A Non-overlapping Quasi-optimal Optimized Schwarz Domain Decomposition Algorithm for the Helmholtz Equation

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1 Introduction

In this paper, we present a new non-overlapping domain decomposition algorithm for the Helmholtz equation. We are particularly interested in the method introduced by P.-L. Lions [6] for the Laplace equation and extended to the Helmholtz equation by B. Després [3]. However, this latest approach provides slow convergence of the iterative method due to the choice of the transmission conditions. Thus, in order to improve the convergence, several methods were developed [4, 5, 9, 10]. The main idea in [5, 9] consists in computing a more accurate approximation of the Dirichlet-to-Neuman (DtN) operator than the one proposed in [3] by using particular local transmission conditions. We propose in this work a different approach to approximate the DtN map. We mainly use Padé approximants to suitably localize the nonlocal representation of the DtN operator [8, 11]. This results in an algorithm with quasi-optimal convergence properties.

2 Model Problem and Non-overlapping Domain Decomposition Method

For the sake of simplicity, we limit ourselves to the evaluation of the two-dimensional time-harmonic scattering wave by an obstacle denoted by K. The three-dimensional case is treated similarly without adding any difficulty. We consider the model problem given by the system

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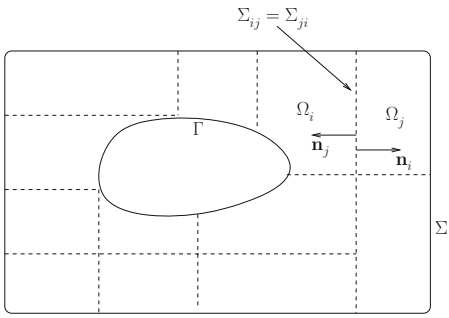

Fig. 1. Example of 2D non-overlapping domain decomposition method

$$\begin{cases} \Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \backslash K, \\ u = f \quad \text{on } \Gamma = \partial K, \\ \lim\limits_{|x|\to\infty} |x|^{1/2}(\partial_{|x|} u - iku) = 0, \end{cases} \tag{1}$$

composed of the Helmholtz equation, the Dirichlet condition on Γ (TE polarization in electromagnetics) where $f = -e^{ik\alpha\cdot x}$ describes the incident plane wave with $|\alpha| = 1$ and k is the wavenumber, and the Sommerfeld radiation condition. To solve (1), we combine the absorbing boundary condition method [1, 2] with non-overlapping domain decomposition methods. The absorbing boundary conditions method consists of truncating the computational domain using an artificial interface Σ, and reducing the system (1) to the following one

$$\begin{cases} \Delta u + k^2 u = 0 \quad \text{in } \Omega, \\ u = f \quad \text{on } \Gamma, \\ \partial_{\mathbf{n}} u + \mathscr{B} u = 0 \quad \text{on } \Sigma, \end{cases} \tag{2}$$

where Ω is the bounded domain enclosed by Σ and Γ, $\mathscr{B}$ indicates the approximation of the Dirichlet-to-Neuman (DtN) operator, and $\mathbf{n}$ is the outward normal to Σ. We are interested in the domain decomposition method introduced in [3, 6]. The first step of this approach consists in splitting Ω into several subdomains Ω_i, $i = 1,\ldots,N$, such that

- $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i$ $(i = 1,\ldots,N)$,
- $\Omega_i \cap \Omega_j = \emptyset$, if $i \neq j$, $(i, j = 1,\ldots,N)$,
- $\partial\Omega_i \cap \partial\Omega_j = \overline{\Sigma}_{ij} = \overline{\Sigma}_{ji}$ $(i, j = 1,\ldots,N)$ is the artificial interface (see Fig. 1) separating Ω_i from Ω_j as long as its interior Σ_{ij} is not empty.

Then, applying the Lions-Després algorithm, the solution of the initial problem (1) is reduced to an iterative procedure, where each iteration is performed by solving the local problems

$$\begin{cases} \Delta u_i^{(n+1)} + k^2 u_i^{(n+1)} = 0 \quad \text{in } \Omega_i, \\ u_i^{(n+1)} = f_i \quad \text{on } \Gamma_i, \\ \partial_{\mathbf{n}_i} u_i^{(n+1)} + \mathscr{B} u_i^{(n+1)} = 0 \quad \text{on } \Sigma_i \end{cases} \tag{3a}$$

$$\partial_{\mathbf{n}_i} u_i^{(n+1)} + \mathscr{S} u_i^{(n+1)} = g_{ij}^{(n)} \quad \text{on } \Sigma_{ij}, \tag{3b}$$

and forming the quantities to be transmitted through the interfaces

$$g_{ij}^{(n+1)} = -\partial_{\mathbf{n}_j} u_j^{(n+1)} + \mathscr{S} u_j^{(n+1)} = -g_{ij}^{(n)} + 2\mathscr{S} u_j^{(n+1)} \quad \text{on } \Sigma_{ij}, \tag{4}$$

where $u_i = u|_{\Omega_i}$, $\mathbf{n}_i$ (resp. $\mathbf{n}_j$) is the outward unit normal of the boundary of Ω_i (resp. Ω_j), $i = 1,\ldots,N$, $j = 1,\ldots,N$, $\Gamma_i = \partial\Omega_i \cap \Gamma$ and $\Sigma_i = \partial\Omega_i \cap \Sigma$. Note that the boundary condition on Γ_i (resp. Σ_i) does not take place if the interior of $\partial\Omega_i \cap \Gamma$ (resp. $\partial\Omega_i \cap \Sigma$) is the empty set.

3 New Transmission Conditions

It is well established that the convergence of the domain decomposition algorithms depends on the choice of the transmission operator $\mathscr{S}$. In the original method proposed by B. Després [3], the usual approximation of the DtN operator $\mathscr{S} u = -\imath k u$ is used. The resulting algorithm does not treat efficiently the evanescent modes of the iteration operator which impairs the iterative method [9]. In order to improve the convergence, two techniques, based on the modification of the operator $\mathscr{S}$, were proposed. First, the optimized Schwarz method introduced by Gander et al. [5]. It consists of using local second-order approximations of the DtN operator $\mathscr{S} u = \delta u + \gamma \partial_s^2 u$, where ∂_s is the tangential derivative operator, and the coefficients δ and γ are optimized using the rate of convergence obtained in the case of the half-plane. The second method, called the "evanescent modes damping algorithm" (EMDA), was introduced by Boubendir et al. [9, 10]. In this case, S is chosen as $\mathscr{S} u = -\imath k u + \mathscr{X} u$ where $\mathscr{X}$ is a self-adjoint positive operator. We only consider here the usual case where $\mathscr{X}$ is a real-valued positive coefficient. In this paper we propose a new "square-root" transmission operator [7, 8, 11] that takes the following form:

$$\mathscr{S} u = -\imath k \mathrm{Op}\left(\sqrt{1 - \frac{\xi^2}{k_\varepsilon^2}}\right) u, \tag{5}$$

where

$$k_\varepsilon = k + \imath\varepsilon \tag{6}$$

is a complexified wavenumber, and the notation $\sqrt{z}$ designates the principal determination of the square-root of a complex number z with branch-cut along the negative real axis. This choice of the square-root operator is motivated by developments of absorbing boundary conditions (ABC) for scattering problems [1, 2]. Generally speaking, the usual techniques to develop absorbing boundary conditions consists

mainly in using Taylor expansions to approximate the symbol of the DtN operator. However, these approximations prevent the modelling of the three parts describing the wave (propagating, evanescent and transition) at the same time, which affects, in return, the final accuracy of the solution. This problem can be solved by high-order local ABC introduced in [7, 8], which uses (5) to model all the scattering modes: propagating, evanescent as well as (in an approximate way) grazing. The localization is performed with complex Padé approximants, and the coefficient ε in (6) can then be chosen to minimize spurious reflections at the boundary. In the context of domain decomposition methods, this optimization of ε improves the spectrum of the iteration operator on these grazing modes. As it is shown in [8], the optimal value of this parameter is given by $\varepsilon = 0.4k^{1/3}\mathscr{H}^{2/3}$, where $\mathscr{H}$ is the mean curvature on the interface.

4 Localization of the Square-Root Operator Using Padé Approximants

Because the square-root operator (5) is nonlocal, its use in the context of finite element method is ineffective since it would lead to consider full matrices for the transmission boundaries. A localization process of this operator can be efficiently done by using partial differential (local) operators and obtain sparse matrices. This is performed [7, 8, 11] in rotating branch-cut approximation of the square-root and then applying complex Padé approximants of order N_p,

$$\begin{aligned}\sqrt{1-\frac{\xi^2}{k_\varepsilon^2}}u &\approx R^{\alpha}_{N_p}(-\frac{\xi^2}{k_\varepsilon^2})u \\ &= C_0 u + \sum_{\ell=1}^{N_p} A_\ell(\frac{-\xi^2}{k_\varepsilon^2})(1+B_\ell(\frac{-\xi^2}{k_\varepsilon^2}))^{-1}u,\end{aligned} \tag{7}$$

which correspond to the complex Padé approximation

$$\sqrt{1+z} \approx R^{\alpha}_{N_p}(z) = C_0 + \sum_{\ell=1}^{N_p} \frac{A_\ell z}{1+B_\ell z}, \tag{8}$$

and where the complex coefficients C_0, A_ℓ and B_ℓ are given by

$$C_0 = e^{\imath\frac{\alpha}{2}} R_{N_p}(e^{-\imath\alpha}-1), A_\ell = \frac{e^{-\frac{\imath\alpha}{2}}a_\ell}{(1+b_\ell(e^{-\imath\alpha}-1))^2}, B_\ell = \frac{e^{-\imath\alpha}b_\ell}{1+b_\ell(e^{-\imath\alpha}-1)}.$$

Here, α is the angle of rotation, (a_ℓ, b_ℓ), $\ell = 1,\ldots,N_p$, are the standard real Padé coefficients

$$a_\ell = \frac{2}{2N_p+1}\sin^2(\frac{\ell\pi}{2N_p+1}),\ b_\ell = \cos^2(\frac{\ell\pi}{2N_p+1}), \tag{9}$$

and R_{N_p} is the real Padé approximant of order N_p

$$\sqrt{1+z} \approx R_{N_p}(z) = 1 + \sum_{\ell=1}^{N_p} \frac{a_\ell z}{1+b_\ell z}. \tag{10}$$

For a variational representation, the approximation of the Padé-localized square-root transmission operators is realized by using auxiliary coupled functions [7, 11]

$$\mathscr{S} u = -\imath k (C_0 u + \sum_{\ell=1}^{N_p} A_\ell \mathrm{div}_{\Sigma_d}(\frac{1}{k_\varepsilon^2} \nabla_{\Sigma_d} \varphi_\ell)) \quad \text{on } \Sigma_d, \tag{11}$$

where the functions φ_ℓ, $\ell = 1,..,N_p$, are defined on any artificial interface Σ_d as the solutions of the surface PDEs

$$(1 + B_\ell \mathrm{div}_{\Sigma_d}(\frac{1}{k_\varepsilon^2} \nabla_{\Sigma_d}))\varphi_\ell = u. \tag{12}$$

The resulting transmitting condition is a Generalized Impedance Boundary Condition, and is denoted by GIBC(N_p, α, ε) for the Padé approximation with N_p auxiliary functions, for an angle of rotation α and a damping parameter ε. The lowest-order approximation $\mathscr{S} = -\imath k I$ (resp. $\mathscr{S} = -\imath k u + \mathscr{X} u$) is denoted by IBC(0) (resp. IBC($\mathscr{X}$)).

5 Numerical Results

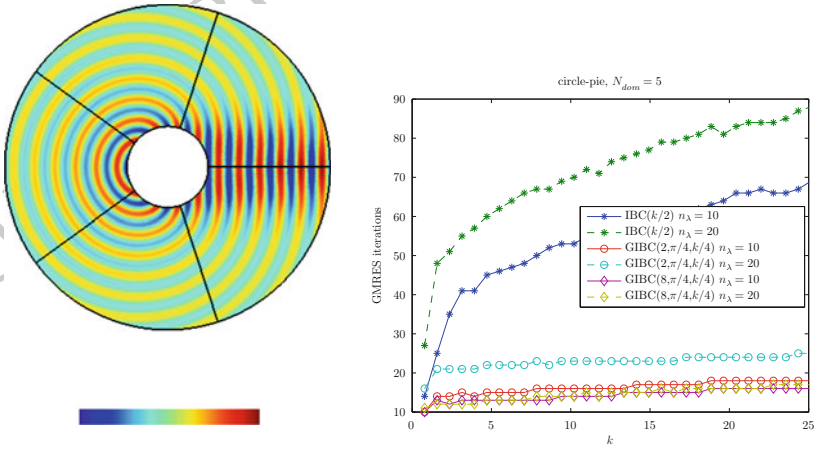

Fig. 2. *Left*: decomposition of the computational domain. *Right*: iteration number with respect to the wavenumber k for two densities of discretization n_λ

The numerical tests presented here concern the scattering of a plane wave by a unit sound-soft circular cylinder. We truncate the computational domain using a circle of radius equal to 4, on which the second-order Bayliss-Turkel absorbing condition [1] is set (see problem (2)). We perform these numerical tests on partitions of the type displayed in Fig. 2, and we refer to them as "circle-pie". We use a finite element method with linear (P1) basis functions to approximate the solution in each subdomain. The implementation of this method with Padé approximants is described in [11]. The iterative problem is solved using GMRES and the iterations are stopped when the initial residual has decreased by a factor of 10^{-6}.

We begin by testing the iterative method with respect to the wavenumber k. Let us consider the number of subdomains $N_{\text{dom}} = 5$. Because the interfaces are straight, as depicted on the left picture of Fig. 2, ε cannot be optimized as described in Sect. 3. However, numerical simulations show that $\varepsilon = k/4$ is an appropriate choice for this kind of interfaces. On the right picture of Fig. 2, we represent the behavior of the number of iterations. We choose two densities of discretization points per wavelength n_λ. We compare the new algorithm noted GIBC(N_p, $\pi/4$, ε), where N_p is the Padé number and $\pi/4$ the angle of rotation, with the EMDA algorithm designated by IBC($k/2$). In this latest case, the number of iterations clearly increases with respect to k and n_λ. However, for GIBC(N_p, $\pi/4$, ε), the convergence rate is almost independent of both the wavenumber and density of discretization points per wavelength. In particular, the convergence for $N_p = 2$ and $N_p = 8$ is similar. This means that the cost of the solution when solving local problems is comparable to the other methods with usual local transmission conditions (see [11] for more details).

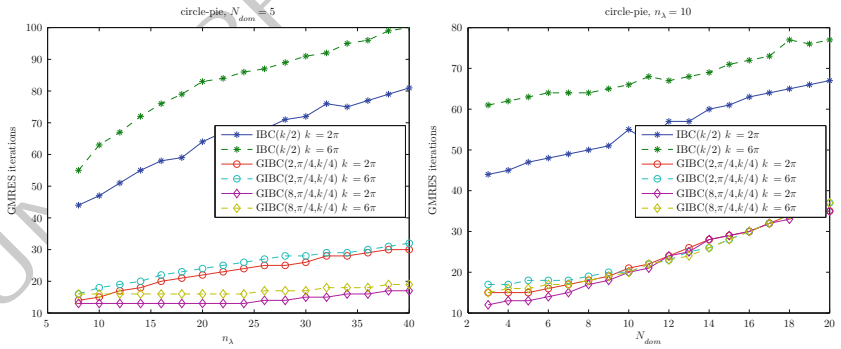

Fig. 3. Number of iterations with respect to the density of discretization n_λ and the number of subdomains N_{dom}

In Fig. 3, we show the number of iterations with respect to: (i) the density of discretization points per wavelength n_λ for two wavenumbers k, and (ii) the number of subdomains N_{dom}. We can see that for a small Padé number ($N_p = 2$), the convergence is almost independent of the mesh size. A larger choice of N_p will provide an optimal result. We also see that the number of iterations with respect to the number

of subdomains does not deteriorate with increasing values of N_p or k, contrary to IBC($k/2$).

6 Conclusion

We designed in this paper a new non-overlapping domain decomposition algorithm for the Helmholtz equation with quasi-optimal convergence properties. It is based on a suitable approach which consists in using Padé approximants to approximate the DtN operator. The analysis of this new approach can be found in [11], as well as several numerical tests including the three-dimensional case.

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