Alternating and Linearized Alternating Schwarz Methods for Equidistributing Grids

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1 Introduction

The solution of partial differential equations (PDEs) with disparate space and time scales often benefit from the use of nonuniform meshes and adaptivity to successfully track local solution features.

In this paper we consider the problem of grid generation using the so–called equidistribution principle (EP) [3] and domain decomposition (DD) strategies. In the time dependent case, the EP is used to evolve an initial (often uniform) grid by **r**elocating a fixed number of mesh nodes. This leads to a class of adaptive methods known as **r**–refinement or moving mesh methods. A thorough recent review of moving mesh methods for PDEs can be found in the book [11].

In general, the appropriate grid for a particular problem depends on features of the (typically unknown) solution of the PDE. Here we will focus on the grid generation problem for the time independent, given function $u(x)$ of a single spatial variable $x \in [0,1]$. Given some positive measure $M(x)$ of the error or difficulty in the solution $u(x)$, the EP requires that the mesh points are chosen so that the error contribution on each interval $[x_{i-1}, x_i]$ is the same. The function M is known as the monitor or mesh density function. Mathematically, we may write this as

$$\int_{x_{i-1}}^{x_i} M(\tilde{x})\,d\tilde{x} \equiv \frac{1}{N}\int_0^1 M(\tilde{x})d\tilde{x} \quad \text{or} \quad \int_0^{x(\xi_i)} M(\tilde{x})\,d\tilde{x} = \frac{i}{N}\theta \equiv \xi_i\theta, \qquad \text{(EP)}$$

where $x(\xi_i) = x_i$ and $\theta \equiv \int_0^1 M(\tilde{x})\,d\tilde{x}$ is the total error in the solution. The EP defines a one–to–one co-ordinate transformation between the physical co–ordinate x and underlying computational co–ordinate ξ. This will naturally concentrate mesh points where the error in the solution is large.

Differentiating the continuous formulation of EP gives the required mesh transformation, $x(\xi)$, as the solution of the nonlinear boundary value problem

$$\frac{d}{d\xi}\left\{M(x(\xi))\frac{d}{d\xi}x(\xi)\right\} = 0, \quad x(0) = 0 \quad \text{and} \quad x(1) = 1. \qquad (1)$$

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If M is chosen properly, we expect the solution $u(x)$ to be easy to represent on a uniform grid in ξ. In general, the physical solution u is not known and instead satisfies a PDE. In that case, the mesh transformation, satisfying (1), and the physical PDE, are coupled and often solved in an iterative fashion.

We will assume (1) has a unique solution, see [8] for details. In [8], the authors consider the solution of (1) and time dependent extensions using classical parallel, optimized and optimal Schwarz methods. In this paper we continue the work of [8] by providing details of the nonlinear and linearized alternating Schwarz approaches. The reader is also referred to the experimental papers [7, 9, 10], which proposed various strategies to couple DD and moving meshes. See [1, 2, 4–6, 12–15] for a discussion of DD methods applied to other nonlinear PDEs.

In Sect. 2 we propose a new nonlinear alternating Schwarz method to solve (1) and prove convergence in L^∞. In Sect. 3 we avoid the nonlinear subdomain problems and propose and analyze a linearized alternating Schwarz algorithm. Brief numerical results are presented in the final section.

2 A Nonlinear Alternating Schwarz Method

In [8] we consider the solution of (1) by a parallel, classical nonlinear Schwarz iteration. On each subdomain a nonlinear BVP is solved and Dirichlet transmission conditions are used at the subdomain interfaces. Convergence of the iteration can be accelerated if we are willing to compute sequentially. Consider the nonlinear alternating Schwarz iteration

$$\begin{aligned} (M(x_1^n)x_{1,\xi}^n)_\xi &= 0,\ \xi\in\Omega_1, & (M(x_2^n)x_{2,\xi}^n)_\xi &= 0,\ \xi\in\Omega_2,\\ x_1^n(0) &= 0, & x_2^n(\alpha) &= x_1^n(\alpha),\\ x_1^n(\beta) &= x_2^{n-1}(\beta), & x_2^n(1) &= 1, \end{aligned} \tag{2}$$

where $\Omega_1=(0,\beta)$ and $\Omega_2=(\alpha,1)$ with $\alpha<\beta$.

Direct integration and enforcing the boundary conditions gives the following implicit representation of the subdomain solutions.

Lemma 1. *The subdomain solutions on Ω_1 and Ω_2 of (2) are given implicitly as*

$$\int_0^{x_1^n(\xi)} M(\tilde{x})\,d\tilde{x} = \frac{\xi}{\beta}\int_0^{x_2^{n-1}(\beta)} M(\tilde{x})\,d\tilde{x}, \tag{3}$$

and

$$\int_0^{x_2^n(\xi)} M(\tilde{x})\,d\tilde{x} = \frac{1-\xi}{1-\alpha}\int_0^{x_1^n(\alpha)} M(\tilde{x})\,d\tilde{x} + \frac{\xi-\alpha}{1-\alpha}\int_0^1 M(\tilde{x})\,d\tilde{x}. \tag{4}$$

Let $\|\cdot\|_\infty$ denote the usual L^∞ norm. We now relate $x_{1,2}^n$ to $x_{1,2}^{n-1}$ and obtain the following result.

Theorem 1. *Assume M is differentiable and there exist positive constants a and A satisfying $0 < a \le M(x) \le A < \infty$. Then the alternating Schwarz iteration (2) converges for any initial guess $x_2^0(\beta)$ and we have the error estimates*

$$||x - x_1^{n+1}||_\infty \le \rho^n \frac{A}{a}|x(\beta) - x_2^0(\beta)|, \quad ||x - x_2^{n+1}||_\infty \le \rho^n \frac{A}{a}|x(\alpha) - x_1^0(\alpha)|, \tag{5}$$

with contraction factor $\rho := \frac{\alpha}{\beta}\frac{1-\beta}{1-\alpha} < 1$.

Proof. Evaluating (3) at $\xi = \alpha$ and using the expression for $x_2^{n-1}(\beta)$ from (4) we have

$$\int_0^{x_1^n(\alpha)} M\,d\tilde{x} = \frac{\alpha}{\beta}\left\{\frac{\beta-1}{\alpha-1}\int_0^{x_1^{n-1}(\alpha)} M\,d\tilde{x} + \frac{\beta-\alpha}{1-\alpha}\int_0^1 M\,d\tilde{x}\right\}.$$

Defining the two quantities

$$K_1^n = \int_0^{x_1^n(\alpha)} M(\tilde{x})\,d\tilde{x} \qquad \text{and} \qquad C = \int_0^1 M(\tilde{x})\,d\tilde{x},$$

we obtain the linear iteration

$$K_1^n = \frac{\alpha}{\beta}\frac{\beta-1}{\alpha-1}K_1^{n-1} + \frac{\alpha}{\beta}\frac{\beta-\alpha}{1-\alpha}C. \tag{6}$$

This iteration converges with rate $\rho := \frac{\alpha}{\beta}\frac{1-\beta}{1-\alpha} < 1$, and has the limit

$$K_1^* = \frac{\alpha}{\beta}\frac{1-\beta}{1-\alpha}K_1^* + \frac{\alpha}{\beta}\frac{\beta-\alpha}{1-\alpha}C \quad \Longrightarrow K_1^* = \alpha C. \tag{7}$$

Since the monodomain solution also satisfies

$$\int_0^{x(\alpha)} M(\tilde{x})\,d\tilde{x} = \alpha C,$$

and $M(x) \ge a > 0$, we have convergence at the interface to the correct limit.

Subtracting (6) from (7) we have

$$\int_{x_1^n(\alpha)}^{x(\alpha)} M(\tilde{x})\,d\tilde{x} = \rho^n \int_{x_1^0(\alpha)}^{x(\alpha)} M(\tilde{x})\,d\tilde{x}. \tag{8}$$

Subtracting (4) from the equivalent expression for the exact solution and using (8) we obtain

$$\int_{x_2^{n+1}(\xi)}^{x(\xi)} M(\tilde{x})\,d\tilde{x} = \frac{1-\xi}{1-\alpha}\int_{x_1^n(\alpha)}^{x(\alpha)} M(\tilde{x})\,d\tilde{x} = \frac{1-\xi}{1-\alpha}\rho^n \int_{x_1^0(\alpha)}^{x(\alpha)} M(\tilde{x})\,d\tilde{x}.$$

Taking the modulus and using the boundedness of M we obtain, for all $\xi \in [\alpha, 1]$,

$$|x(\xi) - x_2^{n+1}(\xi)| \le \frac{1-\xi}{1-\alpha}\rho^n \frac{A}{a}|x(\alpha) - x_1^0(\alpha)|.$$

Taking the supremum gives the second estimate in (5). The estimate on subdomain one is obtained similarly. □

3 A Linearized Alternating Schwarz Method

We may avoid nonlinear solves on each subdomain in (2) by considering a linearized alternating Schwarz iteration,

$$\begin{aligned} (M(x_1^{n-1})x_{1,\xi}^n)_\xi &= 0,\ \xi \in \Omega_1 & \qquad (M(x_2^{n-1})x_{2,\xi}^n)_\xi &= 0,\ \xi \in \Omega_2 \\ x_1^n(0) &= 0, & x_2^n(\alpha) &= x_1^n(\alpha), \\ x_1^n(\beta) &= x_2^{n-1}(\beta), & x_2^n(1) &= 1. \end{aligned} \tag{9}$$

At iteration n we evaluate the nonlinear diffusion coefficient M using the solution obtained from the previous iterate and obtain the updated solution by a single linear BVP solve on each subdomain. A simple calculation yields the following representation of the subdomain solutions.

Lemma 2. *The subdomain solutions of (9) are given by*

$$x_1^n(\xi) = x_2^{n-1}(\beta)\frac{\int_0^\xi \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}}{\int_0^\beta \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}}, \tag{10}$$

and

$$x_2^n(\xi) = x_1^n(\alpha) + (1 - x_1^n(\alpha))\frac{\int_\alpha^\xi \frac{d\tilde{\xi}}{M(x_2^{n-1}(\tilde{\xi}))}}{\int_\alpha^1 \frac{d\tilde{\xi}}{M(x_2^{n-1}(\tilde{\xi}))}}. \tag{11}$$

Convergence of the linearized alternating Schwarz iteration (9) follows by proving convergence at the interior interfaces and showing we have converged to the correct limit.

Theorem 2. *Under the assumptions of Theorem 1, the linearized alternating Schwarz iteration (9) converges for any smooth initial guesses $x_1^0(\xi)$ and $x_2^0(\xi)$.*

Proof. Evaluating the subdomain solutions (10) and (11) at the interfaces, we obtain for the interface values the iterations

$$x_1^n(\alpha) = \mathscr{C}_\alpha^n x_1^{n-1}(\alpha) + \mathscr{D}_\alpha^n \quad \text{and} \quad x_2^n(\beta) = \mathscr{C}_\beta^n x_2^{n-1}(\beta) + \mathscr{D}_\beta^n,$$

where

$$\mathscr{C}_\alpha^n = \frac{\int_\beta^1 \frac{d\tilde{\xi}}{M(x_2^{n-2}(\tilde{\xi}))}}{\int_\alpha^1 \frac{d\tilde{\xi}}{M(x_2^{n-2}(\tilde{\xi}))}} \frac{\int_0^\alpha \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}}{\int_0^\beta \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}}, \quad \mathscr{D}_\alpha^n = \frac{\int_\alpha^\beta \frac{d\tilde{\xi}}{M(x_2^{n-2}(\tilde{\xi}))}}{\int_\alpha^1 \frac{d\tilde{\xi}}{M(x_2^{n-2}(\tilde{\xi}))}} \frac{\int_0^\alpha \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}}{\int_0^\beta \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}},$$

and

$$\mathscr{C}_\beta^n = \frac{\int_\beta^1 \frac{d\tilde{\xi}}{M(x_2^{n-1}(\tilde{\xi}))}}{\int_\alpha^1 \frac{d\tilde{\xi}}{M(x_2^{n-1}(\tilde{\xi}))}} \frac{\int_0^\alpha \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}}{\int_0^\beta \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}}, \quad \mathscr{D}_\beta^n = \frac{\int_\alpha^\beta \frac{d\tilde{\xi}}{M(x_2^{n-1}(\tilde{\xi}))}}{\int_\alpha^1 \frac{d\tilde{\xi}}{M(x_2^{n-1}(\tilde{\xi}))}}.$$

It is possible to show the quantities $\mathscr{C}_\alpha^n, \mathscr{D}_\alpha^n, \mathscr{C}_\beta^n$ and $\mathscr{D}_\beta^n$ satisfy

$$0 < \mathscr{C}_\alpha^n, \mathscr{C}_\beta^n \le \rho < 1, \quad 0 < \mathscr{D}_\alpha^n \le D_\alpha < 1, \quad \text{and} \quad 0 < \mathscr{D}_\beta^n \le D_\beta < 1,$$

where

$$\rho := \frac{1}{1+\frac{a}{A}\frac{\beta-\alpha}{1-\beta}}\frac{1}{1+\frac{a}{A}\frac{\beta-\alpha}{\alpha}}, \quad D_\alpha := \frac{1}{1+\frac{a}{A}\frac{\beta-\alpha}{\alpha}}\frac{1}{1+\frac{a}{A}\frac{1-\beta}{\beta-\alpha}}, \quad \text{and} \quad D_\beta := \frac{1}{1+\frac{a}{A}\frac{1-\beta}{\beta-\alpha}}.$$

To establish these bounds let $F(x) := 1/M(x)$. The assumptions on M imply $\frac{1}{A} \le F(x) \le \frac{1}{a}$. As an example, the upper and lower bounds on F then imply

$$\frac{\int_0^\alpha F(x(\xi))\,d\xi}{\int_0^\beta F(x(\xi))\,d\xi} \le \frac{1}{1+\frac{a}{A}\frac{\beta-\alpha}{\alpha}} \quad \text{and} \quad \frac{\int_\beta^1 F(x(\xi))\,d\xi}{\int_\alpha^1 F(x(\xi))\,d\xi} \le \frac{1}{1+\frac{a}{A}\frac{\beta-\alpha}{1-\beta}}.$$

Consider now the iteration for $x_1^n(\alpha)$ only. Using the recursion, we have

$$x_1^n(\alpha) = \prod_{k=1}^n \mathscr{C}_\alpha^k x_1^0(\alpha) + \sum_{k=1}^n \mathscr{D}_\alpha^k \left(\prod_{l=k+1}^n \mathscr{C}_\alpha^l\right),$$

where the product in the k–th term of the sum is assumed to be one if the lower index of the product exceeds the upper index. Since $\rho < 1$, the product multiplying $x_1^0(\alpha)$ must go to zero as $n \to \infty$. The infinite series converges by direct comparison with $\sum_{k=1}^\infty D_\alpha \rho^{k-1}$. A corresponding argument applies to show convergence of $x_2^n(\beta)$.

Denote the limits of $\{x_1^n(\alpha)\}$ and $\{x_2^n(\beta)\}$ as $\tilde{x}_\alpha$ and $\tilde{x}_\beta$ respectively. Since the interface values converge, the subdomain solutions defined by (9) converge to functions $\tilde{x}_1$ and $\tilde{x}_2$ both satisfying the nonlinear PDE. Since $\tilde{x}_1(\alpha) = \tilde{x}_2(\alpha)$ and $\tilde{x}_1(\beta) = \tilde{x}_2(\beta)$, both $\tilde{x}_1$ and $\tilde{x}_2$ satisfy the same PDE in the overlap with the same two boundary conditions, and by assumption of uniqueness, $\tilde{x}_1$ and $\tilde{x}_2$ must coincide in the overlap. One can therefore simply glue these two solutions together in order to obtain a function which satisfies the PDE everywhere, and also the two original boundary conditions at 0 and 1. Again by uniqueness, this must now be the desired solution. □

4 Numerical Results

In this section we numerically demonstrate the results above using a simple finite difference discretization of the BVP (1) and iterations (2) and (9). We also include results using nonlinear and linearized parallel Schwarz algorithm from [8] for comparison. Details of the numerical approach and convergence of the discrete DD algorithm will be considered elsewhere.

We solve EP for $u(x) = (1-e^{\lambda x})/(1-e^{\lambda})$ on the interval $x \in [0,1]$. For large values of λ this function exhibits a boundary layer at $x = 1$. We use the arc–length monitor function $M(x,u(x)) = \sqrt{1+u_x^2}$ and choose $\lambda = 20$. The errors reported in Figs. 1 and 2 are the differences between the single domain numerical solution and the domain decomposition solution over the first subdomain.

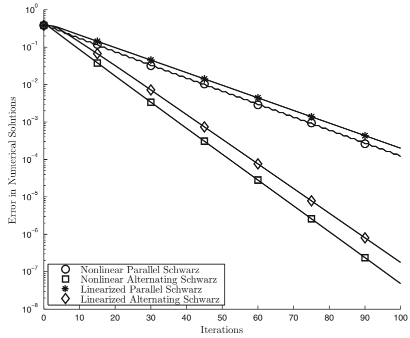

Fig. 1. Error versus # of DD iterations

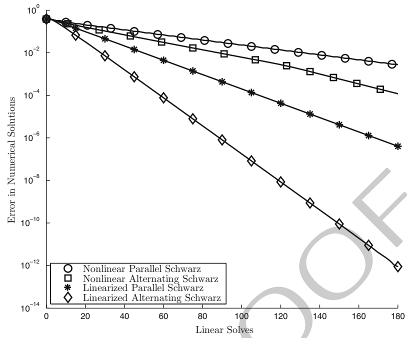

Fig. 2. Error versus # of linear solves

In Fig. 1 we solve (1) on two subdomains with a 5 % overlap using linearized and nonlinear, parallel and alternating Schwarz iterations. We see that the convergence of the alternating iteration is faster than the parallel algorithms for both the nonlinear and linearized versions of the algorithms. In terms of number of iterations the nonlinear algorithms outperform the linearized variants. It is important, however, to keep in mind that each nonlinear DD iteration is more expensive than its linearized counterpart. In Fig. 2 we repeat the convergence history as a function of a *work unit* which we take to be the cost of a linear solve. Each iteration of a linearized Schwarz algorithm requires one linear solve while each iteration of a nonlinear Schwarz algorithm requires many linear solves – one for each Newton step. Each linear solve required by both algorithms has roughly the same cost due to the structure of the Jacobian matrix. As a function of the work effort the efficacy of the linearized Schwarz algorithms is obvious for this example.

In Table 1 we demonstrate the quality of the computed grids by calculating the $\|\cdot\|_\infty$ error between $u(x)$ and the piecewise linear interpolant for $u(x)$ on grids obtained by the nonlinear and linearized alternating Schwarz algorithms, as a function of the number of iterations. The last column shows the interpolation error obtained with the single domain grid: the solution of (1) computed on a uniform ξ grid consisting of 101 points. All interpolation errors are computed using a very fine grid. The results show that the nonlinear Schwarz method is quickly able to find an appropriate grid transformation after a few DD iterations. The linearized Schwarz algorithm, as expected, requires more DD iterations but is able to find a quality grid efficiently due to the smaller relative cost per iteration.

Iterations	1	3	5	7	9	11	∞
Nonlinear	0.3625	0.0498	0.0462	0.0436	0.0449	0.0517	0.0366
Linearized	0.3625	0.1290	0.1019	0.0625	0.0453	0.0435	0.0366

Table 1. Interpolation errors for the grids obtained by Schwarz iterations.

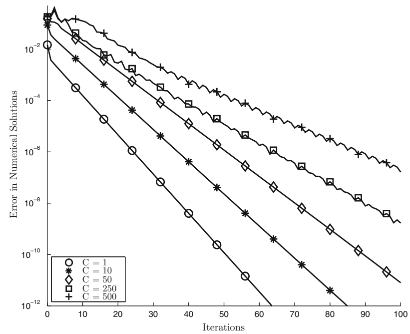

Fig. 3. Linearized Schwarz: error for varying C

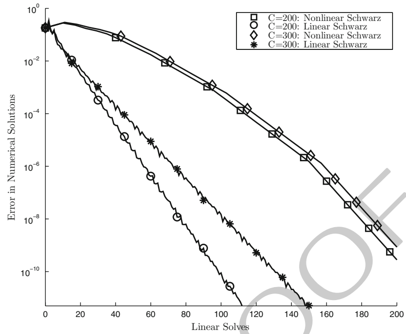

Fig. 4. Non-linear versus linearized Schwarz with varying C

The quantities ρ, D_α and D_β corresponding to iteration (9) and the error estimates in Theorem 1 indicate a dependence on the shape of M for the linearized alternating Schwarz iteration. To test this effect, we consider the performance of (9) for $M(x) = C(x-0.5)^2+1$. The parameter C controls the ratio a/A. As $C \to \infty$, $a/A \to 0$, and the contraction rate could diminish. This is demonstrated in Fig. 3. Figure 4 illustrates the effect of changing the value of C on both the nonlinear and linearized Schwarz algorithms. We see that the linearized Schwarz algorithm is affected more by an increase in C.

In summary, we have proposed, analyzed and provided brief numerical comparisons for two alternating Schwarz algorithms to solve the steady grid generation problem using the EP. Ongoing work includes the analysis of DD approaches to moving mesh PDEs for the time dependent mesh generation problem, the discrete analysis and extensions to higher dimensions.

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