Scalable Domain Decomposition Algorithms for Contact Problems: Theory, Numerical Experiments, and Real World Problems

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Summary. We review our results related to the development of theoretically supported scalable algorithms for the solution of large scale contact problems of elasticity. The algorithms combine the Total FETI/BETI based domain decomposition method adapted to the solution of 2D and 3D multibody contact problems of elasticity, both frictionless and with friction, with our in a sense optimal algorithms for the solution of resulting quadratic programming and QPQC problems. Rather surprisingly, the theoretical results are qualitatively the same as the classical results on scalability of FETI/BETI for linear elliptic problems. The efficiency of the method is demonstrated by results of parallel numerical experiments for contact problems of linear elasticity discretized by more than 11 million variables in 3D and 40 million variables in 2D.

1 Introduction

Contact problems are in the heart of mechanical engineering. Solving large multibody contact problems of linear elastostatics is complicated by the inequality boundary conditions, which make them strongly non-linear, and, if the system of bodies includes "floating" bodies, by the positive semi-definite stiffness matrices resulting from the discretization of such bodies. Observing that the classical Dirichlet and Neumann boundary conditions are known only after the solution has been found, it is natural to assume the solution of contact problems to be more costly than the solution of a related linear problem with the classical boundary conditions. Since the cost of the solution of any problem increases at least linearly with the number of the unknowns, it follows that the development of a scalable algorithm for contact problems is a challenging task which requires to identify the contact interface in a sense for free.

The first promising results, at least for the frictionless problems, were obtained by the researchers who tried to modify the methods that were known to be scalable

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for linear problems, in particular multigrid and domain decomposition. Experimental evidence of scalability was achieved with the monotonic multigrid (see [11] and the references therein). In spite of these nice results, the necessity to keep the coarse grid away from the contact interface prevented the authors to prove the optimality results similar to the classical results for linear problems. However, such result was obtained by Schöberl who has developed an approximate variant of the projection method using a domain decomposition preconditioner and a linear multigrid solver on the interior nodes. An experimental evidence of scalability for the frictionless problems was presented by Avery and Farhat [1]. The point of this paper is to report our optimality results for contact problems of linear elasticity, both frictionless and with friction.

The results are based on a combination of several ingredients. The first one is the application of the TFETI (Total FETI) [8] or TBETI (Total BETI) [14] methods, variants of the duality based domain decomposition methods introduced by Farhat and Roux [9] (finite elements) and Langer and Steinbach [13] (boundary elements). Since the TFETI/TBETI methods treat all the subdomains as "floating", the kernels of the stiffness matrices of the subdomains are a priori known. This makes the method very flexible and simplifies implementation of the multiplication of a vector by a generalized inverse of the stiffness matrix. As any duality based method, TFETI/TBETI reduces general inequality constraints to special separable ones.

The second ingredient is the "natural coarse grid preconditioning" introduced for linear problems by Farhat, Mandel, and Roux [10] and Langer and Steinbach [13]. This preconditioned cost function has the spectrum of the Hessian confined to a positive interval independent of the discretization parameter h and the decomposition parameter H provided the ratio H/h is uniformly bounded. Since our preconditioning uses a projector to the subspace with the solution, it follows that its application to the solution of variational inequalities does not turn the separable constraints into general constraints and can be interpreted as a variant of the multigrid method with the coarse grid on the interface. This unique feature, as compared with the standard multigrid preconditioning for the primal problem, reduces the development of scalable algorithms for the solution of variational inequalities to the solution of bound and equality constrained quadratic programming or QPQC (quadratic programming with quadratic constraints) problems with the rate of convergence in terms of bounds on the spectrum.

The resulting QP and QPQC problems, arising in the solution of the frictionless contact problems and the problems with the Tresca friction (an auxiliary problem for Coulomb friction), respectively, are solved by our algorithms with the rate of convergence in terms of the bounds on the spectrum, the third ingredient of our development (see [7]). Putting the three ingredients together with a few simple observations, we get theoretically supported algorithms for contact problems. The theoretical results are illustrated by the results of numerical experiments which show that both numerical and parallel scalability can be observed in practice. Finally we report the solutions of some real world problems. More details can be found in Dostál et al. [3–5], and Sadowská et al. [14].

2 Dual Formulation of Frictionless Contact Problems

To simplify our presentation, let us assume that the bodies are assembled from N_s subdomains $\Omega^{(s)}$ which are "glued" together by suitable equality constraints. After the standard finite element discretization, the equilibrium of the system is described as a solution u of the problem

$$\min J(v) \quad \text{subject to} \quad \sum_{s=1}^{N_s} B_N^{(s)} v^{(s)} \leq g_N \quad \text{and} \quad \sum_{s=1}^{N_s} B_E^{(s)} v^{(s)} = o, \tag{1}$$

where o denotes the zero vector and $J(v)$ is the energy functional defined by

$$J(v) = \sum_{s=1}^{N_s} \frac{1}{2} v^{(s)T} K^{(s)} v^{(s)} - v^{(s)T} f^{(s)},$$

$v^{(s)}$ and $f^{(s)}$ denote the admissible subdomain displacements and the subdomain vector of prescribed forces, $K^{(s)}$ is the subdomain stiffness matrix, $B_N^{(s)} \in \mathbb{R}^{m_C \times n}$ and $B_E^{(s)} \in \mathbb{R}^{m_E \times n}$ are the blocks of the matrix $B = \left[B_N^T, B_E^T\right]^T$ that correspond to $\Omega^{(s)}$, and g_N is a vector collecting the normal gaps between the bodies in the reference configuration. The matrix B_N and the vector g_N arise from the nodal or mortar description of the non-penetration conditions, while B_E describes the "gluing" of the subdomains into the bodies and the Dirichlet boundary conditions. Recall that if the problem is discretized by the TBETI method, then we get the potential energy minimization problem of the very same structure as (1), where all the objects correspond only to the boundaries $\Gamma^{(s)}$ of $\Omega^{(s)}$ except the term with the prescribed volume forces (if there is some); see [14] for more details. By contrast with TFETI, when the matrices $K^{(s)}$ are sparse, in the case of TBETI these are fully populated.

To simplify the presentation of basic ideas, we can describe the equilibrium in terms of the global stiffness matrix K, the vector of global displacements u, and the vector of global loads f. In the TFETI/TBETI methods, we have

$$K = \operatorname{diag}(K^{(1)}, \ldots, K^{(N_s)}), \quad u = \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N_s)} \end{bmatrix}, \quad \text{and} \quad \mathrm{f} = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N_s)} \end{bmatrix},$$

where $K^{(s)}$, $s = 1, \ldots, N_s$, is a positive semidefinite matrix. The energy function reads

$$j(v) = \frac{1}{2} v^T K v - f^T v$$

and the vector of global displacements u solves

$$\min j(v) \quad \text{s.t.} \quad B_N v \leq g_N \quad \text{and} \quad B_E v = o.$$

Alternatively, the global equilibrium may be described by the Karush–Kuhn–Tucker conditions (see, e.g., [6])

$$Ku = f - B^T\lambda, \quad \lambda_N \geq o, \quad \lambda^T(Bu - g) = o, \tag{2}$$

where $g = \left[g_N^T, o^T\right]^T$ and $\lambda = \left[\lambda_N^T, \lambda_E^T\right]^T$ denotes the vector of Lagrange multipliers which may be interpreted as the reaction forces. The problem (2) differs from the linear problem by the non-negativity constraint on the components of reaction forces λ_N and by the complementarity condition.

We can use the first equation of (2) to eliminate the displacements. We shall get the problem to find

$$\min \Theta(\lambda) \quad \text{s.t.} \quad \lambda_N \geq o \quad \text{and} \quad R^T(f - B^T\lambda) = o, \tag{3}$$

where

$$\Theta(\lambda) = \frac{1}{2}\lambda^T BK^+B^T\lambda - \lambda^T(BK^+f - g) + \frac{1}{2}fK^+f, \tag{4}$$

K^+ denotes a generalized inverse that satisfies $KK^+K = K$, and R denotes the full rank matrix whose columns span the kernel of K. The action of K^+ can be evaluated at the cost comparable with that of Cholesky's decomposition applied to the regularized K (see [2]). Denoting $\mathscr{F} = \|BK^+B^T\|$,

$$F = \mathscr{F}^{-1}BK^+B^T, \quad e = SR^Tf, \quad G = SR^TB^T, \quad \widetilde{d} = \mathscr{F}^{-1}(BK^\dagger f - g),$$

with S denoting a nonsingular matrix that defines the orthonormalization of the rows of R^TB^T, we can modify (3) to

$$\min \widetilde{\theta}(\lambda) \quad \text{s.t.} \quad \lambda_N \geq 0 \quad \text{and} \quad G\lambda = e, \tag{5}$$

where

$$\widetilde{\theta}(\lambda) = \frac{1}{2}\lambda^T F\lambda - \lambda^T\widetilde{d}. \tag{6}$$

Our next step is to replace the equality constraint in (5) by a homogeneous one. To this end, it is enough to find any $\widetilde{\lambda}$ such that

$$G\widetilde{\lambda} = e,$$

denote $\lambda = \mu + \widetilde{\lambda}$, and substitute into (5). We get

$$\widetilde{\theta}(\lambda) = \frac{1}{2}\mu^T F\mu - \mu^T(\widetilde{d} - F\widetilde{\lambda}) + const.$$

After returning to the old notation, problem (5) is reduced to

$$\min \frac{1}{2}\lambda^T F\lambda - \lambda^T d \quad \text{s.t.} \quad G\lambda = o \quad \text{and} \quad \lambda_N \geq \ell_N \tag{7}$$

with $\ell = -\widetilde{\lambda}$ and $d = \widetilde{d} - F\widetilde{\lambda}$. Since G has orthonormal rows, we can use the least square solution

$$\widetilde{\lambda} = G^Te. \tag{8}$$

3 Dual Formulation of Contact Problems with Tresca Friction

If the Tresca friction is prescribed on the contact interface, then the equilibrium of the system is described as a solution u of the problem

$$\min J_T(v) \quad \text{subject to} \quad \sum_{s=1}^{N_s} B_N^{(s)} v^{(s)} \le g_N \quad \text{and} \quad \sum_{s=1}^{N_s} B_E^{(s)} v^{(s)} = o, \tag{9}$$

where $J_T(v)$ is the energy functional defined by

$$J_T(v) = J(v) + j(v), \quad j(v) = \sum_{i=1}^{m_C} \Psi_i \|T_i u\|,$$

Ψ_i denotes an a priori defined slip bound at node i, and $T_i u$ denotes the jump of the tangential displacement due to the displacement u. Using the standard procedure to modify the non-differentiable term j (see [3, 5]), we get

$$j(v) = \sum_{i=1}^{m_C} \Psi_i \|T_i u\| = \sum_{i=1}^{m_C} \max_{\|\tau_i\| \le \Psi_i} \tau_i^T T_i u,$$

where τ_i can be considered as Lagrange multipliers. We assume that B_N, B_E, and T are full rank matrices.

Let $\overline{d}$ denote the spatial dimension and let us introduce the Lagrangian with three types of Lagrange multipliers, namely $\lambda_N \in \mathbb{R}^{m_C}$ associated with the non-interpenetration condition, $\lambda_E \in \mathbb{R}^{m_E}$ associated with the "gluing" and prescribed displacements, and

$$\tau = [\tau_1^T, \tau_2^T, \ldots, \tau_{m_C}^T]^T \in \mathbb{R}^{(\overline{d}-1)m_C}$$

which regularizes the non-differentiability. The Lagrangian associated with problem (1) reads

$$L(u, \lambda_N, \lambda_E, \tau) = J(u) + \tau^T T u + \lambda_N^T (B_N u - c_N) + \lambda_E^T (B_E u - c_E). \tag{10}$$

Using the convexity of the cost function and constraints, we can use the classical duality theory [6] to reformulate problem (9) to get

$$\min_u \sup_{\substack{\lambda_E \in \mathbb{R}^{m_E},\ \lambda_N \ge \mathbf{o} \\ \|\tau_i\| \le \Psi_i,\ i=1,\ldots,m_C}} L(u, \lambda_N, \lambda_E, \tau) = \max_{\substack{\lambda_E \in \mathbb{R}^{m_E},\ \lambda_N \ge o \\ \|\tau_i\| \le \Psi_i,\ i=1,\ldots,m_C}} \min_u L(u, \lambda_N, \lambda_E, \tau).$$

To simplify the notation, we denote

$$\lambda = \begin{bmatrix} \lambda_E \\ \lambda_N \\ \tau \end{bmatrix}, \quad B = \begin{bmatrix} B_E \\ B_N \\ T \end{bmatrix}, \quad c = \begin{bmatrix} c_E \\ c_N \\ o \end{bmatrix},$$

and

$$\Lambda(\Psi) = \left\{(\lambda_E^T, \lambda_N^T, \tau^T)^T \in \mathbb{R}^{m_E+\overline{d}m_C} : \lambda_N \ge o, \|\tau_i\| \le \Psi_i,\ i = 1,\dots,m_C\right\},$$

so that we can write the Lagrangian briefly as

$$L(u,\lambda) = \frac{1}{2}u^T K u - f^T u + \lambda^T (Bu - c)$$

and problem (9) is equivalent to the saddle point problem

$$L(\widehat{u},\widehat{\lambda}) = \max_{\lambda \in \Lambda(\Psi)} \min_u L(u,\lambda). \tag{11}$$

Similarly to the frictionless case, we eliminate the primal variables from (11) and carry out the homogenization to reduce the minimization problem to

$$\min\ \frac{1}{2}\lambda^T F \lambda - \lambda^T d \quad \text{s.t.} \quad G\lambda = o \quad \text{and} \quad \lambda \in \Lambda(\Psi) \tag{12}$$

with the notation of Sect. 2. Notice that we minimize exactly the same type of the cost function as in the frictionless case, but with some additional quadratic constraints.

4 Preconditioning by Projector

Our final step is based on the observation that both the frictionless contact problem and the contact problem with Tresca friction are equivalent to

$$\min \theta(\lambda) \quad \text{s.t.} \quad \lambda \in \Omega, \tag{13}$$

where

$$\theta(\lambda) = \frac{1}{2}\lambda^T (PFP + \overline{\rho}Q)\lambda - \lambda^T P d, \quad Q = G^T (GG^T)^{-1} G, \quad P = I - Q,$$

$\overline{\rho} > 0$, and $\Omega = \{\lambda : G\lambda = o$ and $\lambda_N \ge o\}$ (without friction) or $\Omega = \{\lambda : G\lambda = o$ and $\lambda \in \Lambda(\Psi)\}$ (Tresca). A good choice of the regularization parameter is given by

$$\overline{\rho} = \|PFP\|,$$

as this is the largest value for which

$$\|PFP\| \ge \|PFP + \overline{\rho}Q\|.$$

Problem (13) turns out to be a suitable starting point for development of an efficient algorithm for variational inequalities due to the following classical estimates [10] of the extreme eigenvalues.

Theorem 1. *If the decompositions and the discretizations of given contact problems are sufficiently regular, then there are constants $C_1 > 0$ and $C_2 > 0$ independent of the discretization parameter h and the decomposition parameter H such that*

$$C_1 \frac{h}{H} \le \lambda_{\min}(PFP|\mathrm{Im}P) \quad \text{and} \quad \lambda_{\max}(PFP|\mathrm{Im}P) = \|PFP\| \le C_2, \tag{14}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the extremal eigenvalues of the corresponding matrices.

5 Optimality

Theorem 1 states that if we fix the regularization parameter $\overline{\rho}$ and keep H/h uniformly bounded, then problem (13) resulting from the application of various discretizations and decompositions has the spectrum of the Hessian matrices confined to a positive interval. It follows that to develop a scalable algorithm for the contact problems, it is enough to find an algorithm that is able to find an approximate solution of (13) in a number of matrix–vector multiplications uniformly bounded in terms of bounds on the spectrum of the cost function.

Here we propose to use SMALSE (semi-monotonic augmented Lagrangian method for separable and equality constraints), our variant of the augmented Lagrangian method [7]. SMALSE enforces the equality constraints by the Lagrange multipliers generated in the outer loop, while the auxiliary QPQC problems with separable constraints are solved approximately in the inner loop by the MPGP algorithm proposed by Dostál and Kozubek [7]. MPGP is an active set based algorithm which uses the conjugate gradient method to explore the current face, the fixed steplength gradient projection to change the active set, and the adaptive precision control for the solution of auxiliary linear problems. The unique feature of SMALSE with the inner loop implemented by MPGP when used to (13) is the bound on the number of iterations whose cost is proportional to the number of variables, so that it can return an approximate solution for the cost proportional to the number of variables. It follows that SMALSE/MPGP is a scalable algorithm for the solution of (13) provided the cost of decomposition of K and application of the projectors P and Q is not too large.

Theorem 2. *If the decompositions and the discretizations of a given contact problem are sufficiently regular, then there is a constant $C > 0$ independent of the discretization parameter h and the decomposition parameter H such that the algorithm SMALSE/MPGP (or SMALBE/MPRGP for the frictionless problems) with fixed parameters specified in [7] can find the solution of (13) in a number of iterations bounded by C provided the initial approximation satisfies*

$$\|\lambda^0\| \leq c\|Pd\|,$$

where $c > 0$ is an a priori chosen constant.

6 Numerical Experiments

The algorithms reported in this paper were implemented into our MatSol software [12] and tested with the aim to verify their optimality and capability to solve the real world problems.

6.1 Scalability of TFETI: 2D Cantilever Beams with Tresca Friction

We first tested the scalability on a 2D problem of Fig. 1 with varying discretizations and decompositions using structured grids. We kept the ratio H/h of the decomposition and the discretization parameters approximately constant so that the assumptions of Theorem 1 were satisfied.

The results of computations carried out to the relative precision 10^{-4} are in Table 1. We can observe that the number of matrix–vector multiplications varies only mildly with the increasing dimension of the problem in agreement with the theory. We conclude that the scalability can be observed in practice.

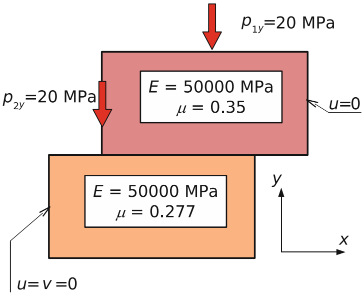

Fig. 1. Geometry of 2D cantilever beams

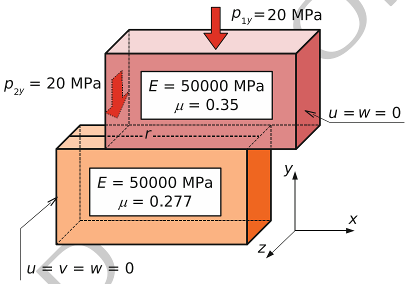

Fig. 2. Geometry of 3D cantilever beams

Table 1. Numerical scalability of TFETI: 2D cantilever beams.

Number of subdomains	1936	4096	7744
Number of CPUs	48	48	48
Primal variables	10,071,072	21,307,392	40,284,288
Dual variables	384,473	817,793	1,551,089
Null space dimension	5808	12,288	23,232
SMALSE-M iterations	8	8	8
Hessian multiplications	119	134	180
Solution time [s]	839	1665	7825

6.2 Scalability of TFETI/TBETI: 3D Cantilever Beams with Tresca Friction

The second problem was a 3D alternative to the previous example (see Fig. 2). The results of computations carried out for both TFETI and TBETI methods are in Tables 2 and 3, respectively. We can see that the number of matrix–vector multiplications again varies only mildly with the increasing problem size as predicted by the theory.

Table 2. Numerical scalability of TFETI: 3D cantilever beams.

Number of subdomains	108	500	1372	2916
Number of CPUs	48	48	48	48
Primal variables	431,244	1,996,500	5,478,396	11,643,588
Dual variables	88,601	444,927	1,261,493	2,728,955
Null space dimension	648	3000	8232	17,496
SMALSE-M iterations	3	4	4	4
Hessian multiplications	78	97	93	119
Solution time [s]	60	374	1663	7745

Table 3. Numerical scalability of TBETI: 3D cantilever beams.

Number of subdomains	108	500	1372	2916
Number of CPUs	48	48	48	48
Primal variables	195,045	903,000	2,477,830	5,266,300
Dual variables	88,601	444,927	1,261,493	2,728,955
Null space dimension	648	3000	8232	17,496
SMALSE-M iterations	7	8	9	9
Hessian multiplications	160	161	160	260
Solution time [s]	46	301	2211	7949

6.3 Applications of TFETI/TBETI to Real World Problems

We have also tested our algorithms on real world problems. First we consider the analysis of the stress in the roller bearings of Fig. 3. The problem is difficult because it consists of 73 bodies in mutual contact and only one is fixed in space. The solution of the problem discretized by 2,730,000/459,800 primal/dual variables and decomposed into 700 subdomains required 4,270 matrix–vector multiplications. The von Mises stress distribution is in Fig. 3.

Second we consider the analysis of the yielding clamp connection of steel arched supports depicted in Fig. 4. This type of construction is used to support the mining openings. It is a typical multibody contact, where the yielding connection plays the role of the mechanical protection against destruction, i.e., against the total deformation of the supporting arches. We consider contact with the Coulomb friction, where the coefficient of friction was $\mathscr{F} = 0.5$. The problem was decomposed into 250 subdomains using METIS and discretized by 1,592,853 and 216,604 primal and dual variables, respectively. The total displacements for both TFETI and TBETI are depicted in Fig. 4. The solution required 1,922 matrix-vector multiplications.

7 Comments and Conclusions

The TFETI method turns out to be a powerful engine for the solution of contact problems of elasticity. The results of numerical experiments comply with the theoretical results and indicate high efficiency of the method reported here. Future research will include adaptation of the standard preconditioning strategies.

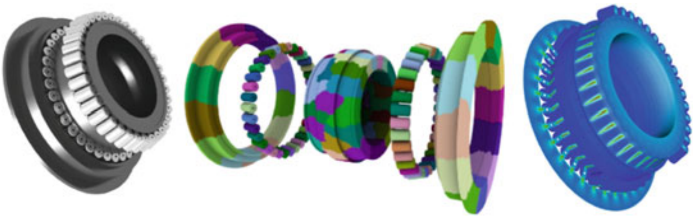

Fig. 3. Frictionless roller bearing of wind generator

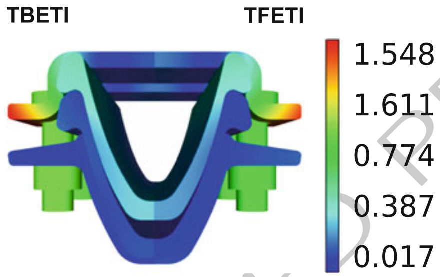

Fig. 4. Steel support with Coulomb friction

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Bibliography

[1] Philip Avery and Charbel Farhat. The FETI family of domain decomposition methods for inequality-constrained quadratic programming: application to contact problems with conforming and nonconforming interfaces. *Comput. Methods Appl. Mech. Engrg.*, 198(21–26):1673–1683, 2009. ISSN 0045-7825. doi: 10.1016/j.cma.2008.12.014. URL http://dx.doi.org/10.1016/j.cma.2008.12.014.

[2] T Brzobohatý, Z Dostál, T Kozubek, P Kovář, and A Markopoulos. Cholesky decomposition with fixing nodes to stable computation of a generalized inverse of the stiffness matrix of a floating structure. Accepted, 2011.

[3] Z. Dostál, T. Kozubek, P. Horyl, T. Brzobohatý, and A. Markopoulos. A scalable TFETI algorithm for two-dimensional multibody contact problems with friction. *J. Comput. Appl. Math.*, 235(2):403–418, 2010. ISSN 0377-0427. doi: 10.1016/j.cam.2010.05.042. URL http://dx.doi.org/10.1016/j.cam.2010.05.042.

[4] Z. Dostál, T. Kozubek, V. Vondrák, T. Brzobohatý, and A. Markopoulos. Scalable TFETI algorithm for the solution of multibody contact problems of elasticity. *Internat. J. Numer. Methods Engrg.*, 82(11):1384–1405, 2010. ISSN 0029-5981.

[5] Z. Dostál, T. Kozubek, A. Markopoulos, T. Brzobohatý, V. Vondrák, and P. Horyl. A theoretically supported scalable tfeti algorithm for the solution of multibody 3d contact problems with friction. In Press, 2011.

[6] Zdeněk Dostál. *Optimal quadratic programming algorithms*, volume 23 of *Springer Optimization and Its Applications*. Springer, New York, 2009. ISBN 978-0-387-84805-1. With applications to variational inequalities.

[7] Zdeněk Dostál and Tomáš Kozubek. An optimal algorithm and superrelaxation for minimization of a quadratic function subject to separable convex constraints with applications. *Mathematical Programming*, pages 1–26, 2011. ISSN 0025-5610. doi: 10.1007/s10107-011-0454-2.

[8] Zdeněk Dostál, David Horák, and Radek Kučera. Total FETI—an easier implementable variant of the FETI method for numerical solution of elliptic PDE. *Comm. Numer. Methods Engrg.*, 22(12):1155–1162, 2006. ISSN 1069-8299. doi: 10.1002/cnm.881. URL http://dx.doi.org/10.1002/cnm.881.

[9] Charbel Farhat and François-Xavier Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Internat. J. Numer. Methods Engrg.*, 32(6):1205–1227, 1991. ISSN 1097-0207. doi: 10.1002/nme.1620320604. URL http://dx.doi.org/10.1002/nme.1620320604.

[10] Charbel Farhat, Jan Mandel, and François-Xavier Roux. Optimal convergence properties of the FETI domain decomposition method. *Comput. Methods Appl. Mech. Engrg.*, 115(3–4):365–385, 1994. ISSN 0045-7825. doi: 10.1016/0045-7825(94)90068-X. URL http://dx.doi.org/10.1016/0045-7825(94)90068-X.

[11] Ralf Kornhuber. *Adaptive monotone multigrid methods for nonlinear variational problems*. Advances in Numerical Mathematics. B. G. Teubner, Stuttgart, 1997. ISBN 3-519-02722-4.

[12] T. Kozubek, A. Markopoulos, T. Brzobohatý, R. Kučera, V. Vondrák, and Z. Dostál. Matsol - matlab efficient solvers for problems in engineering. "http://matsol.vsb.cz/", 2009.

[13] U. Langer and O. Steinbach. Boundary element tearing and interconnecting methods. *Computing*, 71(3):205–228, 2003. ISSN 0010-485X. doi: 10.1007/s00607-003-0018-2. URL http://dx.doi.org/10.1007/s00607-003-0018-2.

[14] M. Sadowská, Z. Dostál, T. Kozubek, A. Markopoulos, and J. Bouchala. Scalable total beti based solver for 3d multibody frictionless contact problems in mechanical engineering. *Eng. Anal. Bound. Elem.*, 35(3):330–341, 2011. ISSN 0955-7997. doi: 10.1016/j.enganabound.2010.09.015.