Fast Summation Techniques for Sparse Shape Functions in Tetrahedral *hp*-FEM

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Summary. This paper considers the *hp*-finite element discretization of an elliptic boundary value problem using tetrahedral elements. The discretization uses a polynomial basis in which the number of nonzero entries per row is bounded independently of the polynomial degree. The authors present an algorithm which computes the nonzero entries of the stiffness matrix in optimal complexity. The algorithm is based on sum factorization and makes use of the nonzero pattern of the stiffness matrix.

1 Introduction

hp-finite element methods (*hp*-FEM), see e.g. [6, 9], have become very popular for the approximation of solutions of boundary value problems with more regularity. In order to obtain the approximate finite element solution numerically stable and fast, the functions have to be chosen properly in *hp*-FEM. For quadrilateral and hexahedral elements, tensor products of integrated Legendre polynomials are the prefered basis functions, see [2]. For triangular and tetrahedral elements, the element can be considered as collapsed quadrilateral or hexahedron. This allows us to use tensor product functions. In order to obtain sparsity in the element matrices and a moderate increase of the condition number, integrated Jacobi polynomials can be used, see [3, 5, 7]. Then, it has been shown that the element stiffness and mass matrix have a bounded number of nonzero entries per row, see [3–5] which results in a total number of $\mathscr{O}(p^d)$, $d = 2,3$, nonzero entries in two and three space dimensions, respectively. However, the explicit computation of the nonzero entries is very involved.

This paper presents an algorithm which computes the element stiffness and mass matrices in $\mathscr{O}(p^3)$ operations in two and three space dimensions. The algorithm combines ideas based on sum factorization, [8], with the sparsity pattern of the matrices. One other important ingredient is the fast evaluation of the Jacobi polynomials.

The outline of this paper is as follows. Section 2 defines H^1-conforming, i.e. globally continuous piecewise polynomials, basis functions on the tetrahedron. The

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sum factorization algorithm is presented in Sect. 3. Section 4 is devoted to the evaluation of the Jacobi polynomials. The complexity of the algorithm is estimated in Sect. 5.

2 Definition of the Basis Functions

For the definition of our basis functions Jacobi polynomials are required. Let

$$p_n^\alpha(x) = \frac{1}{2^n n!(1-x)^\alpha} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((1-x)^\alpha (x^2-1)^n\right) \quad n \in \mathbb{N}_0,\ \alpha, \beta > -1 \tag{1}$$

be the nth Jacobi polynomial with respect to the weight function $(1-x)^\alpha$. The function p_n^α is a polynomial of degree n, i.e., $p_n^\alpha \in \mathbb{P}_n((-1,1))$, where $\mathbb{P}_n(I)$ is the space of all polynomials of degree n on the interval I. In the special case $\alpha = 0$, the functions $p_n^0(x)$ are called Legendre polynomials. Moreover, let

$$\hat{p}_n^\alpha(x) = \int_{-1}^{x} p_{n-1}^\alpha(y)\,\mathrm{d}y \quad n \geq 1, \quad \hat{p}_0^\alpha(x) = 1 \tag{2}$$

be the nth integrated Jacobi polynomial. Several relations are known between the different families of Jacobi polynomials, see e.g. [1]. In this paper, the relations

$$p_n^{\alpha-1}(x) = \frac{1}{\alpha+2n}\left[(\alpha+n)p_n^\alpha(x) - n p_{n-1}^\alpha(x)\right], \tag{3}$$

$$\begin{aligned}\hat{p}_{n+1}^\alpha(x) = {} & \frac{2n+\alpha-1}{(2n+2)(n+\alpha)(2n+\alpha-2)} \\ & \times\left((2n+\alpha-2)(2n+\alpha)x + \alpha(\alpha-2)\right)\hat{p}_n^\alpha(x) \\ & - \frac{(n-1)(n+\alpha-2)(2n+\alpha)}{(n+1)(n+\alpha)(2n+\alpha-2)}\hat{p}_{n-1}^\alpha(x), \quad n \geq 1.\end{aligned} \tag{4}$$

are required.

Let $\hat{\triangle}$ be the reference tetrahedron with the four vertices at $(-1,-1,-1)$, $(1,-1,-1)$, $(0,1,-1)$ and $(0,0,1)$. On this element, the interior bubble functions

$$\phi_{ijk}(x,y,z) = u_i(x,y,z)v_{ij}(y,z)w_{ijk}(z), \quad i \geq 2,\ \ j,k \geq 1, i+j+k \leq p \tag{5}$$

are proposed for H^1 elliptic problems in [3, (29)], where the auxiliary functions are

$$\begin{aligned} u_i(x,y,z) &= \hat{p}_i^0\left(\frac{4x}{1-2y-z}\right)\left(\frac{1-2y-z}{4}\right)^i, \\ v_{ij}(y,z) &= \hat{p}_j^{2i-1}\left(\frac{2y}{1-z}\right)\left(\frac{1-z}{2}\right)^j, \\ w_{ijk}(z) &= \hat{p}_k^{2i+2j-2}(z). \end{aligned}$$

In addition, there are vertex, face and edge based basis functions which can be regarded as special cases of the above functions (5) for limiting cases of the indices i, j and k, see [3] for more details.

Then, the element stiffness matrix for the Laplacian on the reference element $\hat{\triangle}$ with respect to the interior bubbles reads as

$$\mathscr{K} = \left[\int_{\hat{\triangle}} \nabla \phi_{ijk}(x,y,z) \cdot \nabla \phi_{i'j'k'}(x,y,z) \, \mathrm{d}(x,y,z) \right]_{i,j,k \le p, i'+j'+k' \le p} . \tag{6}$$

The transformation to the unit cube $(-1,1)^3$ (Duffy trick) and the evaluation of the nabla operation results in the integration of 21 different summands. More precisely,

$$\mathscr{K} = \sum_{m=1}^{21} \kappa_m \hat{\mathscr{I}}^{(m)}$$

with known numbers $\kappa_m \in \mathbb{R}$ and

$$\begin{aligned} \hat{\mathscr{I}}^{(m)} = & \left[\int_{-1}^{1} p_{x,1}(x) p_{x,2}(x) \, dx \right. \\ & \times \int_{-1}^{1} \left(\frac{1-y}{2} \right)^{\gamma_y} p_{y,1}(y) p_{y,2}(y) \, \mathrm{d}y \\ & \times \left. \int_{-1}^{1} \left(\frac{1-z}{2} \right)^{\gamma_z} p_{z,1}(z) p_{z,2}(z) \, \mathrm{d}z \right]_{i+j+k<p; i'+j'+k'<p} . \end{aligned}$$

The structure of the functions and coefficients is displayed in Table 1.

One summand is the term

$$\hat{\mathscr{I}}^{(6)} = \left(m_{ijk,i'j'k'} \right)_{i+j+k \le p, i'+j'+k' \le p} \tag{7}$$

which corresponds (before the Duffy trick) to

$$\begin{aligned} m_{ijk,i'j'k'} = & \int_{\hat{\triangle}} \hat{p}_i^0 \left(\frac{4x}{1-2y-z} \right) \hat{p}_{i'}^0 \left(\frac{4x}{1-2y-z} \right) \left(\frac{1-2y-z}{4} \right)^{i+i'} \\ & \times \hat{p}_j^{2i-1} \left(\frac{2y}{1-z} \right) \hat{p}_{j'}^{2i'-1} \left(\frac{2y}{1-z} \right) \left(\frac{1-z}{2} \right)^{j+j'} \\ & \times p_{k-1}^{2i+2j-2}(z) p_{k'-1}^{2i'+2j'-2}(z) \, \mathrm{d}(x,y,z). \end{aligned}$$

The Duffy transformation applied to (7) gives

$$\begin{aligned} m_{ijk,i'j'k'} = & \int_{-1}^{1} \hat{p}_i^0(x) \hat{p}_{i'}^0(x) \, dx \int_{-1}^{1} \left(\frac{1-y}{2} \right)^{i+i'+1} \hat{p}_{j'}^{2i'-1}(y) \hat{p}_j^{2i-1}(y) \, \mathrm{d}y \\ & \times \int_{-1}^{1} \left(\frac{1-z}{2} \right)^{i+j+i'+j'+2} p_{k-1}^{2i+2j-2}(z) p_{k'-1}^{2i'+2j'-2}(z) \, \mathrm{d}z. \end{aligned} \tag{8}$$

It has been shown in [3], this matrix has the sparsity pattern

$$m_{ijk,i'j'k'} = 0 \quad \text{if } (i,j,k,i',j',k') \in \mathfrak{S}^p_{ref}(ijk,i'j'k') \tag{9}$$

	$p_{x,1}$	$p_{x,2}$	γ_y	$p_{y,1}$	$p_{y,2}$	γ_z	$p_{z,1}$	$p_{z,2}$
$\hat{\mathscr{I}}^{(1)}$	p_{i-1}^0	$p_{i'-1}^0$	$i+i'-1$	$\hat{p}_j^{2i-1}$	$\hat{p}_{j'}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(2)}$	$\hat{p}_i^0$	$\hat{p}_{i'}^0$	$i+i'+1$	p_{j-1}^{2i-1}	$p_{j'-1}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(3)}$	p_{i-2}^0	$\hat{p}_{i'}^0$	$i+i'$	$\hat{p}_j^{2i-1}$	$p_{j'-1}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(4)}$	$\hat{p}_i^0$	$p_{i'-2}^0$	$i+i'$	p_{j-1}^{2i-1}	$\hat{p}_{j'}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(5)}$	p_{i-2}^0	$p_{i'-2}^0$	$i+i'-1$	$\hat{p}_j^{2i-1}$	$\hat{p}_{j'}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(6)}$	$\hat{p}_i^0$	$\hat{p}_{i'}^0$	$i+i'+1$	$\hat{p}_j^{2i-1}$	$\hat{p}_{j'}^{2i'-1}$	$\beta+\beta'+2$	$p_{k-1}^{-2+2\beta}$	$p_{k'-1}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(7)}$	$\hat{p}_i^0$	$\hat{p}_{i'}^0$	$i+i'+1$	p_{j-2}^{2i-1}	$\hat{p}_{j'}^{2i'-1}$	$\beta+\beta'+1$	$\hat{p}_k^{-2+2\beta}$	$p_{k'-1}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(8)}$	$\hat{p}_i^0$	$\hat{p}_{i'}^0$	$i+i'+1$	p_{j-1}^{2i-1}	$\hat{p}_{j'}^{2i'-1}$	$\beta+\beta'+1$	$\hat{p}_k^{-2+2\beta}$	$p_{k'-1}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(9)}$	p_{i-2}^0	$\hat{p}_{i'}^0$	$i+i'$	$\hat{p}_j^{2i-1}$	$\hat{p}_{j'}^{2i'-1}$	$\beta+\beta'+1$	$\hat{p}_k^{-2+2\beta}$	$p_{k'-1}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(10)}$	$\hat{p}_i^0$	$\hat{p}_{i'}^0$	$i+i'+1$	$\hat{p}_j^{2i-1}$	$p_{j'-2}^{2i'-1}$	$\beta+\beta'+1$	$p_{k-1}^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(11)}$	$\hat{p}_i^0$	$\hat{p}_{i'}^0$	$i+i'+1$	$\hat{p}_j^{2i-1}$	$p_{j'-1}^{2i'-1}$	$\beta+\beta'+1$	$p_{k-1}^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(12)}$	$\hat{p}_i^0$	$\hat{p}_{i'}^0$	$i+i'+1$	p_{j-2}^{2i-1}	$p_{j'-2}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(13)}$	$\hat{p}_i^0$	$\hat{p}_{i'}^0$	$i+i'+1$	p_{j-1}^{2i-1}	$p_{j'-2}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(14)}$	$\hat{p}_i^0$	$\hat{p}_{i'}^0$	$i+i'+1$	p_{j-2}^{2i-1}	$p_{j'-1}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(15)}$	$\hat{p}_i^0$	$\hat{p}_{i'}^0$	$i+i'+1$	p_{j-1}^{2i-1}	$p_{j'-1}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(16)}$	p_{i-2}^0	$\hat{p}_{i'}^0$	$i+i'$	$\hat{p}_j^{2i-1}$	$p_{j'-2}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(17)}$	p_{i-2}^0	$\hat{p}_{i'}^0$	$i+i'$	$\hat{p}_j^{2i-1}$	$p_{j'-1}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(18)}$	$\hat{p}_i^0$	$p_{i'-2}^0$	$i+i'$	$\hat{p}_j^{2i-1}$	$\hat{p}_{j'}^{2i'-1}$	$\beta+\beta'+1$	$p_{k-1}^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(19)}$	$\hat{p}_i^0$	$p_{i'-2}^0$	$i+i'$	p_{j-2}^{2i-1}	$\hat{p}_{j'}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(20)}$	$\hat{p}_i^0$	$p_{i'-2}^0$	$i+i'$	p_{j-1}^{2i-1}	$\hat{p}_{j'}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$
$\hat{\mathscr{I}}^{(21)}$	p_{i-2}^0	$p_{i'-2}^0$	$i+i'-1$	$\hat{p}_j^{2i-1}$	$\hat{p}_{j'}^{2i'-1}$	$\beta+\beta'$	$\hat{p}_k^{-2+2\beta}$	$\hat{p}_{k'}^{-2+2\beta'}$

Table 1. Integrands for $\mathscr{K}$, where $\beta = i+j$, $\beta' = i'+j'$

where

$$\mathfrak{S}^p_{ref}(ijk,i'j'k') = \{i+j+k \le p, i'+j'+k' \le p, |i-i'| \notin \{0,2\} \\ \vee \quad |i-i'+j-j'| > 4 \qquad \vee \quad |i-i'+j-j'+k-k'| > 4\}$$

cf. [3, Theorem 3.3]. In the following the more general case

$$\mathfrak{S}^p(ijk,i'j'k') = \{i+j+k \le p, i'+j'+k' \le p, |i-i'| > 2 \\ \vee \quad |i-i'+j-j'| > 4 \qquad \vee \quad |i-i'+j-j'+k-k'| > 4\} \tag{10}$$

is considered, e.g. the orthogonalities for $|i-i'| = 1$ are not assumed.

All 21 integrals give rise to a similar band structure as detailed above for $\hat{\mathscr{I}}^{(6)}$ and can thus be treated in the same way as explained below for the particular case

m	κ_m
$1,6,9,19$	1
$5,21$	$\frac{5}{4}$
$4,8,20$	$c_1(i,j)$
$7,19$	$c_2(i,j)$
$3,11,17$	$c_1(i',j')$
$2,15$	$c_1(i,j)c_1(i',j')$
13	$c_1(i,j)c_2(i',j')$
$10,16$	$c_2(i',j')$
14	$c_1(i',j')c_2(i,j)$
21	$c_2(i,j)c_2(i',j')$

Table 2. Coefficients κ_m for $\mathscr{K}$, where $c_1(i,j) = -\frac{1}{2}\frac{2i-1}{2i+2j-3}$ and $c_2(i,j) = \frac{j-1}{2i+2j-3}$.

of $\hat{\mathscr{I}}^{(6)}$. The only difference are shifts in the weights α of the Jacobi polynomials or changes of the weight functions (Table 2).

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3 Sum Factorization

In this section, we present an algorithm for the fast numerical generation of the local element matrices (6) for tetrahedra. The methods are based on fast summation techniques presented in [7, 8] and are carried out in detail for the example of the matrix $\hat{\mathscr{I}}^{(6)}$ (8).

All one dimensional integrals in (8) are computed numerically by a Gaussian quadrature rule with points x_k, $k = 1,\ldots,p+1$ and corresponding weights ω_k. The points and weights are chosen such that

$$\int_{-1}^{1} f(x)\,dx = \sum_{l=1}^{p+1} \omega_l f(x_l) \quad \forall f \in \mathscr{P}_{2p}. \tag{11}$$

Since only polynomials of maximal degree $2p$ are integrated in (8), these integrals are evaluated exactly. Therefore, we have to compute

$$\begin{aligned} m_{ijk,i'j'k'} &= \sum_{l=1}^{p+1} \omega_l \hat{p}_i^0(x_l)\hat{p}_{i'}^0(x_l) \\ &\times \sum_{m=1}^{p+1} \omega_m \left(\frac{1-x_m}{2}\right)^{i+i'+1} \hat{p}_{j'}^{2i'-1}(x_m)\hat{p}_j^{2i-1}(x_m) \\ &\times \sum_{n=1}^{p+1} \omega_n \left(\frac{1-x_n}{2}\right)^{i+j+i'+j'+2} p_k^{2i+2j-2}(x_n)p_{k'}^{2i'+2j'-2}(x_n), \end{aligned}$$

i.e., for all $(i,j,k,i',j',k') \notin \mathfrak{S}^p(ijk,i'j'k')$, cf. (10), (9). This is done by the following algorithm.

Algorithm 3.1 *1. Compute*

$$h^{(1)}_{i;i'} = \sum_{l=1}^{p+1} \omega_l \hat{p}_i^0(x_l)\hat{p}_{i'}^0(x_l)$$

for all $i,i' \in \mathbb{N}$ satisfying $|i-i'| \leq 2$ and $i,i' \leq p$.

2. Compute

$$h^{(2)}_{i,j;i',j'} = \sum_{m=1}^{p+1} \omega_m \left(\frac{1-x_m}{2}\right)^{i+i'+1} \hat{p}_j^{2i-1}(x_m)\hat{p}_{j'}^{2i'-1}(x_m)$$

for all $i,j,i',j' \in \mathbb{N}$ satisfying $|i-i'| \leq 2$, $|i+j-i'-j'| \leq 4$, $i+j \leq p$ and $i'+j' \leq p$.

3. Compute

$$h^{(3)}_{\beta,k;\beta,'k'} = \sum_{n=1}^{p+1} \omega_n \left(\frac{1-x_n}{2}\right)^{\beta+\beta'+2} p_k^{2\beta-2}(x_n)p_{k'}^{2\beta'-2}(x_n)$$

for all $k,k',\beta,\beta' \in \mathbb{N}$ satisfying $|\beta-\beta'| \leq 4$, $|\beta+k-\beta'-k'| \leq 4$, $\beta+k \leq p$ and $\beta'+k' \leq p$.

4. For all $(i,j,k,i',j',k') \notin \mathfrak{S}^p(ijk,i'j'k')$, set

$$m_{ijk,i'j'k'} = h^{(1)}_{i;i'} h^{(2)}_{i,j;i',j'} h^{(3)}_{i+j,k;i'+j',k'}.$$

The algorithm requires the numerical evaluation of Jacobi and integrated Jacobi polynomials at the Gaussian points x_l, $l = 1,\dots,p+1$. In the next subsection, we present an algorithm which computes the required values $\hat{p}_k^\alpha(x_l)$, $m = 1,\dots,p+1$, $k = 1,\dots,p$, $\alpha = 1,\dots,2p$ in $\mathcal{O}(p^3)$ operations.

4 Fast Evaluation of Integrated Jacobi Polynomials

The integrated Jacobi polynomials needed in the computation of $m_{ijk,i'j'k'}$ (7) are $\hat{p}_i^0(x), \hat{p}_j^{2i-1}(x)$ (progressing in odd steps with respect to the parameter α) and

$\hat{p}_k^{2i+2j-2}(x)$ (progressing in even steps with respect to the parameter α). For $i+j+k \leq p$ with $i \geq 2$ and $j,k \geq 1$ this means that

$$\left[\hat{p}_i^0(x)\right]_{2\leq i\leq p}, [\hat{p}_j^3(x)]_{1\leq j\leq p}, \ldots, [\hat{p}_j^{2p-3}(x)]_{1\leq j\leq p},$$
$$[\hat{p}_k^4(x)]_{1\leq k\leq p}, \ldots, [\hat{p}_k^{2p-4}(x)]_{1\leq k\leq p}$$

are needed. Since one group proceeds in even, the other one in odd steps, the total of integrated Jacobi polynomials that are needed is

$$\hat{p}_n^a(x), \qquad 1 \leq n \leq p-3, \quad 3 \leq a \leq 2p-3,$$

if we consider the integrated Legendre polynomials separately. However, integrating both sides of (3) yields

$$\hat{p}_{n+1}^{\alpha-1}(x) = \frac{1}{2n+\alpha}\left((n+\alpha)\hat{p}_{n+1}^{\alpha}(x) - n\hat{p}_n^{\alpha}(x)\right),$$

valid for all $n \geq 0$. Using this relation starting from the integrated Jacobi polynomials of highest degree, i.e., $\alpha = 2i-1 = 2p-3$, the remaining Jacobi polynomials can be computed using only two elements of the previous row. Note that for the initial values $n = 1$ we have $\hat{p}_1^{\alpha}(x) = 1+x$ for all α. For assembling the polynomials of highest degree the three term recurrence (4) is used. Summarizing, the evaluation of the functions at the Gaussian points can be done in $\mathcal{O}(p^3)$ operations. This is optimal in the three-dimensional case, but not in the two-dimensional case.

5 Complexity of the Algorithm

The cost of the last three steps is $\mathcal{O}(p^3)$, the first step requires $\mathcal{O}(p^2)$ operations. Together with the evaluation of the Jacobi polynomials, the algorithm requires in total $\mathcal{O}(p^3)$ flops.

This algorithm uses only the sparsity structure (10). Since all matrices $\hat{\mathscr{I}}^{(m)}$, $m = 1,\ldots,21$, have a similar sparsity structure of the form (10), this algorithm can be extended to all ingredients which are required for assembling/computing the element stiffness matrix (6) for the Laplacian, see [3]. The algorithm can also be extended to mass matrices or matrices arising from the discretization of elliptic problems in $H(\mathrm{curl}\,)$ and $H(\mathrm{div})$, see [4]. For two-dimensional problems, the third step of the algorithm is not necessary. However, the values $h^{(2)}_{i,j;i',j'}$ have to be computed. Since this requires $\mathcal{O}(p^3)$ floating point operations, the total cost in 2D is also $\mathcal{O}(p^3)$.

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Bibliography

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] I. Babuska, M. Griebel, and J. Pitkäranta. The problem of selecting the shape functions for a p-type finite element. *Internat. J. Numer. Methods Engrg.*, 28(8):1891–1908, 1989.

[3] S. Beuchler and V. Pillwein. Sparse shape functions for tetrahedral p-FEM using integrated Jacobi polynomials. *Computing*, 80(4):345–375, 2007.

[4] S. Beuchler, V. Pillwein, and S. Zaglmayr. Sparsity optimized high order finite element functions for $h(div)$ on simplices. Technical Report 2010-07, RICAM, 2010. submitted.

[5] S. Beuchler and J. Schöberl. New shape functions for triangular p-FEM using integrated Jacobi polynomials. *Numer. Math.*, 103(3):339–366, 2006.

[6] Leszek Demkowicz, Jason Kurtz, David Pardo, Maciej Paszyński, Waldemar Rachowicz, and Adam Zdunek. *Computing with hp-adaptive finite elements. Vol. 2*. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. Chapman & Hall/CRC, Boca Raton, FL, 2008. Frontiers: three dimensional elliptic and Maxwell problems with applications.

[7] George Em Karniadakis and Spencer J. Sherwin. *Spectral/hp element methods for computational fluid dynamics*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, second edition, 2005.

[8] J.M. Melenk, K. Gerdes, and C. Schwab. Fully discrete hp-finite elements: Fast quadrature. *Comp. Meth. Appl. Mech. Eng.*, 190:4339–4364, 1999.

[9] Ch. Schwab. *p- and hp-finite element methods*. Numerical Mathematics and Scientific Computation. The Clarendon Press Oxford University Press, New York, 1998. Theory and applications in solid and fluid mechanics.