Robust Parameter-Free Multilevel Methods for Neumann Boundary Control Problems

Etereldes Gonçalves[1] and Marcus Sarkis[2]

[1] Universidade Federal do Espírito Santo - UFES - etereldes@gmail.com
[2] Mathematical Sciences Department, Worcester Polytechnic Institute, USA and Instituto de Matemática Pura e Aplicada - IMPA, Brazil msarkis@wpi.edu

Summary. We consider a linear-quadratic elliptic control problem (LQECP). For the problem we consider here, the control variable corresponds to the Neumann data on the boundary of a convex polygonal domain. The optimal control unknown is the one for which the harmonic extension approximates best a specified target in the interior of the domain. We propose a multilevel preconditioner for the reduced Hessian resulting from the application of the Schur complement method to the discrete LQECP. In order to derive robust stabilization parameters-free preconditioners, we first show that the Schur complement matrix is associated to a linear combination of negative Sobolev norms and then propose preconditioner based on multilevel methods. We also present numerical experiments which agree with the theoretical results.

1 Introduction

The problem of solving linear systems is central in numerical analysis. Systems arising from the discretization of PDEs and control problems have received special attention since they appear in many applications, such as in fluid dynamics and structural mechanics. Typically, as the dimension of the discrete space increases, the resulting system becomes very ill-conditioned. To avoid the large cost of LU factorizations of KKT saddle point linear systems, we consider instead the reduced Hessian systems. To build efficient solvers, the spectral properties of these systems must be taken into account. In this paper, we develop the mathematical tools necessary to analyze and to design solvers for a model control problem. We believe that the proposed framework can be extended to more complex control problems.

2 Setting Out the Problem

Consider the following LQECP:

$$\text{Minimize } J(u,\lambda) := \|u-u_*\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\lambda\|^2_{H^{-1/2}(\Gamma)} + \frac{\beta}{2}\|\lambda\|^2_{L^2(\Gamma)}$$
$$\text{subject to} \quad \begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega \subset \mathbb{R}^2, \\ \gamma \frac{\partial u}{\partial \eta}(s) = -\lambda(s) & \text{on } \Gamma := \partial\Omega, \end{cases} \tag{1}$$

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where u_* and f are given functions in $L^2(\Omega)\backslash\mathbb{R}$, γ is the trace operator on Γ, and α and β are nonnegative given stabilization parameters. The minimization is taken on $u \in H^1(\Omega)\backslash\mathbb{R}$ and $\lambda \in L^2(\Gamma)\backslash\mathbb{R}$. Here, "$\backslash\mathbb{R}$" stands for functions with zero average on Ω or Γ. We assume that the domain Ω is a convex polygonal domain, hence, H^2-regularity of u is assumed. The norm $H^{-1/2}(\Gamma)$ is defined as

$$\|\lambda\|^2_{H^{-1/2}(\Gamma)} := |v_\lambda|^2_{H^1(\Omega)}, \tag{2}$$

where $v_\lambda \in H^1(\Omega)\backslash\mathbb{R}$ is the harmonic extension of λ in Ω. We remark that the assumption $\alpha + \beta > 0$ is necessary for the well-posedness of the problem (1), see [7, 9, 11] and references therein. The case $\alpha = \beta = 0$ can also be treated by enlarging the minimizing space for λ from $H^{-1/2}(\Gamma)\backslash\mathbb{R}$ to $H^{-3/2}_{t,00}(\Gamma)\backslash\mathbb{R}$; see [6] for details. To make the notation less cumbersome, we sometimes drop "$\backslash\mathbb{R}$" below.

We consider the following discretization for the LQECP (1). We consider the space of piecewise linear and continuous functions $V_h(\Omega) \subset H^1(\Omega)$ to approximate u and p, and $\Lambda_h(\Gamma) \subset H^{1/2}(\Gamma)$ (the restriction of $V_h(\Omega)$ to Γ) to approximate λ. The underlying triangulation $\mathcal{T}_h(\Omega)$ is assumed to be quasi-uniform with mesh size $O(h)$. Let $\{\phi_1(x), \ldots, \phi_n(x)\}$ and $\{\varphi_1(x), \ldots, \varphi_m(x)\}$ denote the standard hat nodal basis functions for $V_h(\Omega)$ and $\Lambda_h(\Gamma)$, respectively. The corresponding discrete problem associated to (1) results in

$$\begin{bmatrix} M & 0 & A^T \\ 0 & G & Q^T E^T \\ A & EQ & 0 \end{bmatrix} \begin{bmatrix} \mathrm{u} \\ \lambda \\ \mathrm{p} \end{bmatrix} = \begin{bmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \\ \mathrm{f}_3 \end{bmatrix}, \tag{3}$$

where the matrices M and A are the mass and stiffness matrices on Ω, and Q is the mass matrix on Γ. We define $Q_{ext_{ij}} = (\phi_i, \varphi_j)_{L^2(\Gamma)}$; $\phi_i \in V_h(\Omega)$ and $\varphi_j \in \Lambda_h(\Gamma)$. It is easy to see that $Q_{ext} = EQ$, where $E \in \mathbb{R}^{n\times m}$ is the trivial zero discrete extension operator defined from $\Lambda_h(\Gamma)$ to $V_h(\Omega)$. We define $G \in \mathbb{R}^{m\times m}$ as be the matrix associated to the norm $\frac{\alpha}{2}\|\cdot\|^2_{H_h^{-1/2}(\Gamma)} + \frac{\beta}{2}\|\cdot\|^2_{L^2(\Gamma)}$ on $\Lambda_h(\Gamma)$, where $\|\lambda\|_{H_h^{-1/2}(\Gamma)} := |v^h_\lambda|_{H^1(\Omega)}$ with $v^h_\lambda := A^\dagger Q_{ext}\lambda$, i.e., v^h_λ is the discrete harmonic extension version of (2) with $\lambda \in \Lambda_h(\Gamma)$. Hence, we have $G = \alpha(Q^T_{ext}A^\dagger)A\,(A^\dagger Q_{ext}) + \beta Q = Q^T(\alpha E^T A^\dagger E + \beta Q^{-1})Q$. Here and the following $A^\dagger$ is the pseudo inverse of A. The discrete forcing terms are defined by $(\mathrm{f}_1)_i = \int_\Omega u_*(x)\,\phi_i(x)\,dx$, for $1 \le i \le n$, $\mathrm{f}_2 = 0$ and $(\mathrm{f}_3)_i = \int_\Omega f(x)\,\phi_i(x)\,dx$.

3 The Reduced Hessian $\mathscr{H}$

In this paper we propose and analyze preconditioners for the reduced Hessian associated to (3). Eliminating the variables u and p from Eq. (3), and denoting $S_1^\dagger := E^T A^\dagger E$ and $S_3^\dagger := E^T A^\dagger M A^\dagger E$, we obtain

$$\mathscr{H}\lambda := Q(\alpha S_1^\dagger + \beta Q^{-1} + S_3^\dagger)Q\lambda = b := Q^T_{ext}A^\dagger M A^\dagger \mathrm{f}_3 - Q^T_{ext}A^\dagger \mathrm{f}_1. \tag{4}$$

The matrix $\mathscr{H}$ is known as the Schur complement (reduced Hessian) with respect to the discrete control variable λ. We observe that the state variable u can be obtained by solving (4) and using the third equation of (3). We note that the Reduced matrix $\mathscr{H}$ is a symmetric positive definite matrix on

$$\Lambda_h(\Gamma)\backslash_Q\mathbb{R} := \{\lambda \in \Lambda_h(\Gamma); (\lambda,1)_{L^2(\Gamma)} = (Q\lambda, 1_m)_{\ell^2} = 0\},$$

hence, we consider the *Preconditioned Conjugate Gradient* (PCG) with a preconditioner acting on $\Lambda_h(\Gamma)\backslash_Q\mathbb{R}$. Note also that $A^\dagger$ is also symmetric positive definite matrix on

$$V_h(\Omega)\backslash_M\mathbb{R} := \{u \in V_h(\Omega); (u,1)_{L^2(\Omega)} = (Mu, 1_n)_{\ell^2} = 0\}.$$

The main goal of this paper is to develop robust preconditioned multilevel methods for the matrix $\mathscr{H}$ such that the condition number estimates that do not depend on α and β, and depend on $\log^2(h)$.

We point out that several block preconditioners for solving systems like (3) were proposed in the past; see [1, 8, 11, 14] and references therein. These preconditioners depend heavily on the availability of a good preconditioner for the Schur complement matrix. To the best of our knowledge, no robust and mathematically sounded preconditioner was systematically carried out for the reduced Hessian (4). Most of the existing work is toward problems where the control variable is f rather than λ, and even for these cases, condition number estimates typically deteriorate when all the stabilization parameters go to zero. Related work to ours is developed in [13] where it is proposed a preconditioner for the first biharmonic problem discretized by the mixed finite element method introduced by Ciarlet and Raviart [4]. Using techniques developed in [5], Peisker transforms the discrete problem to an interface problem and a preconditioner based on FFT is proposed and analyzed. This approach can also be interpreted as a control problem like (1), however, replacing the Neumann control by a Dirichlet control. We note that Dirichlet control problems are much easier to handle and to study since in (4) the operator $S_3^\dagger$ is replaced by $S_1^\dagger$, and therefore, a multilevel method such as in [2], can be applied. An attempt to precondition the Neumann control problem via FFT was considered in [7], however, such as in Peisker's work, it holds only for special meshes where the Schur complement matrix and the mass matrix on Γ share the same set of eigenvectors.

4 Theoretical Remarks on the Reduced Hessian $\mathscr{H}$

In this section we associate the Reduced Hessian $\mathscr{H}$ to a linear combination of Sobolev norms. Here and below we use the notation $a \preceq (\succeq)\ b$ to indicate that $a \le (\ge) Cb$, where the positive constant C depends only on the shape of Ω and $\mathcal{T}_h(\Omega)$. When $a \preceq b \preceq a$, we say $a \asymp b$.

First we observe that G is associated to the norm $\frac{\alpha}{2}\|\cdot\|^2_{H_h^{-1/2}(\Gamma)} + \frac{\beta}{2}\|\cdot\|^2_{L^2(\Gamma)}$ in $\Lambda_h(\Gamma)$. It is well known that for $\lambda \in \Lambda_h(\Gamma)\backslash_Q\mathbb{R}$ we have

$$\lambda^T Q S_1^{\dagger} Q \lambda = \|\lambda\|^2_{H_h^{-1/2}(\Gamma)} \asymp \|\lambda\|^2_{H^{-1/2}(\Gamma)}. \tag{5}$$

What is not obvious is how to associate the matrix $QS_3^{\dagger}Q$ to a Sobolev norm, and this is given in the following result (see [6]):

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. Let $v_\lambda^h := A^{\dagger} Q_{ext} \lambda \in V_h(\Omega)\backslash_M \mathbb{R}$ be the discrete harmonic function with Neumann data $\lambda \in \Lambda_h(\Gamma)\backslash_Q \mathbb{R}$. Then,*

$$\lambda^T Q S_3^{\dagger} Q \lambda = \|v_\lambda^h\|^2_{L^2(\Omega)} \asymp \|\lambda\|^2_{H_{t,00}^{-3/2}(\Gamma)} + h^2 \|\lambda\|^2_{H^{-1/2}(\Gamma)}. \tag{6}$$

Using these results we conclude that $\mathscr{H}$ is associated to the following linear combination of Sobolev norms

$$\lambda^T \mathscr{H} \lambda \asymp (\alpha + h^2) \|\lambda\|^2_{H^{-1/2}(\Gamma)} + \beta \|\lambda\|^2_{L^2(\Gamma)} + \|\lambda\|^2_{H_{t,00}^{-3/2}(\Gamma)}. \tag{7}$$

Remark 1. We next hint why the norm $\|\cdot\|^2_{H_{t,00}^{-3/2}(\Gamma)}$ is fundamental for this problem. Let $\{\Gamma_k\}_{1\le k\le K}$ and $\{\delta_k\}_{1\le k\le K}$ be the edges and the vertices of the polygonal Γ, respectively. Let $C_{t,00}^{\infty}(\Gamma_k) := \{\lambda \in C^{\infty}(\Gamma_k); \partial\lambda/\partial\tau_k \in C_0^{\infty}(\Gamma_k)\}$, where τ_k stands for the tangential unit vector on Γ_k. Define $H_{t,00}^2(\Gamma_k)$ by the closure of $C_{t,00}^{\infty}(\Gamma_k)$ in the $H^2(\Gamma_k)$-norm, that is,

$$H_{t,00}^2(\Gamma_k) := \{\lambda \in H^2(\Gamma_k); \frac{\partial\lambda}{\partial\tau_k}(\delta_{k-1}) = \frac{\partial\lambda}{\partial\tau_k}(\delta_k) = 0\}. \tag{8}$$

Using interpolation theory of operators and a characterization of $H_{t,00}^{3/2}(\Gamma_k)$, see [10], it is possible to show that

$$H_{t,00}^{3/2}(\Gamma_k) := \left[H_{t,00}^2(\Gamma_k), H^1(\Gamma_k)\right]_{1/2} = \left\{\lambda \in H^{3/2}(\Gamma_k); \partial\lambda/\partial\tau_k \in H_{00}^{1/2}(\Gamma_k)\right\}.$$

We define $H_{t,00}^{3/2}(\Gamma) = H^{1/2}(\Gamma) \cap \prod_{k=1}^{K} H_{t,00}^{3/2}(\Gamma_k)$ endowed with the norm

$$\|\lambda\|_{H_{t,00}^{3/2}(\Gamma)} := \|\lambda\|^2_{H^{1/2}(\Gamma)} + \sum_{k=1}^{K} \|\frac{\partial\lambda}{\partial\tau_k}\|^2_{H_{00}^{1/2}(\Gamma_k)}, \tag{9}$$

and define $H_{t,00}^{-3/2}(\Gamma) = (H_{t,00}^{3/2}(\Gamma))'$. The fundamental property of this space is that

$$\|\lambda\|_{H_{t,00}^{-3/2}(\Gamma)} \asymp \|v_\lambda\|_{L^2(\Omega)},$$

where v_λ is defined by (2); see [6].

5 Preconditioning Sobolev Norms Using Multilevel Methods

In this section, using multilevel based preconditioners, we develop spectral approximations for matrices associated to several Sobolev norms; see [2, 3, 12, 15], and references therein.

5.1 Notation and Technical Tools

From now on, we assume that the triangulation $\mathcal{T}_h$ of Γ has a *multilevel* structure. More precisely, denoting $\mathcal{T}_h$ as the restriction of $\mathcal{T}_h(\Omega)$ to Γ, we assume that the triangulation $\mathcal{T}_h$ is obtained from $(L-1)$ successive refinements of an initial coarse triangulation $\mathcal{T}_0$ with initial grid size h_0. We assume also that $h_\ell = h_{\ell-1}/2$ is the grid size on the ℓ-th triangulation $\mathcal{T}_\ell$ and associate the standard P_1 finite element space $V_\ell(\Gamma)$ generated by continuous and piecewise linear basis functions $\{\varphi_i^\ell\}_{i=1}^{m_\ell}$. Hence, we have

$$V_0(\Gamma) \subset V_1(\Gamma) \subset \cdots \subset V_L(\Gamma) := V_h(\Gamma) \subset L^2(\Gamma).$$

Let P_ℓ denote the $L^2(\Gamma)$-orthogonal projection onto $V_\ell(\Gamma)$, and let $\Delta P_\ell := (P_\ell - P_{\ell-1})$, that is, the $L^2(\Gamma)$-orthogonal projection onto $V_\ell(\Gamma) \cap V_{\ell-1}(\Gamma)^\perp$. We have that P_0, $(P_1 - P_0), \ldots, (P_L - P_{L-1})$ restricted to $V_L(\Gamma)$ are mutually L^2-orthogonal projections which satisfy:

$$I = P_0 + (P_1 - P_0) + \cdots + (P_L - P_{L-1}). \tag{10}$$

Note that $P_L = I$. The matrix form of P_ℓ restricted to $V_L(\Gamma)$ is given by

$$P_\ell = R_\ell^T Q_\ell^{-1} R_\ell Q, \tag{11}$$

where R_ℓ is the $m_\ell \times m_L$ restriction matrix, that is, the i-th row of R_ℓ is obtained by interpolating the basis function $\varphi_i^\ell \in V_\ell := V_\ell(\Gamma)$ at the nodes of the finest triangulation $\mathcal{T}_L := \mathcal{T}_h$.

It follows from [2, 12], that for $-3/2 < s < 3/2$

$$\|\mathrm{v}\|^2_{H^s(\Gamma)} \asymp \sum_{\ell=0}^{L} h_\ell^{-2s} \|(P_\ell - P_{\ell-1})\mathrm{v}\|^2_{L^2(\Gamma)}, \ \text{ for all } \ \mathrm{v} \in V_L. \tag{12}$$

This constraint for s comes from the fact that for $s \geq 3/2$ we have $V_h(\Gamma) \not\subset H^s(\Gamma)$, therefore, the equivalence deteriorates when s tends to $3/2$. Results for negative norms are obtained by duality.

We now describe how to represent the splitting $\sum_{\ell=0}^{L} \mu_\ell \|(P_\ell - P_{\ell-1})\mathrm{v}\|^2_{L^2(\Gamma)}$ into a matrix form. Let $\Delta_\ell := (P_\ell - P_{\ell-1})Q^{-1} = R_\ell^T Q_\ell^{-1} R_\ell - R_{\ell-1}^T Q_{\ell-1}^{-1} R_{\ell-1}$. Then we have

$$\Delta_k Q \Delta_\ell = \delta_{k\ell}\Delta_\ell \ \text{ and } \ \sum_{\ell=0}^{L} \mu_\ell \|(P_\ell - P_{\ell-1})\mathrm{v}\|^2_{L^2(\Gamma)} = \sum_{\ell=0}^{L} \mu_\ell \mathrm{v}^T Q (P_\ell - P_{\ell-1})\mathrm{v}, \tag{13}$$

where $P_{-1} = 0$. We observe that $Q(P_\ell - P_{\ell-1}) = Q\Delta_\ell Q$ is symmetric semi-positive definite. By (12) and (13), for all $\mathrm{v} \in V_L$ we have

$$\|\mathrm{v}\|^2_{H^{-1/2}(\Gamma)} \asymp \Big(\sum_{\ell=0}^{L} h_\ell \Delta_\ell Q \mathrm{v}, Q\mathrm{v}\Big). \tag{14}$$

To invert a matrix of the form $\sum_{k=0}^{L} \mu_k^{-1} \Delta_k Q$, we first assume that $\mu_k > 0$, $0 \leq k \leq L$. Then, from (10) and (13) we obtain

$$\Big(\sum_{k=0}^{L} \mu_k^{-1} \Delta_k Q\Big)\Big(\sum_{\ell=0}^{L} \mu_\ell \Delta_\ell Q\Big) = I. \tag{15}$$

5.2 Multilevel Preconditioner for the Reduced Hessian $\mathscr{H}$

In this subsection we analyze a multilevel preconditioner for Reduced Hessian $\mathscr{H}$. We first present a preconditioner for G as follows. Using (2), (14) and (15) we obtain

$$\begin{cases} S_1 \asymp Q\sum_{\ell=0}^{L} h_\ell^{-1}\Delta_\ell Q, \\ QS_1^\dagger Q \asymp Q\sum_{\ell=0}^{L} h_\ell \Delta_\ell Q. \end{cases} \tag{16}$$

The above equivalences yield simultaneous approximation for the spectral representations of $G := \beta Q + \alpha Q S_1^\dagger Q$ in terms of the Δ_ℓ and Q. More precisely,

$$G \asymp Q\sum_{\ell=1}^{L} (\beta + \alpha h_\ell)\Delta_\ell Q, \tag{17}$$

and using (15) and (17), the following spectral equivalency holds

$$G^{-1} \asymp \sum_{\ell=0}^{L} (\beta + \alpha h_\ell)^{-1}\Delta_\ell. \tag{18}$$

We next establish that $\sum_{\ell=0}^{L}(h_\ell^{-3})\Delta_\ell$ is a quasi-optimal preconditioner for $QS_3^\dagger Q$. More precisely, we have the following result (see [6]):

Theorem 2. *For all* $\mathrm{v}_L \in \mathbf{V}_L$*, the following inequalities hold:*

$$\|\mathrm{v}_L\|^2_{H^{-3/2}_{t,00}(\Gamma)} \preceq \sum_{\ell=1}^{L} h_\ell^3 \|\Delta P_\ell \mathrm{v}_L\|^2_{L^2} \preceq (L+1)^2 \|\mathrm{v}_L\|^2_{H^{-3/2}_{t,00}(\Gamma)}. \tag{19}$$

From Theorems 1 and 2 and (15), we establish the main result, the quasi-optimality for a preconditioner for $\mathscr{H}$.

Theorem 3. *Let* $\mathscr{PC} := \sum_{\ell=0}^{L}(\alpha h_\ell + \beta + h_\ell^3)^{-1}\Delta_\ell$. *Then*

$$(L+1)^{-2}\mathscr{PC} \;\preceq\; \mathscr{H}^{-1} \;\preceq\; \mathscr{PC}. \tag{20}$$

6 Numerical Results

In this section we show numerical results conforming the theory developed. For all tests presented, Ω is the square domain $[0,1]\times[0,1]$. The triangulation of Ω is constructed as follows. We divide each edge of $\partial\Omega$ into 2^N parts of equal length, where N is an integer denoting the number of refinements. In all tests (*cond*) means condition number, (*it*) indicates the number of iterations of the PCG, (*eig min*) means the lowest eigenvalue for preconditioned system. To calculate the eigenvalues we build the preconditioned system and use the function *eig* of MATLAB. We can see from tables below the asymptotic $\log^2(h)$ behavior for the case $\alpha = \beta = 0$, i.e., $\mathrm{cond}(N+1) - \mathrm{cond}(N)$ grows linearly with N. As expected, larger is α or β, better conditioned are the preconditioned systems (Tables 1–4).

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Remark 2. Numerical experiments show (not reported here) that the largest eigenvalue of $\left(\sum_{\ell=0}^{L}\Delta_\ell\right) * Q$ divided by the largest eigenvalue of $\left(\sum_{\ell=0}^{L} h_\ell^{-3}\Delta_\ell\right) * QS_3^\dagger Q$ converges to 36 when h decreases to zero. In tables above, we considered the rescaled preconditioner

	$\mathscr{PC}_r * \mathscr{H}$ with $\beta = 1$			$\mathscr{PC}_r * \mathscr{H}$ with $\beta = (0.1)^3$		
N ↓	cond	eig min	it	cond	eig min	it
4	1.04237	0.02756	2	4.94294	0.01622	7
5	1.04222	0.02757	2	4.87258	0.01655	7
6	1.04218	0.02757	2	4.85515	0.01663	7
7	1.04217	0.02757	2	4.85084	0.01665	7

Table 1. Equivalence between $\mathscr{H}$ and $\mathscr{PC}_r$ with $r = 36$ and $\alpha = 0$.

	$\mathscr{PC}_r * \mathscr{H}$ with $\beta = (0.1)^6$			$\mathscr{PC}_r * \mathscr{H}$ with $\beta = 0$		
N ↓	cond	eig min	it	cond	eig min	it
4	28.1662	0.004747	15	33.5522	0.004016	16
5	24.3303	0.005739	20	41.9737	0.003407	25
6	20.3042	0.006984	22	50.5193	0.002930	35
7	18.9576	0.007514	20	59.2085	0.002550	44

Table 2. Equivalence between $\mathscr{H}$ and $\mathscr{PC}_r$ with $r = 36$ and $\alpha = 0$.

	$\mathscr{PC}_r * \mathscr{H}$ with $\alpha = 1$			$\mathscr{PC}_r * \mathscr{H}$ with $\alpha = (0.1)^3$		
N ↓	cond	eig min	it	cond	eig min	it
4	4.62312	0.11893	10	13.7601	0.010698	14
5	5.12018	0.11826	10	18.3917	0.012503	19
6	5.33402	0.11798	11	26.2878	0.013139	22
7	5.45327	0.11788	12	35.6393	0.013312	26

Table 3. Equivalence between $\mathscr{H}$ and $\mathscr{PC}_r$ with $r = 36$ and $\beta = 0$.

	$\mathscr{PC}_r * \mathscr{H}$ with $\alpha = (0.1)^6$			$\mathscr{PC}_r * \mathscr{H}$ with $\alpha = 0$		
4	33.4363	0.004031	16	33.5522	0.0040164	16
5	41.4318	0.003452	25	41.9737	0.0034074	25
6	48.1852	0.003073	33	50.5193	0.0029301	35
7	50.8326	0.002973	43	59.2085	0.0025501	44

Table 4. Equivalence between $\mathscr{H}$ and $\mathscr{PC}_r$ with $r = 36$ and $\beta = 0$.

$$\mathscr{PC}_r := \sum_{\ell=0}^{L} (\alpha h_\ell + r\beta + h_\ell^3)^{-1} \Delta_\ell,$$

with $r = 36$, instead of $\mathscr{PC} := \sum_{\ell=0}^{L} (\alpha h_\ell + \beta + h_\ell^3)^{-1} \Delta_\ell$. This change improves considerably the condition number of preconditioners and improve slightly the number of iterations.

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