A Nonoverlapping DD Preconditioner for a Weakly Over-Penalized Symmetric Interior Penalty Method

Andrew T. Barker[1], Susanne C. Brenner[2], Eun-Hee Park[3], and Li-Yeng Sung[4]

[1] Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA andrewb@math.lsu.edu
[2] Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA brenner@math.lsu.edu
[3] Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA epark2@math.lsu.edu
[4] Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA sung@math.lsu.edu

1 Introduction

In this paper we present a nonoverlapping domain decomposition preconditioner for a weakly over-penalized symmetric interior penalty method that is based on balancing domain decomposition by constraints (BDDC) methodology (cf. [2, 5, 7, 8]). The full analysis of the preconditioner can be found in [4].

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ and $f \in L_2(\Omega)$. Consider the following model problem:
Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in H_0^1(\Omega). \tag{1}$$

Let $\mathscr{T}_h$ be a quasi-uniform triangulation of Ω, where the mesh parameter h measures the maximum diameter of the triangles in $\mathscr{T}_h$, and let

$$V_h = \{v \in L_2(\Omega) : v|_T \in P_1(T) \quad \forall T \in \mathscr{T}_h\}$$

be the discontinuous P_1 finite element function space associated with $\mathscr{T}_h$. The model problem (1) can be discretized by the following weakly over-penalized symmetric interior penalty (WOPSIP) method (cf. [3, 9]):
Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad v \in V_h,$$

where

R. Bank et al. (eds.), *Domain Decomposition Methods in Science and Engineering XX*,
Lecture Notes in Computational Science and Engineering 91,
DOI 10.1007/978-3-642-35275-1_28, © Springer-Verlag Berlin Heidelberg 2013

$$a_h(v,w) = \sum_{T\in\mathscr{T}_h}\int_T \nabla v\cdot\nabla w\,dx + \sum_{e\in\mathscr{E}_h}\frac{1}{|e|^3}\int_e \Pi_e^0[\![v]\!]\cdot\Pi_e^0[\![w]\!]\,ds, \qquad (2)$$

$\mathscr{E}_h$ is the set of the edges of $\mathscr{T}_h$, $|e|$ is the length of the edge e, $[\![v]\!]$ denotes the jump of v across the edges, and Π_e^0 is the orthogonal projection from $[L_2(e)]^2$ onto $[P_0(e)]^2$. $P_0(e)$ denotes the space of constant functions on the edge e.

For simplicity in presentation, we consider the Poisson model on conforming meshes. But the results can be extended to heterogeneous elliptic problems on nonconforming meshes (cf. [4]). We note that BDDC technique was used in [6] to couple conforming finite element spaces from different subdomains that allows nonmatching meshes across subdomain boundaries, where condition number estimates independent of the coefficients were obtained for heterogeneous elliptic problems. The main difference between [6] and this paper is that the finite element functions in this paper can be discontinuous at the element boundaries.

The rest of the paper is organized as follows. In Sect. 2 we introduce a subspace decomposition. We then design a BDDC preconditioner for the reduced problem in Sect. 3. The condition number estimate is also presented. In Sect. 4 we report numerical results that illustrate the performance of the proposed preconditioner and confirm the theoretical estimates.

Throughout the paper we will use $A \lesssim B$ and $A \gtrsim B$ to represent the statements that $A \le (\text{constant})B$ and $A \ge (\text{constant})B$, where the positive constant is independent of the mesh size, the subdomain size, and the number of subdomains. The statement $A \approx B$ is equivalent to $A \lesssim B$ and $A \gtrsim B$.

2 A Subspace Decomposition

In this section we propose an intermediate preconditioner for the WOPSIP method, which is based on a subspace decomposition.

Let $\Omega_1,\ldots,\Omega_J$ be a nonoverlapping partition of Ω aligned with $\mathscr{T}_h$ and $\Gamma = \left(\bigcup_{j=1}^J \partial\Omega_j\right)\setminus\partial\Omega$ be the interface of the subdomains. We assume that the subdomains are shape regular polygons (cf. [1, Sect. 7.5]). We denote the diameter of Ω_j by H_j and define H to be $\max_{1\le j\le J} H_j$. $\mathscr{E}_{h,\Gamma}$ is the subset of $\mathscr{E}_h$ containing the edges on Γ.

First we decompose V_h into two subspaces as follows:

$$V_h = V_{h,C}\oplus V_{h,D},$$

where

$$\begin{aligned} V_{h,C} &= \{v\in V_h : [\![v]\!] = 0 \text{ at the midpoints of the edges on the boundaries}\\ &\qquad \text{of the subdomains}\},\\ V_{h,D} &= \big\{v\in V_h : \{\!\{v\}\!\} = 0 \text{ at the midpoints of the edges in } \mathscr{E}_{h,\Gamma} \text{ and}\\ &\qquad v = 0 \text{ at the midpoints of the edges in } \Omega\setminus\Gamma\big\}. \end{aligned}$$

Here $\{\!\{v\}\!\}$ denotes the average of v from the two sides of an edge in $\mathscr{E}_{h,\Gamma}$.

Let $A_h : V_h \longrightarrow {V_h}'$ be the symmetric positive-definite (SPD) operator defined by

$$\langle A_h v, w\rangle = a_h(v,w) \qquad \forall\, v,w \in V_h,$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form between a vector space and its dual. Similarly, we define $A_{h,D} : V_{h,D} \longrightarrow V'_{h,D}$ and $A_{h,C} : V_{h,C} \longrightarrow V'_{h,C}$ by

$$\langle A_{h,D} v, w\rangle = a_h(v,w) \qquad \forall\, v,w \in V_{h,D}, \tag{3}$$
$$\langle A_{h,C} v, w\rangle = a_h(v,w) \qquad \forall\, v,w \in V_{h,C}. \tag{4}$$

Given any $v \in V_h$, we have a unique decomposition $v = v_D + v_C$ where $v_D \in V_{h,D}$ and $v_C \in V_{h,C}$. Then based on the definitions of the subspaces $V_{h,D}$ and $V_{h,C}$, it can be shown that

$$\langle A_h v, v\rangle \approx \langle A_{h,D} v_D, v_D\rangle + \langle A_{h,C} v_C, v_C\rangle \qquad \forall\, v \in V_h. \tag{5}$$

Remark 1. Since functions in $V_{h,C}$ are continuous at the midpoints of the edges in $\mathscr{E}_{h,\Gamma}$, we have

$$a_h(v,w) = \sum_{j=1}^{J} a_{h,j}(v_j, w_j) \qquad \forall\, v,w \in V_{h,C}, \tag{6}$$

where $v_j = v\big|_{\Omega_j}$, $w_j = w\big|_{\Omega_j}$ and

$$a_{h,j}(v_j, w_j) = \sum_{\substack{T\in\mathscr{T}_h\\ T\subset\Omega_j}} \int_T \nabla v_j \cdot \nabla w_j \, dx + \sum_{\substack{e\in\mathscr{E}_h\\ e\subset\Omega_j}} \frac{1}{|e|^3} \int_e \Pi_e^0[\![v_j]\!] \cdot \Pi_e^0[\![w_j]\!] \, ds. \tag{7}$$

Note that the second sum on the right-hand side of (7) is over the edges interior to Ω_j and therefore $a_{h,j}(\cdot,\cdot)$ is a localized bilinear form. The introduction of the subspace decomposition where the bilinear form can be localized as shown in (6) and (7) is the key ingredient in designing our preconditioner in Sect. 3.

Next we decompose $V_{h,C}$ into two subspaces $V_{h,C}(\Omega\setminus\Gamma)$ and $V_{h,C}(\Gamma)$ defined as follows:

$$V_{h,C}(\Omega\setminus\Gamma) = \{v \in V_{h,C} : v = 0 \text{ at all the midpoints of the edges in } \mathscr{E}_{h,\Gamma}\},$$
$$V_{h,C}(\Gamma) = \{v \in V_{h,C} : a_h(v,w) = 0 \quad \forall w \in V_{h,C}(\Omega\setminus\Gamma)\}.$$

The space $V_{h,C}(\Gamma)$ is the space of discrete harmonic functions, which are uniquely determined by their values at the midpoints of the edges in $\mathscr{E}_{h,\Gamma}$.

Let the SPD operators $A_{h,\Omega\setminus\Gamma} : V_{h,C}(\Omega\setminus\Gamma) \longrightarrow V_{h,C}(\Omega\setminus\Gamma)'$ and $S_h : V_{h,C}(\Gamma) \longrightarrow V_{h,C}(\Gamma)'$ be defined by

$$\langle A_{h,\Omega\setminus\Gamma} v, w\rangle = a_h(v,w) \qquad \forall\, v,w \in V_{h,C}(\Omega\setminus\Gamma),$$
$$\langle S_h v, w\rangle = a_h(v,w) \qquad \forall\, v,w \in V_{h,C}(\Gamma).$$

Note that given any $v_C \in V_{h,C}$, we have a unique decomposition $v_C = v_{C,\Omega\setminus\Gamma} + v_{C,\Gamma}$ where $v_{C,\Omega\setminus\Gamma} \in V_{h,C}(\Omega \setminus \Gamma)$ and $v_{C,\Gamma} \in V_{h,C}(\Gamma)$. It follows from the definitions of $V_{h,C}(\Omega \setminus \Gamma)$ and $V_{h,C}(\Gamma)$ that

$$\langle A_{h,C} v_C, v_C\rangle = \langle A_{h,\Omega\setminus\Gamma} v_{C,\Omega\setminus\Gamma}, v_{C,\Omega\setminus\Gamma}\rangle + \langle S_h v_{C,\Gamma}, v_{C,\Gamma}\rangle \qquad \forall\, v_C \in V_{h,C}. \tag{8}$$

Based on the relations (5) and (8), we define a preconditioner $B_1 : V_h{}' \longrightarrow V_h$ for A_h by

$$B_1 = I_D A_{h,D}^{-1} I_D^t + I_{h,\Omega\setminus\Gamma} A_{h,\Omega\setminus\Gamma}^{-1} I_{h,\Omega\setminus\Gamma}^t + I_\Gamma S_h^{-1} I_\Gamma^t,$$

where $I_D : V_{h,D} \longrightarrow V_h$, $I_{h,\Omega\setminus\Gamma} : V_{h,C}(\Omega \setminus \Gamma) \longrightarrow V_h$, and $I_\Gamma : V_{h,C}(\Gamma) \longrightarrow V_h$ are natural injections.

It follows from (5) and (8) that

$$\kappa(B_1 A_h) = \frac{\lambda_{\max}(B_1 A_h)}{\lambda_{\min}(B_1 A_h)} \approx 1. \tag{9}$$

Remark 2. Let us observe the properties of the preconditioner B_1 from the implementational point of view. First it is easy to implement the solve $A_{h,D}^{-1}$ because $A_{h,D}$ is a block diagonal matrix with small blocks. Next in view of (6) and (7), the solve $A_{h,\Omega\setminus\Gamma}^{-1}$ can be implemented by solving independent subdomain problems in parallel. On the other hand, noting that S_h is a global solve, we need to design a good preconditioner for S_h in order to obtain a good parallel preconditioner for A_h.

3 A BDDC Preconditioner

In this section we propose a preconditioner for the Schur complement operator S_h based on the BDDC methodology.

Let $V_{h,j}$ be the space of discontinuous P_1 finite element functions on Ω_j that vanish at the midpoints of the edges on $\partial\Omega_j \cap \partial\Omega$, and $V_h(\Omega_j)$ be the subspace of $V_{h,j}$ whose members vanish at the midpoints of the edges on $\partial\Omega_j$. We denote by $\mathscr{H}_j$ the space of local discrete harmonic functions defined by

$$\mathscr{H}_j = \left\{ v \in V_{h,j} : a_{h,j}(v,w) = 0 \quad \forall\, w \in V_h(\Omega_j) \right\}.$$

The space $\mathscr{H}_m$ is defined by gluing the spaces $\mathscr{H}_j$ together along the interface Γ through enforcing the continuity of the mean values on the common edges of subdomains:

$$\mathscr{H}_m = \{ v \in L_2(\Omega) : v_j = v|_{\Omega_j} \in \mathscr{H}_j \text{ for } 1 \le j \le J \\ \text{and} \int_{\partial\Omega_j \cap \partial\Omega_k} v_j\, ds = \int_{\partial\Omega_j \cap \partial\Omega_k} v_k\, ds \text{ for } 1 \le j,k \le J \},$$

and we equip $\mathscr{H}_m$ with the bilinear form

$$a_h^m(v,w) = \sum_{1 \le j \le J} a_{h,j}(v_j, w_j).$$

Let $\mathscr{E}_H$ be the set of the edges of the subdomains $\Omega_1, \cdots, \Omega_J$. The BDDC preconditioner is based on a decomposition of $\mathscr{H}_m$ into orthogonal subspaces with respect to $a_h^m(\cdot,\cdot)$:

$$\mathscr{H}_m = \mathring{\mathscr{H}} \oplus \mathscr{H}_0, \tag{10}$$

where

$$\mathring{\mathscr{H}} = \left\{ v \in \mathscr{H}_m : \int_E v\, ds = 0 \quad \forall E \in \mathscr{E}_H \right\}$$

and

$$\mathscr{H}_0 = \left\{ v \in \mathscr{H}_m : a_h^m(v,w) = 0 \quad \forall w \in \mathring{\mathscr{H}} \right\}. \tag{11}$$

Then we equip $\mathscr{H}_0$ and the localized subspaces $\mathring{\mathscr{H}}_j$ $(1 \le j \le J)$ of $\mathring{\mathscr{H}}$:

$$\mathring{\mathscr{H}}_j = \left\{ v \in \mathscr{H}_j : \int_E v\, ds = 0 \text{ for all the edges } E \text{ of } \Omega_j \right\},$$

with the SPD operators $S_0 : \mathscr{H}_0 \longrightarrow \mathscr{H}_0'$ and $S_j : \mathring{\mathscr{H}}_j \longrightarrow \mathring{\mathscr{H}}_j'$ defined by

$$\langle S_0 v, w \rangle = a_h^m(v,w) \qquad \forall v, w \in \mathscr{H}_0, \tag{12}$$

$$\langle S_j v, w \rangle = a_{h,j}(v,w) \qquad \forall v, w \in \mathring{\mathscr{H}}_j. \tag{13}$$

Note that $V_{h,C}(\Gamma)$ is a subspace of $\mathscr{H}_m$ and there exists a projection $P_\Gamma : \mathscr{H}_m \to V_{h,C}(\Gamma)$ defined by averaging:

$$(P_\Gamma v)(m_e) = \{\!\{v\}\!\}\,(m_e) \quad \forall e \in \mathscr{E}_{h,\Gamma},$$

where m_e is the midpoint of e. The operator P_Γ connects the BDDC preconditioner based on $\mathscr{H}_m$ to the Schur complement operator S_h on $V_{h,C}(\Gamma)$.

We can now define the BDDC preconditioner $B_{BDDC} : V_{h,C}(\Gamma)' \longrightarrow V_{h,C}(\Gamma)$ for the Schur complement operator $S_h : V_{h,C}(\Gamma) \longrightarrow V_{h,C}(\Gamma)'$ as follows:

$$B_{BDDC} = (P_\Gamma I_0)\, S_0^{-1} \,(P_\Gamma I_0)^t + \sum_{j=1}^{J} (P_\Gamma \mathbb{E}_j)\, S_j^{-1}\, (P_\Gamma \mathbb{E}_j)^t,$$

where I_0 is the natural injection of $\mathscr{H}_0$ into $\mathscr{H}_m$ and $\mathbb{E}_j : \mathring{\mathscr{H}}_j \longrightarrow \mathring{\mathscr{H}}$ is the trivial extension defined by

$$\mathbb{E}_j \mathring{v}_j = \begin{cases} \mathring{v}_j & \text{on } \Omega_j \\ 0 & \text{on } \Omega \setminus \Omega_j \end{cases} \qquad \forall \mathring{v}_j \in \mathring{\mathscr{H}}_j.$$

We then obtain the preconditioner $B_2 : V_h' \longrightarrow V_h$ for A_h by replacing the global solve S_h^{-1} in (2) with the preconditioner B_{BDDC}:

$$B_2 = I_D A_{h,D}^{-1} I_D^t + I_{h,\Omega\setminus\Gamma} A_{h,\Omega\setminus\Gamma}^{-1} I_{h,\Omega\setminus\Gamma}^t + I_\Gamma B_{BDDC} I_\Gamma^t.$$

We can analyze the condition number of $B_{BDDC} S_h$ by the theory of additive Schwarz preconditioners (cf. [1, 10, 11], and the references therein). The proof of the following result can be found in [4].

Lemma 1. *We have the following bounds for the eigenvalues of* $B_{BDDC}S_h$

$$\lambda_{\min}(B_{BDDC}S_h) \geq 1,$$
$$\lambda_{\max}(B_{BDDC}S_h) \lesssim \left(1+\ln\frac{H}{h}\right)^2.$$

Combining (5), (8) and Lemma 1, we have the following estimate of the condition number of the preconditioned system B_2A_h.

Theorem 1. *There exists a positive constant* C*, independent of* h, H *and* J*, such that*

$$\kappa(B_2A_h) = \frac{\lambda_{\max}(B_2A_h)}{\lambda_{\min}(B_2A_h)} \leq C\left(1+\ln\frac{H}{h}\right)^2.$$

4 Numerical Results

In this section we present some numerical results that illustrate the performance of the preconditioners B_1 and B_2.

We consider the model problem (1) on the unit square $(0,1)^2$ with the exact solution $u(x,y) = y(1-y)\sin(\pi x)$. We use a uniform triangulation $\mathscr{T}_h$ of isosceles right triangles, where the mesh parameter h represents the length of the horizontal/vertical edges. The domain Ω is divided into J nonoverlapping squares aligned with $\mathscr{T}_h$ and the length of the horizontal/vertical edges of the squares is denoted by H. The discrete problem obtained by the WOPSIP method is solved by the preconditioned conjugate gradient method. The iteration is stopped when the relative residual is less than 10^{-6}.

Numerical results for the preconditioners B_1 and B_2 are presented in Table 1, which confirm the theoretical estimates in (9) and Theorem 1.

Table 1. Results for the preconditioners B_1 and B_2 with $J = 4^2$

h	H/h	B_1A_h			B_2A_h		
		κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$
2^{-3}	2	1.4206	8.2624e-1	1.1738	1.4478	8.2623e-1	1.1962
2^{-4}	4	1.1916	9.1258e-1	1.0874	1.7782	9.1300e-1	1.6235
2^{-5}	8	1.0919	9.5608e-1	1.0439	2.3215	9.5673e-1	2.2211
2^{-6}	16	1.0433	9.7880e-1	1.0212	3.0490	9.7994e-1	2.9879

We present in Table 2 the iteration counts and total time to solution for a parallel implementation of our preconditioner. For comparison, results on a single processor of the same machine without preconditioning are also presented for $J = 1$. The three operations $A_{h,D}^{-1}, A_{h,\Omega\setminus\Gamma}^{-1}$, and B_{BDDC} are performed one after the other, sequentially, but each of these operators is evaluated in parallel on the decomposed domain with one subdomain per processor. Iteration counts are consistent with our theory and confirm again that the method is scalable, and the running times show good parallel speedup for large problems.

Table 2. Parallel performance of the preconditioner B_2

h	$J=1$		$J=4^2, H=2^{-2}$		$J=8^2, H=2^{-3}$		$J=16^2, H=2^{-4}$	
	Its	Wall clock time	Its	Wall clock time	Its	Wall clock time	Its	Wall clock time
2^{-6}	235	0.46	7	0.37	7	0.5	5	1.14
2^{-7}	450	3.75	8	2.22	8	1.06	6	1.96
2^{-8}	884	35.45	9	20.12	8	4.35	6	2.71
2^{-9}	1786	319.0	8	126.15	8	27.15	7	7.81

The numbers $\kappa(B_2A_h)/(1+\ln(H/h))^2$ and $\kappa(B_{BDDC}S_h)/(1+\ln(H/h))^2$ are plotted against H/h in Fig. 1. As H/h increases these two numbers settle down to around 0.2, which indicates that the estimates in Lemma 1 and Theorem 1 are sharp.

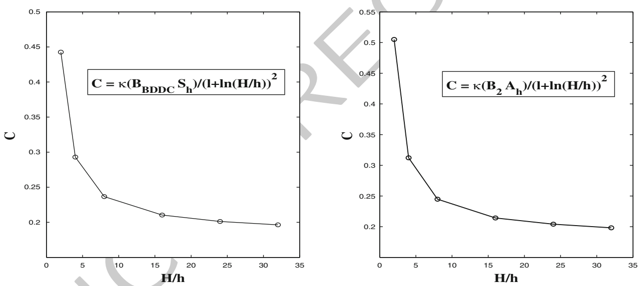

Fig. 1. *Left figure*: the behavior of $C=\kappa(B_{BDDC}S_h)/(1+\ln(H/h))^2$ for the BDDC preconditioner; *right figure*: the behavior of $C=\kappa(B_2A_h)/(1+\ln(H/h))^2$ for the preconditioner B_2

Acknowledgments The work of the first author was supported in part by the National Science Foundation under Grant No. DMS-07-39382. The work of the second and forth authors was supported in part by the National Science Foundation under Grant No. DMS-10-16332. The work of the third author was supported in part by the National Research Foundation of Korea under Grant No. NRF-2009-352-C00009. The work of all four authors was supported in part by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation.

Bibliography

[1] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods (Third Edition)*. Springer, 2008.

[2] S. C. Brenner and L.-Y. Sung. BDDC and FETI-DP without matrices or vectors. *Comput. Methods Appl. Mech. Engrg.*, 196:1429–1435, 2007.

[3] S. C. Brenner, L. Owens, and L.-Y. Sung. A weakly over-penalized symmetric interior penalty method. *Electron. Trans. Numer. Anal.*, 30:107–127, 2008.

[4] S. C. Brenner, E.-H. Park, and L.-Y. Sung. A BDDC preconditioner for a weakly over-penalized symmetric interior penalty method. *preprint*, 2011.

[5] C. R. Dohrmann. A preconditioner for substructuring based on constrained energy minimization. *SIAM J. Sci. Comput.*, 25:246–258, 2003.

[6] M. Dryja, J. Galvis, and M. Sarkis. BDDC methods for discontinuous Galerkin discretization of elliptic problems. *J. Complex.*, 23:715–739, 2007.

[7] J. Li and O. B. Widlund. FETI-DP, BDDC, and block Cholesky methods. *Internat. J. Numer. Methods Engrg.*, 66:250–271, 2006.

[8] J. Mandel and C. R. Dohrmann. Convergence of a balancing domain decomposition by constraints and energy minimization. *Numer. Linear Algebra Appl.*, 10:639–659, 2003.

[9] L. Owens. *Multigrid methods for weakly over-penalized interior penalty methods*. Ph.D. thesis, University of South Carolina, 2007.

[10] B. Smith, P. Bjørstad, and W. Gropp. *Domain Decomposition*. Cambridge University Press, Cambridge, 1996.

[11] A. Toselli and O. B. Widlund. *Domain Decomposition Methods - Algorithms and Theory*. Springer-Verlag, Berlin, 2005.