BDDC for Higher-Order Discontinuous Galerkin Discretizations

Laslo Diosady[1] and David Darmofal[2]

[1] Massachusetts Institute of Technology diosady@alum.mit.edu
[2] Massachusetts Institute of Technology darmofal@mit.edu

Summary. The BDDC algorithm is extended to a large class of discontinuous Galerkin (DG) discretizations of second order elliptic problems in two spatial dimensions. An estimate of $C(1+\log(p^2H/h))^2$ is obtained for the condition number of the preconditioned system where C is a constant independent of p, h or H. Numerical simulations are presented which confirm the theoretical results

1 Introduction

A Balancing Domain Decomposition by Constraints (BDDC) method is presented for the solution of a discontinuous Galerkin (DG) discretization of a second-order elliptic problem in two dimensions. BDDC was originally introduced in [8] for the solution of continuous finite element discretizations. Mandel and Dohrmann [13] later proved a condition number bound of $\kappa \leq C(1+\log(H/h))^2$ for preconditioned system of a continuous finite element discretization of second order elliptic problems. Pavarino [15] and Klawonn et al. [11] extended the BDDC algorithm to higher-order finite element methods and proved a condition number bound of $\kappa \leq C(1+\log(p^2H/h))^2$. Further analysis of BDDC methods and their connection to FETI methods has been presented in [12, 14].

While domain decomposition methods have been widely studied for continuous finite element discretizations, relatively little work has been performed for discontinuous Galerkin discretizations. Previous work on domain decomposition methods for DG discretizations include [1, 10] and [9]. This work presents a BDDC method applied to a large class of DG methods considered in the unified analysis of [2]. A key component for the development and analysis of the BDDC algorithm involves presenting the DG discretization as the sum of element-wise "local" bilinear forms. The element-wise perspective leads naturally to the appropriate choice for the subdomain-wise local bilinear forms. Additionally, this perspective enables a connection to be drawn between the DG discretization and a related continuous finite element discretization. As a result of this connection, the condition number bound

R. Bank et al. (eds.), *Domain Decomposition Methods in Science and Engineering XX*,
Lecture Notes in Computational Science and Engineering 91,
DOI 10.1007/978-3-642-35275-1_66,

for the BDDC preconditioned system for a large class of conservative and consistent DG methods is identical to that for continuous finite element methods.

2 DG Discretization

Consider the second order elliptic equation in a domain $\Omega \subset \mathscr{R}^2$:

$$-\nabla\cdot(\rho\nabla u) = f \qquad \text{in } \Omega, \qquad\qquad u = 0 \qquad \text{on } \partial\Omega \tag{1}$$

with positive $\rho > 0 \in L^\infty(\Omega)$, $f \in L^2(\Omega)$. Let the triangulation $\mathscr{T}$ be a partition of Ω into triangles or quadrilaterals. In order to simplify the presentation we assume that ρ takes on a constant value, ρ_κ on each element κ. Define $\mathscr{E}$ to be the union of edges of elements κ. Additionally, define $\mathscr{E}^i \subset \mathscr{E}$ and $\mathscr{E}^\partial \subset \mathscr{E}$ to be the set of interior, respectively boundary edges. Note that any edge $e \in \mathscr{E}^i$ is shared by two adjacent elements κ^+ and κ^- with corresponding outward pointing normal vectors $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$. Let $\mathscr{P}^p(\kappa)$ denote the space of polynomials of order at most p on κ and define the following finite element space $W_h^p := \{w_h \in \mathbf{L}^2(\Omega) : w_h|_\kappa \in \mathscr{P}^p(\kappa) \quad \forall \kappa \in \Omega\}$. Note that traces of functions $u_h \in W_h^p$ are in general double valued on each edge, $e \in \mathscr{E}^i$, with values u_h^+ and u_h^- corresponding to traces from elements κ^+ and κ^- respectively. On $e \in \mathscr{E}^\partial$, associate u_h^+ with the trace taken from the element, $\kappa^+ \in \mathscr{T}_h$, neighbouring e. The weak form of (1) on each element is given by: $\forall w_h \in \mathscr{P}^p(\kappa)$

$$(\rho\nabla u_h, \nabla w_h)_\kappa - \left\langle \rho(u_h^+ - \hat{u}_h)\boldsymbol{n}^+, \nabla w_h^+ \right\rangle_{\partial\kappa} + \left\langle \hat{\boldsymbol{q}}_h, w_h^+ \boldsymbol{n}^+ \right\rangle_{\partial\kappa} = (f, w_h)_\kappa \tag{2}$$

where $(\cdot,\cdot)_\kappa := \int_\kappa$ and $\langle\cdot,\cdot\rangle_{\partial\kappa} := \int_{\partial\kappa}$. Superscript $^+$ is used to explicitly denote values on $\partial\kappa$, taken from κ. For all $w_h \in W_h^p$, $\hat{w}_h = \hat{w}_h(w_h^+, w_h^-)$ is a single valued numerical trace on $e \in \mathscr{E}^i$, while $\hat{w}_h = 0$ for $e \in \mathscr{E}^\partial$. Note that $\hat{u}_h = 0$ on $e \in \mathscr{E}^\partial$, corresponds to weakly enforced homogeneous boundary conditions on $\partial\Omega$. Similarly $\hat{\boldsymbol{q}} = \hat{\boldsymbol{q}}(\rho^+, \rho^-, \nabla u_h^+, \nabla u_h^-, u_h^+, u_h^-)$ is a single valued numerical flux approximating $\boldsymbol{q} = \rho\nabla u$ on $e \in \mathscr{E}$. Summing over all elements gives:

$$a(u_h, w_h) = (f, w_h)_\Omega \qquad\qquad \forall w_h \in W_h^p \tag{3}$$

A key component, required for the development and analysis of the algorithms presented, is to express the global bilinear form $a(u_h, w_h)$ as the sum of element-wise contributions $a_\kappa(u_h, w_h)$ such that

$$a(u_h, w_h) = \sum_{\kappa\in\mathscr{T}} a_\kappa(u_h, w_h) \tag{4}$$

where $a_\kappa(u_h, w_h)$ is a symmetric, positive semi-definite "local bilinear form". In particular, the local bilinear form should have a compact stencil, such that $a_\kappa(u_h, w_h)$ is a function of only u_h, ∇u_h in κ, and u_h^+, ∇u_h^+ and $\hat{u}_h$ on $\partial\kappa$. The local bilinear form is written as:

$$\begin{aligned} a_\kappa(u_h, w_h) &= (\rho\nabla u_h, \nabla w_h)_\kappa - \left\langle \rho(u_h^+ - \hat{u}_h)\boldsymbol{n}^+, \nabla w_h^+ \right\rangle_{\partial\kappa} + \left\langle \hat{\boldsymbol{q}}_h^+, (w_h^+ - \hat{w}_h)\boldsymbol{n}^+ \right\rangle_{\partial\kappa} \\ &= (\rho\nabla u_h, \nabla w_h)_\kappa - \left\langle \rho\,[\![u]\!]_h^+, \nabla w_h^+ \right\rangle_{\partial\kappa} + \left\langle \hat{\boldsymbol{q}}_h^+, [\![w_h]\!]^+ \right\rangle_{\partial\kappa} \end{aligned} \tag{5}$$

where $\hat{\boldsymbol{q}}_h^+ = \hat{\boldsymbol{q}}_h^+(\rho^+, \nabla u_h^+, u_h^+, \hat{u}_h)$ is a "local numerical flux". The choice of the numerical trace $\hat{u}_h$ and flux $\hat{\boldsymbol{q}}_h$ define the particular DG method considered. Table 1 lists the numerical traces and fluxes for the DG methods considered in this paper, while Table 2 lists the corresponding local bilinear forms.

DG Method	$\hat{u}_h$	$\hat{\boldsymbol{q}}_h$	$\hat{\boldsymbol{q}}_h^+$
IP	$\{u_h\}$	$-\{\rho\nabla u_h\} + \frac{\eta_e}{h}\{\rho [\![u_h]\!]^{\pm}\}$	$-\rho^+\nabla u_h^+ + \frac{\eta_e}{h}\rho^+ [\![\rho u_h]\!]^+$
BR2	$\{u_h\}$	$-\{\rho\nabla u_h\} + \eta_e\{\rho r_e([\![u_h]\!]^{\pm})\}$	$-\rho^+\nabla u_h^+ + \eta_e\rho^+ r_e([\![u_h]\!]^+)$
Brezzi	$\{u_h\}$	$\{\boldsymbol{q}_h\} + \eta_e\{\rho r_e([\![u_h]\!]^{\pm})\}$	$\boldsymbol{q}_h^+ + \eta_e\rho^+ r_e([\![u_h]\!]^+)$
LDG	$\{u_h\} - \beta\cdot[\![u_h]\!]$	$\{\boldsymbol{q}_h\} + \beta [\![\boldsymbol{q}_h]\!] + \frac{2\eta_e}{h}\{\rho [\![u_h]\!]^{\pm}\}$	$\boldsymbol{q}_h^+ + \frac{\eta_e}{h}\rho^+ [\![u_h]\!]^+$
CDG	$\{u_h\} - \beta\cdot[\![u_h]\!]$	$\{\boldsymbol{q}_h^e\} + \beta [\![\boldsymbol{q}_h^e]\!] + \frac{2\eta_e}{h}\{\rho [\![u_h]\!]^{\pm}\}$	$\boldsymbol{q}_h^{e+} + \frac{\eta_e}{h}\rho^+ [\![u_h]\!]^+$

Table 1. Numerical fluxes for different DG methods. (IP: Interior Penalty, BR2: [3], Brezzi: [4], LDG: [5] CDG: [16])

Method	$a_\kappa(u_h, w_h)$
IP	$g + \sum_{e\in\partial\kappa}\frac{\eta_e}{h_e}\left\langle \rho [\![u_h]\!]^+, [\![w_h]\!]^+\right\rangle_e$
BR2	$g + \sum_{e\in\partial\kappa}\eta_e\left(\rho r_e([\![u_h]\!]^+), r_e([\![w_h]\!]^+)\right)_\kappa$
Brezzi	$g + \left(\rho r_\kappa([\![u_h]\!]^+), r_\kappa([\![w_h]\!]^+)\right)_\kappa + \sum_{e\in\partial\kappa}\eta_e\left(\rho r_e([\![u_h]\!]^+), r_e([\![w_h]\!]^+)\right)_\kappa$
LDG	$g + \left(\rho r_\kappa([\![u_h]\!]^+), r_\kappa([\![w_h]\!]^+)\right)_\kappa + \sum_{e\in\partial\kappa}\frac{\eta_e}{h_e}\left\langle \rho [\![u_h]\!]^+, [\![w_h]\!]^+\right\rangle_e$
CDG	$g + \sum_{e\in\partial\kappa}\left(\rho r_e([\![u_h]\!]^+), r_e([\![w_h]\!]^+)\right)_\kappa + \sum_{e\in\partial\kappa}\frac{\eta_e}{h_e}\left\langle \rho [\![u_h]\!]^+, [\![w_h]\!]^+\right\rangle_e$

Where $g = (\rho\nabla u_h, \nabla w_h)_\kappa - \left\langle \rho [\![u_h]\!]^+, \nabla w_h^+\right\rangle_{\partial\kappa} - \left\langle \rho\nabla u_h, [\![w_h]\!]^+\right\rangle_{\partial\kappa}$

Table 2. Elementwise bilinear form for different DG methods

In the definition of the different DG methods, $\{u_h\} = \frac{1}{2}(u_h^+ + u_h^-)$ and $[\![u_h]\!] = u_h^+\boldsymbol{n}^+ + u_h^-\boldsymbol{n}^-$ are average and jump operators on $e \in \mathscr{E}^i$. Additionally, a second set of jump operators involving the numerical trace $\hat{u}$ are given by $[\![u_h]\!]^+ = u_h^+\boldsymbol{n}^+ + \hat{u}_h\boldsymbol{n}^-$ and $[\![u_h]\!]^- = \hat{u}_h\boldsymbol{n}^+ + u_h^-\boldsymbol{n}^-$. Define $\boldsymbol{q}_h = -\rho(\nabla u_h - r_\kappa([\![u_h]\!]^+))$ and $\boldsymbol{q}_h^e = -\rho(\nabla u_h - r_e([\![u]\!]^+))$ where $r_\kappa(\phi)$ and $r_e(\phi) \in [\mathscr{P}^p(\kappa)]^n$ are lifting operators defined such that: $(r_\kappa(\phi), \boldsymbol{v}_h)_\kappa = \left\langle \phi, \boldsymbol{v}_h^+\right\rangle_\kappa$ and $(r_e(\phi), \boldsymbol{v}_h)_\kappa = \left\langle \phi, \boldsymbol{v}_h^+\right\rangle_e$, $\forall \boldsymbol{v}_h \in [\mathscr{P}^p(\kappa)]^n$. Additionally, on each edge in $\mathscr{E}$, η_e is a penalty parameter, while $\beta = \frac{1}{2}S_{\kappa^+}^{\kappa^-}\boldsymbol{n}^+ + S_{\kappa^-}^{\kappa^+}\boldsymbol{n}^-$ is a vector where $S_{\kappa^+}^{\kappa^-} \in \{0,1\}$ is a switch defined, such that $S_{\kappa^+}^{\kappa^-} + S_{\kappa^-}^{\kappa^+} = 1$.

Consider using a nodal basis on each element κ to define W_h^p. Figure 1 shows graphically the nodal degrees of freedom involved in defining the local bilinear form. For the IP, BR2 and Brezzi schemes, the numerical trace $\hat{u}_h$ on an edge/face depends on both u_h^+ and u_h^-. Hence the local bilinear form corresponds to all nodal degrees of freedom defining u_h on κ as well as nodal values on all edge/faces of $\partial\kappa \cap \mathscr{E}^i$

corresponding to the trace of u_h from elements neighbouring κ. On the other hand, for the LDG and CDG methods, the numerical trace $\hat{u}_h$ takes on the value of u_h^+ if $S^{\kappa^-}_{\kappa^+} = 0$ or u_h^- if $S^{\kappa^-}_{\kappa^+} = 1$. Hence the local bilinear form corresponds only to degrees of freedom defining u_h on κ and nodal values corresponding to the trace of u_h on neighbouring elements across edge/faces of $\partial\kappa \cap \mathscr{E}^i$ for which $S^{\kappa^-}_{\kappa^+} = 1$.

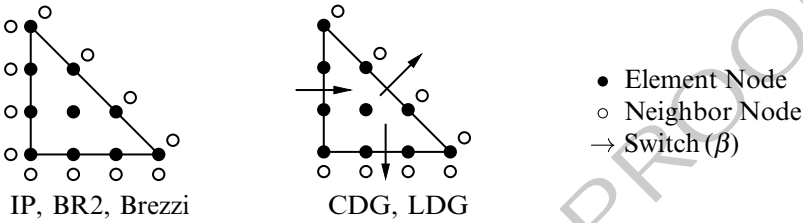

Fig. 1. Degrees of freedom involved in "local" bilinear form

The element-wise bilinear form $a_\kappa(u_h, u_h)$ satisfies

$$a_\kappa(u_h, u_h) \geq 0 \tag{6}$$

with $a_\kappa(u_h, u_h) = 0$ iff $u_h = \hat{u}_h = K$ for some constant K. The proof of (6) closely follows the proof of boundedness and stability of the different DG methods presented in [2]. As a result it is possible to show that the bilinear form is equivalent to a quadratic form based on the value of u_h at the nodes $\boldsymbol{x}$:

$$ca_\kappa(u_h, u_h) \leq \rho_\kappa p^4 h^{n-2} \sum_{\boldsymbol{x}_i, \boldsymbol{x}_j \in \kappa \cup \kappa'} (u_h(\boldsymbol{x}_i) - u_h(\boldsymbol{x}_j))^2 \leq Ca_\kappa(u_h, u_h) \tag{7}$$

where c and C are constants independent of h, p and ρ, while $\boldsymbol{x}_i, \boldsymbol{x}_j$ are the nodes on κ defining the basis for u_h and nodes on $\partial\kappa'$ defining a basis for the trace u_h^- from neighbours κ' of κ. Using the quadratic form in (7) a connection may be drawn between the DG discretization a continuous finite element discretization on a subtriangulation (See for example [6] Lemma 4.3). Further details are given in [7].

3 Domain Decomposition

Consider a partition of the domain Ω into substructures Ω_i such that $\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i$. The substructures Ω_i are disjoint shape regular polygonal regions of diameter $O(H)$, consisting of a union of elements in $\mathscr{T}$. Assume that $\rho(\boldsymbol{x})$ takes on a constant value, ρ_i, within each subdomain Ω_i. Additionally, assume that each element κ in Ω_i with an edge e on $\partial\Omega_i \cap \partial\Omega_j$ has neighbours only in $\Omega_i \cup \Omega_j$.

Define the local interface $\Gamma_i = \partial\Omega_i \backslash \partial\Omega$ and global interface Γ by $\Gamma = \cup_{i=1}^N \Gamma_i$. Denote by $W_\Gamma^{(i)}$ the space of discrete nodal values on Γ_i which correspond to degrees

of freedom shared between Ω_i and neighbouring subdomains Ω_j, while $W_I^{(i)}$ denotes the space of discrete unknowns local to a single substructure Ω_i. In particular, note that for the IP, BR2 and Brezzi et al. methods $W_\Gamma^{(i)}$ includes for each edge $e \in \Gamma_i$ degrees of freedom defining two sets of trace values u^+ from $\kappa^+ \in \Omega_i$ and u^- for $\kappa^- \in \Omega_j$. Thus, $W_I^{(i)}$ corresponds to nodal values strictly interior to Ω_i or on $\partial\Omega_i \backslash \Gamma_i$. On the other hand, for the CDG and LDG methods $W_\Gamma^{(i)}$ includes for each edge $e \in \Gamma_i$ degrees of freedom defining a single trace value corresponding to either u^+ from $\kappa^+ \in \Omega_i$ if $S_{\kappa^+}^{\kappa^-} = 0$ or u^- from $\kappa^- \in \Omega_j$ if $S_{\kappa^+}^{\kappa^-} = 1$. Hence, $W_I^{(i)}$ corresponds to nodal values interior to Ω_i and on $\partial\Omega_i \backslash \Gamma_i$ as well as nodal values defining u^+ on $e \in \Gamma_i$ for which $S_{\kappa^+}^{\kappa^-} = 1$.

Similarly, define $\hat{W}_\Gamma$ as the space of degrees of freedom shared among multiple subdomains and W_I as the space of degrees of freedom which correspond only to a single subdomain. Note that W_I is equal to the product space $W_I := \Pi_{i=1}^N W_I^{(i)}$, while in general $\hat{W}_\Gamma \subset W_\Gamma := \Pi_{i=1}^N W_\Gamma^{(i)}$. Define local operators $R_\Gamma^{(i)} : \hat{W}_\Gamma \rightarrow W_\Gamma^{(i)}$ which extract the local degrees of freedom on Γ_i from those on Γ. Additionally define a global operator $R_\Gamma : \hat{W}_\Gamma \rightarrow W_\Gamma$ which is formed by a direct assembly of $R_\Gamma^{(i)}$. The discrete form of (3) is written as:

$$\begin{bmatrix} A_{II} & A_{\Gamma I}^T \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_I \\ u_\Gamma \end{bmatrix} = \begin{bmatrix} b_I \\ b_\Gamma \end{bmatrix}. \tag{8}$$

where u_I and u_Γ corresponds to degrees of freedom associated with W_I and $\hat{W}_\Gamma$ respectively. Since the degrees of freedom associated with W_I are local to a particular substructure they may be locally eliminated to obtain a system

$$\hat{S}_\Gamma u_\Gamma = g_\Gamma \tag{9}$$

where $\hat{S}_\Gamma = A_{\Gamma\Gamma} - A_{\Gamma I} A_{II}^{-1} A_{\Gamma I}^T$ and $g_\Gamma = b_{\Gamma\Gamma} - A_{\Gamma I} A_{II}^{-1} b_{\Gamma I}$. $\hat{S}_\Gamma$ and g_Γ may be formed by a direct assembly:

$$\hat{S}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} S_\Gamma^{(i)} R_\Gamma^{(i)} \qquad g_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} g_\Gamma^{(i)} \tag{10}$$

where $S_\Gamma^{(i)} = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} A_{\Gamma I}^{(i)^T}$ and $g_\Gamma^{(i)} = b_\Gamma^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} b_I^{(i)}$.

4 BDDC

A BDDC preconditioner is used to solve the Schur complement problem (9). A full description of the BDDC preconditioner is given by Li and Widlund [12]. In order to define the BDDC preconditioner $W_\Gamma^{(i)}$ is reparameterize into two orthogonal spaces $W_\Pi^{(i)}$ and $W_\Delta^{(i)}$. The primal space $W_\Pi^{(i)}$ is the space of discrete unknowns corresponding to functions with a constant value of $\hat{u}$ on each edge of substructure Ω_i. The

dual space, $W_\Delta^{(i)}$ is the space of discrete unknowns corresponding to functions which have zero mean value of $\hat{u}$ on Γ_i. For continuous finite element discretizations, different primal degrees of freedom such as subdomain corners have also been used, however these are not explored in this work. The BDDC algorithm is implemented using a change of basis as described in [12]. The partially assembled space is defined as $\tilde{W}_\Gamma = \hat{W}_\Pi \oplus \left(\Pi_{i=1}^N W_\Delta^{(i)}\right)$, where $\hat{W}_\Pi$, single valued on Γ, is formed by assembling the local primal spaces, $W_\Pi^{(i)}$. Define additional local operators $\bar{R}_\Gamma^{(i)} : \tilde{W}_\Gamma \to W_\Gamma^{(i)}$ which extract the degrees of freedom in $\tilde{W}_\Gamma$ corresponding to Γ_i. The global operator $\bar{R}_\Gamma : \tilde{W}_\Gamma \to W_\Gamma$ is formed by a direct assembly of $\bar{R}_\Gamma^{(i)}$. Also define the global operator $\tilde{R}_\Gamma : \hat{W}_\Gamma \to \tilde{W}_\Gamma$. The partially assembled Schur complement matrix $\tilde{S}$, is given by:

$$\tilde{S}_\Gamma = \sum_{i=1}^{N} \bar{R}_\Gamma^{(i)^T} S_\Gamma^{(i)} \bar{R}_\Gamma^{(i)} \tag{11}$$

The scaled operator $\tilde{R}_{D,\Gamma} : \hat{W}_\Gamma \to \tilde{W}_\Gamma$ is obtained by multiplying the entries of $\tilde{R}_\Gamma$ corresponding to $W_\Delta^{(i)}$ by $\delta_i^\dagger(x)$, where $\delta_i^\dagger(x)$ defined for each nodal degree of freedom in $W_\Gamma^{(i)}$ on $\partial\Omega_i$ and $\partial\Omega_j$ as $\delta_i^\dagger = \frac{\rho_i^\gamma}{\rho_i^\gamma + \rho_j^\gamma}$, $\gamma \in [1/2, \infty)$. The BDDC preconditioner $M_{\text{BDDC}}^{-1} : \hat{W}_\Gamma \to \hat{W}_\Gamma$ is given by:

$$M_{\text{BDDC}}^{-1} = \tilde{R}_{D,\Gamma}^T \tilde{S}_\Gamma^{-1} \tilde{R}_{D,\Gamma} \tag{12}$$

The condition number of the preconditioner operator $M_{\text{BDDC}}^{-1}\hat{S}$ is bounded by $C(1 + \log(p^2 H/h))^2$ where C is a constant independent of p, h, H or ρ. This is the same condition number bound as obtained by Klawonn et al. [11] for a continuous finite element discretization. Proof of this condition number bound closely follows that presented by Tu [17] for mixed finite element methods, which in turn builds upon the work of [6]. The key idea is to connect the DG discretization to a related continuous finite element discretization on a subtriangulation of $\mathscr{T}$. The ability to connect the DG discretization to the continuous finite element discretization is a direct result of (7) (see [6]). The existing theory for continuous finite elements developed in [13, 15] and [11] is then leveraged to obtain the desired condition number bound. Further details are provided in [7].

5 Numerical Results

This section presents numerical results using the BDDC preconditioner introduced in Sect. 4. For each numerical experiment the linear system resulting from the DG discretization is solved iteratively using a Preconditioned Conjugate Gradient (PCG) method, starting from zero initial condition until l_2 norm of the residual is decreased by a factor of 10^{10}. The domain $\Omega = (0,1)^2$ is partitioned into $N \times N$ square subdomains Ω_i with side lengths H such that $N = \frac{1}{H}$. Each subdomain is the union of triangular elements obtained by bisecting squares of side length h. In the first numerical

experiment (1) is solved on Ω with $\rho = 1$ and f chosen such that the exact solution is given by $u = \sin(\pi x)\sin(\pi y)$. Table 3 shows the number of PCG iteration required to converge varying N, $\frac{H}{h}$ and p for each of the DG discretization considered. Table 3 also gives the Lanczos estimate of the maximum eigenvalue of the preconditioned system. The minimum eigenvalue is bounded below by unity as with continuous finite element methods. As expected the number of iterations is independent of the number of subdomains and only weakly dependent on the number of elements per subdomain or the solution order.

$\frac{1}{H}$	$\frac{H}{h}$	p	IP	BR2	Brezzi	LDG	CDG
2			12 (12.1)	15 (12.0)	15 (7.7)	11 (6.1)	12 (5.9)
4			22 (14.3)	27 (14.0)	23 (9.2)	24 (7.4)	24 (7.1)
8	8	4	31 (15.2)	34 (14.8)	30 (9.8)	28 (7.7)	27 (7.5)
16			33 (15.3)	36 (14.9)	32 (9.9)	29 (8.0)	28 (7.8)
32			33 (15.3)	36 (14.9)	32 (9.9)	29 (7.9)	27 (7.7)
	2		25 (10.9)	29 (10.9)	26 (6.9)	23 (5.2)	23 (5.3)
	4		29 (13.0)	34 (12.8)	28 (8.3)	26 (6.4)	25 (6.2)
8	8	4	31 (15.2)	34 (14.8)	30 (9.8)	28 (7.8)	27 (7.5)
	16		33 (17.6)	36 (17.1)	33 (11.5)	29 (9.3)	29 (9.1)
	32		35 (20.2)	38 (19.4)	34 (13.4)	32 (11.0)	31(10.7)
		1	32 (11.1)	36 (13.8)	28 (8.1)	26 (5.9)	25 (5.6)
		2	31 (12.9)	34 (14.1)	29 (8.7)	26 (6.4)	26 (6.3)
8	8	4	31 (15.2)	34 (14.8)	30 (9.8)	28 (7.8)	27 (7.5)
		8	34 (18.4)	37 (16.2)	34 (11.7)	31 (9.9)	32 (9.6)
		16	36 (22.5)	38 (18.6)	38 (14.4)	34 (12.8)	36 (12.2)

Table 3. Iteration count ($\lambda_{\max}$) for BDDC preconditioner using different DG methods

In the second numerical experiment the behaviour of the preconditioner for large jumps in the coefficient ρ is examined. For this numerical experiment only the CDG discretization is used. The domain is partitioned in a checkerboard pattern with $\rho = 1$ on half of the subdomains and $\rho = 1{,}000$ in the remaining subdomains. Initially set $\delta_i^\dagger = \frac{1}{2}$, which corresponds to setting $\gamma = 0$, which does not satisfy the assumption $\gamma \in [1/2, \infty)$. Poor convergence of the BDDC algorithm is seen in Table 4a. Next $\delta_i^\dagger$ is set to $\delta_i^\dagger = \frac{\rho_i}{\rho_i + \rho_j}$ which corresponds to $\gamma = 1$. With this choice of $\delta_i^\dagger$ the good convergence properties of the BDDC algorithm is recovered as shown in Table 4b.

6 Conclusions

The BDDC preconditioner has been extended to a large class of DG discretizations for second-order elliptic problems. The condition number of the BDDC preconditioned system is bounded by $C(1 + \log(p^2 H/h))^2$, with constant C independent of p, h, H or the coefficient ρ. This is the same condition number bound previously proven for continuous finite element methods. Numerical results confirm the theory.

(a) $\delta_i^\dagger = \frac{1}{2}$, $\frac{H}{h} = 8$

	$\frac{1}{H}$				
p	2	4	8	16	32
1	51	119	179	215	232
3	55	133	207	267	316
5	59	153	242	306	361

(b) $\delta_i^\dagger = \frac{\rho_i}{\rho_i+\rho_j}$, $\frac{H}{h} = 8$

	$\frac{1}{H}$				
p	2	4	8	16	32
1	4	7	14	18	19
3	4	7	15	18	19
5	4	7	14	19	20

Table 4. Iteration count for BDDC preconditioner using the CDG method with ρ=1 or 1000.

Bibliography

[1] P. Antonietti and B. Ayuso. Schwarz domain decomposition preconditioners for discontinuous Galerkin approximations of elliptic problems: Non-overlapping case. *Math. Model. Numer. Anal.*, 41(1):21–54, 2007.

[2] D. Arnold, F. Brezzi, B. Cockburn, and D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5): 1749–1779, 2002.

[3] F. Bassi and S. Rebay. A high-order discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations. *J. Comput. Phys.*, 131:267–279, 1997.

[4] F. Brezzi, G. Manzini, D. Marini, P. Pietra, and A. Russo. Discontinuous Galerkin approximations for elliptic problems. *Numer. Meth. Part. D. E.*, 16 (4):365–378, July 2000.

[5] B. Cockburn and C.W. Shu. The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM J. Numer. Anal.*, 35(6):2440–2463, December 1998.

[6] L. Cowsar, J. Mandel, and M. Wheeler. Balancing domain decomposition for mixed finite elements. *Math. Comp.*, 64(211):989–1015, 1995.

[7] Laslo T. Diosady and David L. Darmofal. A unified analysis of balancing domain decomposition by constraints for discontinuous Galerkin discretizations. *Submitted to SIAM J. Numer. Anal.*, 2011.

[8] Clark R. Dohrmann. A preconditioner for substructuring based on constrained energy minimization. *SIAM J. Sci. Comput.*, 25(1):246–258, 2003.

[9] M. Dryja, J. Galvis, and M. Sarkis. BDDC methods for discontinuous Galerkin discretization of elliptic problems. *J. Complexity*, 23(4):715–739, 2007.

[10] X. Feng and O.A. Karakashian. Two-level additive Schwarz methods for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.*, 39(4):1343–1365, 2002.

[11] Axel Klawonn, Luca F. Pavarino, and Oliver Rheinbach. Spectral element FETI-DP and BDDC preconditioners with multi-element subdomains. *Comput. Methods Appl. Mech. Engrg.*, 198:511–523, 2008.

[12] J. Li and O.B. Widlund. FETI-DP, BDDC, and block Cholesky methods. *Internat. J. Numer. Methods Engrg.*, 66:250–271, 2006.

[13] Jan Mandel and Clark R. Dohrmann. Convergence of a balancing domain decomposition by constraints and energy minimization. *Numer. Linear Algebra Appl.*, 10:639–659, 2003.

[14] Jan Mandel and B. Sousedik. BDDC and FETI-DP under minimalist assumptions. *Computing*, 81:269–280, 2007.

[15] L.F. Pavarino. BDDC and FETI-DP preconditioners for spectral element discretizations. *Comput. Methods Appl. Mech. Engrg.*, 196(8):1380–1388, 2007.

[16] J. Peraire and P-O. Persson. The compact discontinuous Galerkin (CDG) method for elliptic problems. *SIAM J. Sci. Comput.*, 30(4):1806–1824, 2008.

[17] X. Tu. A BDDC algorithm for flow in porous media with a hybrid finite element discretization. *Electron. Trans. Numer. Anal.*, 26:146–160, 2007.