5):2524–2550, 2008.

[9] L. Chen and J. Xu. Convergence of adaptive finite element methods. In T. Tang and J. Xu, editors, *Adaptive Computations: Theory and Algorithms*, pages 9–37. Science Press, Beijing, 2007.

[10] R. A. DeVore. Nonlinear approximation. *Acta Numer.*, pages 51–150, 1998.

[11] W. Dörfler. A convergent adaptive algorithm for Poisson's equation. *SIAM J. Numer. Anal.*, 33:1106–1124, 1996.

[12] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman, Boston, 1995.

[13] W. F. Mitchell. *Unified Multilevel Adaptive Finite Element Methods for Elliptic Problems*. PhD thesis, University of Illinois at Urbana-Champaign, 1988.

[14] W. F. Mitchell. A comparison of adaptive refinement techniques for elliptic problems. *ACM Trans. Math. Softw. (TOMS) archive*, 15(4):326–347, 1989.

[15] P. Morin, R. H. Nochetto, and K. G. Siebert. Data oscillation and convergence of adaptive FEM. *SIAM J. Numer. Anal.*, 38(2):466–488, 2000.

[16] P. Morin, R. H. Nochetto, and K. G. Siebert. Convergence of adaptive finite element methods. *SIAM Rev.*, 44(4):631–658, 2002.

[17] R. M. Stein. Note on the class L(logL). *Studia. Math.*, 32:305–310, 1969.

[18] R. Stevenson. Optimality of a standard adaptive finite element method. *Found. Comput. Math.*, 7(2):245–269, 2007.

[19] R. Verfürth. *A Review of A Posteriori Error Estimation and Adaptive Mesh Refinement Tecniques*. B. G. Teubner, 1996.