On the Applicability of Lions' Energy Estimates in the Analysis of Discrete Optimized Schwarz Methods with Cross Points

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1 Introduction

For a bounded open subset $\Omega \subset \mathbb{R}^2$, suppose we want to solve

$$(\eta - \Delta)u = f \qquad \text{on } \Omega, \qquad u = g \qquad \text{on } \partial\Omega, \tag{1}$$

for $\eta \geq 0$ using the optimized Schwarz method (OSM)

$$\begin{aligned} &(\eta - \Delta)u_i^k = f|_{\Omega_i} \quad \text{on } \Omega_i, \qquad u_i^k = g|_{\partial\Omega_i} \quad \text{on } \partial\Omega_i \cap \partial\Omega, \\ &\frac{\partial u_i^k}{\partial n_i} + p_{ij}u_i^k = \frac{\partial u_j^{k-1}}{\partial n_i} + p_{ij}u_j^{k-1} \qquad \text{on } \Gamma_{ij} \text{ for all } \Gamma_{ij} \neq \emptyset, \end{aligned} \tag{2}$$

for $k = 1, 2, \ldots$ and $i = 1, \ldots, n$, where $\Omega_i \subset \Omega$ are non-overlapping subdomains, $\Gamma_{ij} = \partial\Omega_i \cap \overline{\Omega_j}$ is the interface between Ω_i and an adjacent subdomain Ω_j, $j \neq i$, and $p_{ij} > 0$ are Robin parameters along Γ_{ij}. In [7], the powerful technique of energy estimates is used to show convergence of (2) for $\eta = 0$ under very general conditions. Similar techniques have been used to prove convergence results for other types of equations, cf. [2] for the Helmholtz equation and [5] for the time-dependent wave equation. While one often assumes that the proof carries over trivially to finite-element discretizations, it has been reported in the literature (cf. [8, 9]) that discrete OSMs can diverge when the domain decomposition contains cross points, i.e., when more than two subdomains share a common point. This is in apparent contradiction to Lions' proof, and such difficulties contribute to the limited use of OSMs in practice. The goal of this paper is to explain why the presence of cross points makes it possible for the discrete OSM to diverge despite the proof of convergence at the continuous level, and why this difference in behavior is generally unavoidable.

The remainder of the paper proceeds as follows. In Sect. 2, we recall Lions' energy estimate argument. In Sect. 3, we explain why it is impossible to convert the continuous energy estimate into a discrete one in a generic way, without sacrificing continuity of the solutions across subdomain boundaries. In Sect. 4, we show two modifications that preserve continuity of the discrete solutions, but both must be used

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with Krylov methods to avoid divergent iterations. Finally, we show in Sect. 5 that a Lions-type discrete estimate can only hold under very stringent conditions; thus, continuous estimates generally do not predict the behavior of discrete OSMs.

2 Continuous Energy Estimates

We briefly recall the argument in [7] proving the convergence of (2). We assume $p_{ij} = p_{ji}$ to be a positive function that is bounded away from zero and defined on $\Gamma_{ij} = \Gamma_{ji}$. To show that (2) converges for all initial guesses, we first write the error equations

$$\begin{aligned} (\eta - \Delta)e_i^k = 0 \quad \text{on } \Omega_i, \qquad e_i^k = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_i, \\ \frac{\partial e_i^k}{\partial n_i} + p_{ij}e_i^k = \frac{\partial e_j^{k-1}}{\partial n_i} + p_{ij}e_j^{k-1} \quad \text{on } \Gamma_{ij} \text{ for all } \Gamma_{ij} \neq \emptyset, \end{aligned} \tag{3}$$

where $e_i = u_i^k - u|_{\Omega_i}$ with u being the exact solution to (1). We then multiply the first equation in (3) by e_i^k and integrate to get

$$0 = a_i(e_i^k, e_i^k) - \int_{\partial\Omega_i} e_i^k \frac{\partial e_i^k}{\partial n_i} = a_i(e_i^k, e_i^k) - \sum_{(i,j)\in E} \int_{\Gamma_{ij}} e_i^k \frac{\partial e_i^k}{\partial n_i},$$

where the last sum is over all pairs of subdomains (i, j) that share an interface, and $a_i(u_i, v_i) = \int_{\Omega_i} (\nabla u \cdot \nabla v + \eta uv)\,dx$ is the energy bilinear form defined on subdomain Ω_i, so that $a_i(e_i^k, e_i^k) = \int_{\Omega_i} \eta|e_i^k|^2 + |\nabla e_i^k|^2\,dx \geq 0$ is the energy of the error on subdomain Ω_i. We now rewrite the product term as

$$e_i^k \frac{\partial e_i^k}{\partial n_i} = \frac{1}{4p_{ij}} \left[\left(\frac{\partial e_i^k}{\partial n_i} + p_{ij}e_i^k \right)^2 - \left(-\frac{\partial e_i^k}{\partial n_i} + p_{ij}e_i^k \right)^2 \right] =: \left(T_{+ij}^k\right)^2 - \left(T_{-ij}^k\right)^2,$$

where $T_{\pm ij}^k = \frac{1}{\sqrt{4p_{ij}}}(\pm \frac{\partial e_i^k}{\partial n_i} + p_{ij}e_i^k)$. Since $\frac{\partial e_j^k}{\partial n_i} = -\frac{\partial e_j^k}{\partial n_j}$ on Γ_{ij}, the interface condition in (3) can be written as $T_{+ij}^k = T_{-ji}^{k-1}$, which means

$$a_i(e_i^k, e_i^k) = \sum_{(i,j)\in E} \int_{\Gamma_{ij}} \left[\left(T_{+ij}^k\right)^2 - \left(T_{-ij}^k\right)^2 \right] ds = \sum_{(i,j)\in E} \int_{\Gamma_{ij}} \left[\left(T_{-ji}^{k-1}\right)^2 - \left(T_{-ij}^k\right)^2 \right] ds.$$

Thus,

$$a_i(e_i^k, e_i^k) + \sum_{(i,j)\in E} \int_{\Gamma_{ij}} \left(T_{-ij}^k\right)^2 ds = \sum_{(i,j)\in E} \int_{\Gamma_{ij}} \left(T_{-ji}^{k-1}\right)^2 ds. \tag{4}$$

If we sum (4) through all subdomains i, we get

$$\sum_{i=1}^N a_i(e_i^k, e_i^k) + \sum_{i=1}^N \sum_{(i,j)\in E} \int_{\Gamma_{ij}} \left(T_{-ij}^k\right)^2 ds = \sum_{i=1}^N \sum_{(i,j)\in E} \int_{\Gamma_{ij}} \left(T_{-ji}^{k-1}\right)^2 ds. \tag{5}$$

We can now sum (5) over k and simplify to get

$$\sum_{k=0}^K \sum_{i=1}^N a_i(e_i^k, e_i^k) + B^K = B^0, \tag{6}$$

where $B^k := \sum_{i=1}^{N} \sum_{(i,j)\in E} \int_{\Gamma_{ij}} \left(T_{-ij}^k\right)^2 ds \geq 0$. Since $B^K \geq 0$ and each $a_i(e_i^k, e_i^k) \geq 0$, we see that $\sum_{k=0}^{K} a_i(e_i^k, e_i^k) \leq B^0$ for all i and all K; hence $a_i(e_i^k, e_i^k) \to 0$ as $k \to \infty$ for all i. This implies that $\|e_i^k\|_{H^1(\Omega_i)} \to 0$ when $\eta > 0$, so $u_i \to u|_{\Omega_i}$ in the H^1 norm. A similar argument holds for $\eta = 0$. Note that the possible presence of cross points does not cause any difficulty in the proof, since they form a subset of measure zero in $\partial\Omega_i$ and thus do not contribute to the boundary terms when integrating by parts.

3 Finite Element Discretization

We now try to mimic Lions' proof in the finite element case. The finite element method uses the weak form of (2), i.e., we must multiply the PDE by a test function ϕ and integrate by parts. The problem becomes

Find $u_i \in V^h \subset H^1(\Omega_i)$ s.t. for all $\phi \in W^h \subset H_0^1(\Omega) \cap H^1(\Omega_i)$,

$$\int_{\Omega_i} (\nabla\phi \cdot \nabla u_i^k + \eta\phi u_i^k) - \int_{\partial\Omega_i} \phi \frac{\partial u_i^k}{\partial n_i} = \int_{\Omega_i} \phi f. \tag{7}$$

We now suppose that ϕ is a basis function corresponding to a degree of freedom along Γ_{ij}, whose support does not contain any cross points, see Fig. 1a To obtain an expression for $\int_{\partial\Omega_i} \phi \frac{\partial u_i^k}{\partial n_i}$, we multiply the interface condition by ϕ and integrate to get

$$\int_{\Gamma_{ij}} \phi\left(\frac{\partial u_i^k}{\partial n_i} + p u_i^k\right) = \int_{\Gamma_{ij}} \phi\left(\frac{\partial u_j^{k-1}}{\partial n_i} + p u_j^{k-1}\right). \tag{8}$$

Substituting into (7) gives

$$a_i(\phi, u_i^k) + \int_{\Gamma_{ij}} \phi p u_i^k - \int_{\Gamma_{ij}} \phi \frac{\partial u_j^{k-1}}{\partial n_i} = \int_{\Omega_i} \phi f + \int_{\Gamma_{ij}} \phi p u_j^{k-1}. \tag{9}$$

Thus, we are faced with the same problem of finding an expression for $\int_{\Gamma_{ij}} \phi \frac{\partial u_j^{k-1}}{\partial n_i}$. Fortunately, we can use the weak form of the PDE from Ω_j

$$a_j(\phi, u_j^{k-1}) - \int_{\partial\Omega_j} \phi \frac{\partial u_j^{k-1}}{\partial n_j} = \int_{\Omega_j} \phi f. \tag{10}$$

Since $n_i = -n_j$ on Γ_{ij}, adding (9) and (10) and rearranging gives

$$a_i(\phi, u_i^k) + \int_{\Gamma_{ij}} \phi p u_i^k = \int_{\Omega_i} \phi f - a_j(\phi, u_j^{k-1}) + \int_{\Gamma_{ij}} \phi p u_j^{k-1}, \tag{11}$$

which is just the usual block-Jacobi splitting of the stiffness matrix along Γ_{ij}.

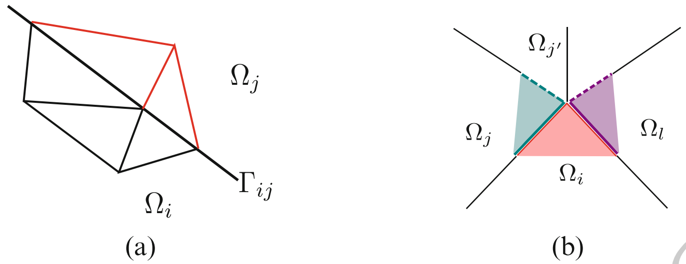


Fig. 1. Finite element discretization (**a**) without cross points and (**b**) with a cross point

Now assume that the support of ϕ contains cross points, see Fig. 1b. Here Ω_i is adjacent to two distinct subdomains Ω_j and Ω_l, $j \neq l$, and ϕ is non-zero on all three subdomains. Since the two parts of the interface, Γ_{ij} and Γ_{il}, must satisfy different interface conditions, we must separate $\int_{\partial\Omega_i} \phi \frac{\partial u_i^k}{\partial n}$ into contributions along Γ_{ij} and Γ_{il},

$$a_i(\phi, u_i^k) - \int_{\Gamma_{ij}} \phi \frac{\partial u_i^k}{\partial n_i} - \int_{\Gamma_{il}} \phi \frac{\partial u_i^k}{\partial n_i} = \int_{\Omega_i} \phi f.$$

The boundary term along Γ_{ij} can be replaced by the interface condition

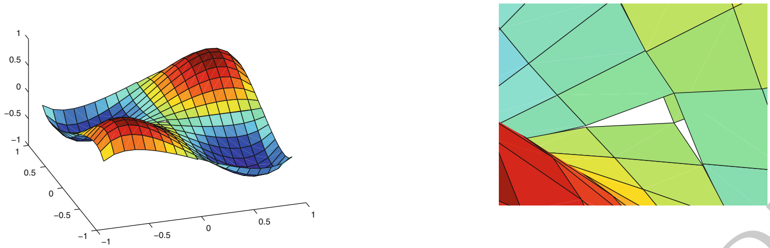
$$\int_{\Gamma_{ij}} \phi \left(\frac{\partial u_i^k}{\partial n_i} + p u_i^k\right) = \int_{\Gamma_{ij}} \phi \left(\frac{\partial u_j^{k-1}}{\partial n_i} + p u_j^{k-1}\right),$$

but now if we try to use the PDE on Ω_j to eliminate the term $\int_{\Gamma_{ij}} \phi \frac{\partial u_j^{k-1}}{\partial n_i}$, we would get

$$\int_{\Gamma_{ij}} \phi \frac{\partial u_j^{k-1}}{\partial n_j} = a_j(\phi, u_j^{k-1}) - \int_{\Gamma_{jj'}} \phi \frac{\partial u_j^{k-1}}{\partial n_j} - \int_{\Omega_j} \phi f,$$

so we get a new term representing the trace along $\Gamma_{jj'}$, where $\Omega_{j'}$ is another subdomain adjacent to j (see Fig. 1b). The same problem occurs when we try to eliminate the trace along Γ_{il}. Note that, in the discrete FEM setting, the Robin traces are integrated along a subset of $\partial\Omega_i$ of non-zero measure straddling both interfaces Γ_{ij} and Γ_{il}, and piecewise interface quantities are not available. Thus, the traces cannot be transmitted separately along Γ_{ij} and Γ_{il}, unlike in the continuous case; one must introduce extra unknowns to represent the piecewise Robin traces (integrated against a test function) for each subdomain at the cross point.

One way of circumventing the problem is to use mortar methods [1, 6], which are designed for non-conforming grids. In these methods, the interface conditions are imposed using mortar functions, which have one degree of freedom less at the ends of intervals. Thus, there is no equation at the cross point, and the problem of unavailable Robin traces goes away. However, since the interface conditions are only enforced weakly, the method does not generally converge to the exact solution of the global FEM problem, but rather to a discontinuous solution (Fig. 2) that is $O(h^p)$-accurate, where p is the order of the finite element method.



(a) (b)

Fig. 2. (**a**) The solution of $-\Delta u = f$ with four subdomains on $\Omega = [-1,1]^2$, with *right-hand side* $f(x,y) = \sin(xy)$. The interface conditions are imposed using a mortar space. (**b**) Discontinuity of the composite solution near the origin

4 Two Lagrange Multiplier and Primal-Dual Methods

If we want to formulate subdomain problems that are equivalent to the *discrete global* FEM problem, we need to introduce extra variables to represent the total Robin traces. Thus, at the cross point, we impose for each Ω_i

$$a_i(\phi, u_i^k) + \int_{\partial\Omega_i} p\phi \cdot u_i^k + \lambda_i^k = \int_{\Omega_i} \phi f, \tag{12}$$

where λ_i^k are Lagrange multipliers for ensuring consistency with the global problem. A cross point touching r subdomains requires r such Lagrange multipliers, so we also need r constraints to be satisfied at convergence:

- Continuity constraints ($r-1$ equations): at the cross point, we must have $u_1 = u_2 = \cdots = u_r$.
- PDE constraint (1 equation): if we sum (12) over the r subdomains and then subtract the global PDE $\sum_{i=1}^r a_i(\phi, u_i) = \int_\Omega \phi f$ from the result, we get

$$\sum_{i=1}^N \int_{\partial\Omega_i} p\phi u_i + \sum_{i=1}^N \lambda_i = 0.$$

This gives two types of algorithms:

1. Primal-Dual methods: the continuity constraints are enforced for every iteration. Thus, it suffices to introduce one extra variable (typically a coarse-grid basis function that has the value one at the cross point), and the PDE constraint is used as part of the coarse problem. This approach is similar to FETI-DP [3], except it is usually formulated with Neumann rather than Robin traces.
2. Two-Lagrange Multiplier methods: the λ_i^k are retained, but the u_i^k are eliminated using the PDE in the interior of the subdomains. This leads to a substructured problem formulated on the interface, which is then solved using a preconditioned Krylov method such as GMRES. This is known as the Two-Lagrange Multiplier (2LM) method and has been studied in detail in [8].

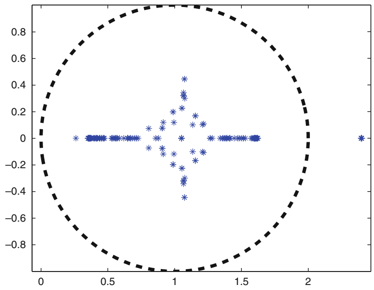


Fig. 3. Eigenvalues of the 2LM-preconditioned system for Poisson's equation ($\eta = 0$), using a 4×4 decomposition of the unit square with mesh size $h = 1/64$ and Robin parameter $p = C/\sqrt{h}$ for all interface nodes

Note that neither formulation is an exact discretization of (2) at cross points; thus, Lions' convergence analysis does not apply there. In fact, one can show [4] that the eigenvalues of the iteration matrix of the 2LM method may lie outside the unit disc when cross points are present, as seen in the 4×4 example shown in Fig. 3. In such cases, the method diverges. However, convergence can be restored if one uses Robin parameters with a different scaling at the cross points [4].

5 Conditions for Existence of Discrete Energy Estimates

To see what conditions are needed for Lions' estimates to hold in the discrete case, let us consider solving $-\Delta u = f$ on $\Omega = [-1,1]^2$ using P^1 finite elements on a structured triangular mesh. This yields the system $Au = f$, where A is identical to the matrix obtained from finite differences. If we now divide Ω into four subdomains corresponding to the four quadrants of the plane, then an optimized Schwarz method must solve

$$(A_i + L_i)u_i^k = g_i^k \qquad \text{on each } \Omega_i.$$

Here, A_i is the partially assembled stiffness matrix for Ω_i, L_i corresponds to transmission conditions, and g_i^k is a function of f and u_j^{k-1} for $j \neq i$. To define the *discrete* error function, let us write $u_i^* = u^*|_{\Omega_i}$, where u^* is the exact solution to $Au = f$. Then the error on Ω_i is $e_i^k = u_i^k - u_i^*$, with discrete energy $a_i(e_i^k, e_i^k) = (e_i^k)^T A_i e_i^k > 0$ whenever $e_i^k \neq 0$, since each subdomain touches a Dirichlet boundary. Now observe that

$$A_i e_i^k = A_i u_i^k - A_i u_i^* = A_i u_i^k - f_i \qquad \text{at interior nodes.}$$

Since the stencils of A_i and A coincide at interior nodes, we see that $A_i e_i^k$ must be zero away from the interfaces. Thus, we in fact have

$$a_i(e_i^k, e_i^k) = \sum_{v\in\partial\Omega_i\setminus\partial\Omega} e_i^k(v)\cdot(A_i e_i^k)(v) = \sum_{v\in\partial\Omega_i\setminus\partial\Omega} [(T_{+i}^k(v))^2 - (T_{-i}^k(v))^2],$$

where $T_{\pm i}^k(v)$ are the "Robin traces" at an interface point v:

$$T_{+i}^k(v) = \tfrac{1}{\sqrt{4p}}\left[(A_i e_i^k)(v) + p e_i^k(v)\right], \qquad T_{-i}^k(v) = \tfrac{1}{\sqrt{4p}}\left[-(A_i e_i^k)(v) + p e_i^k(v)\right].$$

Hence, if we let $T_{+i}^k(v) = T_{-j}^{k-1}(v)$ at every point v on the interface, then the energy estimate holds exactly the same way as in the continuous case, and we have convergence of the method. This allows us to deduce the correct interface conditions for v away from the cross point. Using the definition $e_i^k = u_i^k - u_i^*$, we have

$$(A_i(u_i^k - u_i^*))(v) + p(u_i^k(v) - u_i^*(v)) = -(A_j(u_j^{k-1} - u_j^*))(v) + p(u_j^{k-1}(v) - u_j^*(v)). \tag{13}$$

But since

$$(A_i u_i^*)(v) + (A_j u_j^*)(v) = f(v), \tag{14}$$

we can simplify (13) to get

$$(A_i u_i^k)(v) + p u_i^k(v) = f(v) - (A_j u_j^{k-1})(v) + p u_j^{k-1}(v).$$

In other words, we need

$$(L_i u_i^k)(v) = p u_i^k(v), \qquad g_i^k(v) = f(v) - (A_j u_j^{k-1})(v) + p u_j^{k-1}(v).$$

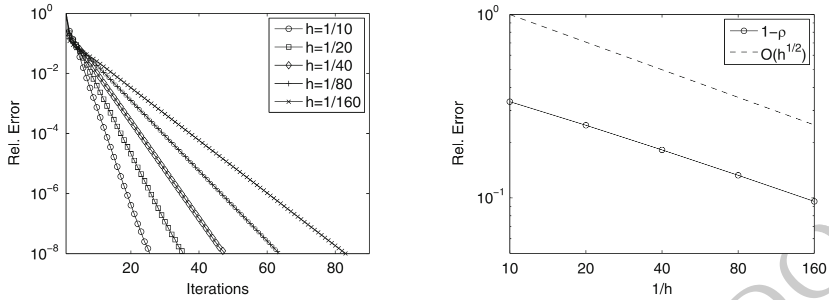
On the other hand, if v is a cross point, then (14) is no longer valid, since $f(v)$ is the sum of many subdomain contributions. Thus, it is in general impossible to find L_i and g_i^k such that the relation $T_{+i}^k(v) = T_{-j}^{k-1}(v)$ holds at the cross point for some j. In our model problem, however, the stencil at the cross point has a special form for the first and third quadrant:

$$\begin{aligned}
(A_1 u_1^*)(0,0) &= u^*(0,0) - \tfrac{1}{2}u^*(0,h) - \tfrac{1}{2}u^*(h,0),\\
(A_3 u_3^*)(0,0) &= u^*(0,0) - \tfrac{1}{2}u^*(0,-h) - \tfrac{1}{2}u^*(-h,0).
\end{aligned}$$

Thus, we actually have $(A_1 u_1^*)(0,0) + (A_3 u_3^*)(0,0) = \tfrac{1}{2}f(0,0)$, a known quantity! A similar relation holds between Ω_2 and Ω_4, so it is actually possible to find transmission conditions at the cross point that satisfy the discrete energy estimate. For Ω_1, this reads

$$(A_1 u_1^k)(v) + p u_1^k(v) = \tfrac{1}{2}f(v) - (A_3 u_3^{k-1})(v) + p u_3^{k-1}(v).$$

Figure 4 shows the convergence of the method for $p = \frac{\pi}{2\sqrt{h}}$, which gives the optimal contraction factor $\rho = 1 - O(\sqrt{h})$, just as in the two-subdomain case. Since the discrete energy estimate holds, the converged subdomain solutions always coincide with the *exact* discrete solution u^*, unlike in the mortar case. In general, discrete energy estimates can only be derived if *for every cross point v, its set of neighbors can be partitioned into disjoint pairs (i,j) such that $(A_i u_i^*)(v) + (A_j u_j^*)(v) = f_{ij}(v)$ can be calculated without knowing u^**. For cross points with wide stencils or an odd number of neighbors, this is not possible. In such cases, the methods in Sect. 4 are still excellent choices in practice, but one cannot use Lions' estimates to deduce convergence for arbitrary positive Robin parameters p.



(a) (b)

Fig. 4. (**a**) Convergence for different grid spacing h; (**b**) Contraction rate versus h

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