A Substructuring Preconditioner for Three-Dimensional Maxwell's Equations

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Summary. We propose a new nonoverlapping domain decomposition preconditioner for the discrete system arising from the edge element discretization of the three-dimensional Maxwell's equations. This preconditioner uses the simplest coarse edge element space induced by the coarse triangulation. We will show that the rate of the PCG convergence with this substructuring preconditioner is quasi-optimal, and is independent of large variations of the coefficients across the local interfaces.

1 Introduction

When the time-dependent Maxwell's equations is solved numerically, we need to solve the following **curlcurl**-system at each time step [4, 6, 8, 12]:

$$\mathbf{curl}(\alpha \mathbf{curlu}) + \beta \mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega \tag{1}$$

where Ω is assumed to be an open polyhedral domain in $\mathbf{R}^3$, and the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are two positive bounded functions in Ω. We shall complement the Eq. (1) with the perfect conductor condition $\mathbf{u} \times \mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$.

Edge finite element methods have been widely applied in the numerical solution of the system (1), see, for example, [5, 6, 8, 11]. Compared to the standard nodal finite element methods, the discrete systems resulting from the edge element discretization are essentially different in nature. The non-overlapping domain decomposition preconditioners have been well developed for the nodal element systems for the standard second order elliptic problems in the past two decades, and proved both numerically and theoretically to perform nearly optimally in terms of the fine mesh size and subdomain size; see, e.g., the monograph [15]. But these preconditioners,

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or their natural generalizations turn out to perform mostly very poorly for the edge element systems for the **curlcurl**-system (1), especially in three dimensions.

A lot of important efforts have been made in the construction of effective domain decomposition methods for the system (1). A substructuring type method was analysed in [16] for two dimensions, and in [2] for three dimensions with two subdomains. In [7], a novel substructuring type method was proposed for general two-dimensional multiple subdomains with quite irregular boundaries, and it was proved to be nearly optimal in terms of a variety of mesh decompositions and distributions of physical material properties. However, it has been a challenge how to construct an efficient non-overlapping domain decomposition preconditioner for the Maxwell's equations in three dimensions with general multiple subdomains. A first important attempt to this problem was made in [9] where a wire basket type algorithm was proposed and analysed. Then a substructuring preconditioner and a dual-primal FETI algorithm were introduced and fully analysed for three dimensions in [10] and [14], respectively. These three methods have their respective advantages and disadvantages: the algorithms in [9] and [14] both involve smaller coarse solvers but they are difficult to implement; the method in [10] is easier to implement but it involves a relatively large coarse solver.

This work intends to construct a new substructuring type preconditioner for the three-dimensioanl **curlcurl**-system (1) for general multiple subdomains. In this preconditioner, the coarse space is chosen to be the edge element space induced by the coarse triangulation, so the resulting coarse solver is very cheap and simple to implement. It is shown that the rate of the PCG convergence with this substructuring preconditioner is quasi-optimal, and more importantly, independent of the large variations of the coefficients in the system (1) across the local interfaces.

2 Domain Decompositions and Discretizations

This section introduces the non-overlapping domain decomposition of domain Ω, the weak form of the system (1) and the edge element spaces.

2.1 Initial Domain Decomposition Based on the Distribution of the Coefficients

We assume that the entire domain Ω is decomposed into N_0 open convex polyhedral subdomains $D_1, D_2, \cdots, D_{N_0}$ such that $\bar{\Omega} = \cup_{r=1}^{N_0} \bar{D}_r$ and $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are positive constants on each subdomain D_r, namely for $r = 1, 2, \ldots, N_0$,

$$\alpha(\mathbf{x}) = \alpha_r, \quad \beta(\mathbf{x}) = \beta_r \quad \forall \mathbf{x} \in D_r\,.$$

Clearly such a decomposition is always possible when the domain Ω is occupied by multiple media. In fact, if for some medium we have an irregular nonconvex subregion in Ω, we can further split each nonconvex medium subregion into smaller convex subdomains. This means that our assumption does cover many practical cases, especially considering the fact that the domain Ω on which we solve the original

Maxwell system (1) by a finite element method is often obtained by approximating the original physical domain by a polyhedral domain. Note that N_0 typically is a *fixed* constant in applications, so $diam(D_r) = O(1)$.

Let F_{nm} denote the common face of two neighboring subdomains D_n and D_m, and set $D_{nm} = D_n \cup D_m \cup F_{nm}$. For simplicity of the analysis, we assume

$$\beta_r \lesssim \alpha_r \lesssim d^{-2}\alpha_r, \quad r = 1, \cdots, N_0. \tag{2}$$

2.2 Domain Decomposition

For a number $d \in (0,\ 1)$, let each polyhedron D_l be decomposed into the union of some non-overlapping tetrahedra (or hexahedra) $\{\Omega_k\}$ of size d (see [3, 15] and [18]), which results in a non-overlapping domain decomposition for Ω: $\bar{\Omega} = \bigcup_{k=1}^{N} \bar{\Omega}_k$. Naturally we further assume that $\Omega_i \cap \Omega_j = \emptyset$ when $i \neq j$; if $i \neq j$ and $\partial\Omega_i \cap \partial\Omega_j \neq \emptyset$, $\partial\Omega_i \cap \partial\Omega_j$ is a common face (or edge or vertex) of Ω_i and Ω_j. Now the subdomains $\Omega_1, \cdots, \Omega_N$ constitute our desired *coarse* triangulation $\mathscr{T}_d$ of Ω. The faces and vertices of the subdomains are always denoted by F and V, while the common (open) face of the subdomains Ω_i and Ω_j are denoted by Γ_{ij}, and the union of all such common faces by Γ, i.e., $\Gamma = \cup\bar{\Gamma}_{ij}$. Γ will be called *the interface*. By Γ_k we denote the intersection of Γ with the boundary of the subdomain Ω_k. So we have $\Gamma_k = \partial\Omega_k$ if Ω_k is an interior subdomain of Ω. We shall set $\Omega_{ij} = \Omega_i \cup \Omega_j \cup \Gamma_{ij}$.

2.3 Weak Formulation

Let $H(\mathbf{curl};\Omega)$ be the Sobolev space consisting of all square integrable functions whose **curl**'s are also square integrable in Ω, and $H_0(\mathbf{curl};\Omega)$ be a subspace of $H(\mathbf{curl};\Omega)$ of all functions whose tangential components vanish on $\partial\Omega$. Then by writing the scalar product in $(L^2(\Omega))^3$ as $(\cdot,\cdot)$, we can state the variational problem for system (1) as follows:

Find $\mathbf{u} \in H_0(\mathbf{curl};\Omega)$ such that

$$\mathscr{A}(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall \mathbf{v} \in H_0(\mathbf{curl};\Omega) \tag{3}$$

where $\mathscr{A}(\cdot,\cdot)$ is a bilinear form given by

$$\mathscr{A}(\mathbf{u},\mathbf{v}) = (\alpha\ \mathbf{curl}\ \mathbf{u}, \mathbf{curl}\ \mathbf{v}) + (\beta\mathbf{u},\mathbf{v}), \quad \mathbf{u},\mathbf{v} \in H(\mathbf{curl};\Omega).$$

2.4 Fine Triangulation and Their Associated Finite Element Spaces

We further divide each Ω_k into smaller tetrahedral elements of size h so that elements from two neighboring subdomains have an intersection which is either empty or a single nodal point or an edge or a face on the interface Γ. Let $\mathscr{T}_h$ be the resulting triangulation of the domain Ω, which we assume is quasi-uniform. Then we introduce the N*é*d*é*lec edge element space of the lowest order defined on $\mathscr{T}_h$ (cf. [12] and [13]):

$$V_h(\Omega) = \Big\{ \mathbf{v} \in H_0(\mathbf{curl};\Omega);\ \mathbf{v}\,|_K \in R(K),\ \forall K \in \mathscr{T}_h \Big\},$$

where $R(K)$ is a subset of all linear polynomials on the element K of the form:

$$R(K) = \Big\{ \mathbf{a} + \mathbf{b} \times \mathbf{x};\ \mathbf{a}, \mathbf{b} \in \mathbf{R}^3,\ \mathbf{x} \in K \Big\}.$$

In an analogous way, we can define the coarse edge element space $V_d(\Omega) \subset V_h(\Omega)$, associated with the *coarse* triangulation $\mathscr{T}_d$.

It is well-known that for any $\mathbf{v} \in V_h(\Omega)$, its tangential components are continuous on all edges of each element in the triangulation $\mathscr{T}_h$. Moreover, each edge element function $\mathbf{v}$ in $V_h(\Omega)$ is uniquely determined by its moments on each edge e of $\mathscr{T}_h$:

$$\Big\{ \lambda_e(\mathbf{v}) = \int_e \mathbf{v} \cdot \mathbf{t}_e ds;\ e \in \mathscr{E}_h \Big\},$$

where $\mathscr{E}_h$ denotes the set of the *fine* edges from the triangulation $\mathscr{T}_h$, and $\mathbf{t}_e$ denotes the unit vector on the edge e.

By $Z_h(\Omega)$ we denote the continuous piecewise linear finite element subspace of $H_0^1(\Omega)$ associated with the triangulation $\mathscr{T}_h$. Similarly, let $Z_d(\Omega)$ denote the continuous piecewise linear finite element subspace of $H_0^1(\Omega)$ associated with the triangulation $\mathscr{T}_d$.

2.5 Discrete Variational Problem

Using the edge element space $V_h(\Omega)$, the system (3) may be approximated as follows: Find $\mathbf{u}_h \in V_h(\Omega)$ such that

$$(\alpha \mathbf{curl}\, \mathbf{u}_h, \mathbf{curl}\, \mathbf{v}_h) + (\beta \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h(\Omega). \tag{4}$$

Define the operator $A : V_h(\Omega) \to V_h(\Omega)$ by

$$(A\mathbf{u}_h, \mathbf{v}_h) = (\alpha \mathbf{curl}\, \mathbf{u}_h, \mathbf{curl}\, \mathbf{v}_h) + (\beta \mathbf{u}_h, \mathbf{v}_h), \quad \forall \mathbf{u}_h, \mathbf{v}_h \in V_h(\Omega),$$

Then, (4) can be written in the operator form

$$A\mathbf{u}_h = \mathbf{f}_h. \tag{5}$$

3 A Nearly Optimal Preconditioner for A

3.1 Construction of the Preconditioner

We first introduce some useful sets and subspaces.

$\mathscr{E}_h$: the set of all edges from the triangulations $\mathscr{T}_h$;

$\mathscr{E}_{\Gamma,h}$: the set of edges which belong to $\mathscr{E}_h$ and have two endpoints on the interface Γ;

$\mathscr{E}_d$: the set of all (coarse) edges from the triangulations $\mathscr{T}_d$;

$\mathscr{W}_{\mathrm{E}}$: the union of all the coarse edges $\mathrm{E}' \in \mathscr{E}_d$, which have a common endpoint with the coarse edge $\mathrm{E} \in \mathscr{E}_d$. And $\mathscr{W}_{\mathrm{E}}$ is called E-*basket*.

$\mathscr{E}^b_{\mathrm{E},h}$: the set of all (fine) edges which belong to $\mathscr{E}_h$ and have at least one endpoint on $\mathscr{W}_{\mathrm{E}}$;

Let D be either a subdomain D_r or a subdomain Ω_k or a subdomain Ω_{ij} or a subdomain D_{mn}. The restrictions of $V_h(\Omega)$ (resp. $Z_h(\Omega)$) on D is denoted by $V_h(D)$ (resp. $Z_h(D)$). The following local subspaces of $V_h(D)$ will be important to our analysis:

$$V_h^0(D) = \Big\{ \mathbf{v} \in V_h(D);\ \mathbf{v} \times \mathbf{n} = 0 \ \text{ on } \ \partial D \Big\},$$

and

$$Z_h^0(D) = \Big\{ \varphi \in Z_h(\Omega);\ supp\ \varphi \subset D \Big\}.$$

We define subspaces of $V_h(\Omega)$:

$$V_h^H(\Omega) = \Big\{ \mathbf{v} \in V_h(\Omega);\ \mathbf{v} \text{ is the discrete } A\text{-extension of } \mathbf{v}|_{\partial\Omega_k} \text{ in each } \Omega_k \Big\},$$

$$V_h^H(\Omega_{ij}) = V_h^H(\Omega) \bigcap V_h^0(\Omega_{ij}),$$

and for $\mathrm{E} \in \mathscr{E}_d$,

$$V_h^{\mathrm{E}}(\Omega) = \Big\{ \mathbf{v} \in V_h^H(\Omega);\ \lambda_e(\mathbf{v}) = 0 \text{ for each } e \in \mathscr{E}_{\Gamma,h} \backslash \mathscr{E}^b_{\mathrm{E},h} \Big\}.$$

It is well known that a suitable *coarse* subspace plays a key role in the construction of an effective domain decomposition preconditioner, and it is generally rather technical and problem-dependent to choose such a *coarse* subspace. Surprisingly we are going to choose the coarse subspace to be the simplest one, namely the subspace $V_d(\Omega)$ induced by the coarse triangulation $\mathscr{T}_d$.

It is easy to see that the space $V_h(\Omega)$ has the (non-direct sum) decomposition

$$V_h(\Omega) = V_d(\Omega) + \sum_{k=1}^{N} V_h^0(\Omega_k) + \sum_{\mathrm{E}} V_h^{\mathrm{E}}(\Omega) + \sum_{\Gamma_{ij}} V_h^H(\Omega_{ij}). \tag{6}$$

Next, we define the corresponding solvers on the subspaces $V_h^0(\Omega_k)$, $V_h^{\mathrm{E}}(\Omega)$, $V_h^H(\Omega_{ij})$ and $V_d(\Omega)$.

As usual, we denote the restriction of A on $V_h^0(\Omega_k)$ by A_k, i.e.,

$$(A_k\mathbf{v}, \mathbf{u})_{\Omega_k} = (A\mathbf{v}, \mathbf{u}) = \mathscr{A}(\mathbf{v}, \mathbf{u}),\ \mathbf{v} \in V_h^0(\Omega_k),\ \forall \mathbf{u} \in V_h^0(\Omega_k).$$

Let $B_k : V_h^0(\Omega_k) \to V_h^0(\Omega_k)$, $B_d : V_d(\Omega) \to V_d(\Omega)$ and $B_{ij} : V_h^H(\Omega_{ij}) \to V_h^H(\Omega_{ij})$ be the symmetric and positive definite operators such that

$$(B_k\mathbf{v}, \mathbf{v}) \approxeq (A_k\mathbf{v}_k, \mathbf{v}_k)_{\Omega_k}, \quad \forall \mathbf{v} \in V_h^0(\Omega_k),$$

where $\mathbf{v}_k = \mathbf{v}|_{\Omega_k}$ for $k = 1, 2, \cdots, N$, and

$$(B_d\mathbf{v}_d,\mathbf{v}_d) \eqsim \mathscr{A}(\mathbf{v}_d,\mathbf{v}_d), \quad \forall \mathbf{v}_d \in V_d(\Omega),$$
$$(B_{ij}\mathbf{v},\mathbf{v}) \eqsim \mathscr{A}(\mathbf{v},\mathbf{v}), \quad \forall \mathbf{v} \in V_h^H(\Omega_{ij}).$$

The symbol $\eqsim$ above means each of the two quantities involved is bounded by the other up to a constant independent of h, d and functions involved in the two quantities.

The local solvers on $V_h^{\mathrm{E}}(\Omega)$ should be solvable in an efficient manner, and their constructions are much more tricky and technical than the others. To do so, we introduce more notation.

For any face F from the triangulations $\mathscr{T}_d$, we use F_b to denote the union of all $\mathscr{T}_h$-induced (closed) triangles on F, which have either one single vertex or one edge lying on $\partial\mathrm{F}$, and F_∂ to denote the open set $\mathrm{F}\backslash\mathrm{F}_b$. For any subdomain Ω_k, define

$$\Delta_k = \bigcup_{\mathrm{F}\subset\Gamma_k} \mathrm{F}_b, \quad k = 1,\cdots,N. \tag{171}$$

We will also need the so-called tangential divergence $\mathrm{div}_\tau \Phi = curl_S \Phi$ for $\Phi \in V_h(\Gamma_k)$, which is defined here as in [1, 2]. Then we can introduce our local solver $B_{\mathrm{E}} : V_h^{\mathrm{E}}(\Omega) \to V_h^{\mathrm{E}}(\Omega)$ as follows:

$$\begin{aligned}(B_{\mathrm{E}}\mathbf{v},\mathbf{u}) = h[1+\log(d/h)]\sum_{k=1}^{N}\Big\{&\alpha_k\langle \mathrm{div}_\tau(\mathbf{v}\times\mathbf{n})|_{\Gamma_k}, \mathrm{div}_\tau(\mathbf{u}\times\mathbf{n})|_{\Gamma_k}\rangle_{\Delta_k}\\ &+ \beta_k\langle \mathbf{v}\times\mathbf{n}, \mathbf{u}\times\mathbf{n}\rangle_{\Delta_k}\Big\}, \qquad \mathbf{v}\in V_h^{\mathrm{E}}(\Omega),\ \forall \mathbf{u}\in V_h^{\mathrm{E}}(\Omega).\end{aligned} \tag{7}$$

For convenience, we call B_{E} an

Let $Q_k : V_h(\Omega) \to V_h^0(\Omega_k)$, $Q_d : V_h(\Omega) \to V_d(\Omega)$, $Q_{\mathrm{E}} : V_h(\Omega) \to V_h^{\mathrm{E}}(\Omega)$ and $Q_{ij} : V_h(\Omega) \to V_h^H(\Omega_{ij})$ be the standard the standard L^2-projections. Then we are ready to propose our new preconditioner for A as follows:

$$B^{-1} = B_d^{-1}Q_d + \sum_{k=1}^{N} B_k^{-1}Q_k + \omega\sum_{\mathrm{E}} B_{\mathrm{E}}^{-1}Q_{\mathrm{E}} + \sum_{\Gamma_{ij}} B_{ij}^{-1}Q_{ij}, \tag{8}$$

where ω is a (constant) relaxation parameter, which is introduced to obtain a balance between the local solvers B_{E} and other remaining solvers.

3.2 Algorithm Based on the New Preconditioner and Main Results

The action of the preconditioner B^{-1} which is needed in each PCG iteration can be described in the following algorithm.

Algorithm 4.1. For $\mathbf{g} \in V_h(\Omega)$, we can compute $\mathbf{u} = B^{-1}\mathbf{g}$ in five steps.

Step 1. Solve the system for $\mathbf{u}_d \in V_d(\Omega)$:

$$(B_d\mathbf{u}_d,\mathbf{v}_d) = (\mathbf{g},\mathbf{v}_d), \quad \forall \mathbf{v}_d \in V_d(\Omega);$$

Step 2. Solve the following system for $\mathbf{u}_k \in V_h^0(\Omega_k)$ in each subdomain in parallel:

$$(B_k\mathbf{u}_k, \mathbf{v}) = (\mathbf{g}, \mathbf{v}), \quad \forall \mathbf{v} \in V_h^0(\Omega_k), \ \ k = 1, \cdots, N;$$

Step 3. Solve the following system for $\mathbf{u}_{ij} \in V_h^0(\Omega_{ij})$ in each subdomain Ω_{ij} in parallel:

$$(B_{ij}\mathbf{u}_{ij}, \mathbf{v}) = (\mathbf{g}, \mathbf{v}) - (A_i\mathbf{u}_i, \mathbf{v})_{\Omega_i} - (A_j\mathbf{u}_j, \mathbf{v})_{\Omega_j}, \quad \forall \mathbf{v} \in V_h^0(\Omega_{ij});$$

Step 4. Solve the system for $\mathbf{u}_{\mathrm{E}} \in V_h^{\mathrm{E}}(\Omega)$:

$$(B_{\mathrm{E}}\mathbf{u}_{\mathrm{E}}, \mathbf{v}) = (\mathbf{g}, \tilde{\mathbf{v}}) - \sum_{k=1}^{N} (A_k\mathbf{u}_k, \tilde{\mathbf{v}}), \ \mathbf{v} \in V_h^{\mathrm{E}}(\Omega),$$

where $\tilde{\mathbf{v}} \in V_h(\Omega)$ is a natural extension of $(\mathbf{v} \times \mathbf{n})|_\Gamma$ by zero.

Step 5. Set $\Phi_h = (\sum_{\Gamma_{ij}} \mathbf{u}_{ij} + \sum_{\mathrm{E}} \mathbf{u}_{\mathrm{E}}) \times \mathbf{n}|_\Gamma$ and compute the A-extension A-extension of Φ_h on each Ω_k to obtain $\mathbf{u}^H \in V_h^H(\Omega)$. This leads to

$$\mathbf{u} = \mathbf{u}_d + \sum_{k=1}^{N} \mathbf{u}_k + \mathbf{u}^H.$$

Remark 1. For the local solver B_{ij} on each face Γ_{ij}, we may use the face extended domain formed by, e.g., one half of each of the two neighboring subdomains Ω_i and Ω_j. Such definition of B_{ij}'s can reduce the computational complexity in their numerical realization.

Let E denote a coarse edge of the subdomain D_r. Define

$$V_h^{\perp}(\Omega) = \{\mathbf{v}_h : \mathbf{v}_h \in V_h(\Omega), \ \int_E \mathbf{v}_h \cdot \mathbf{t}_E ds = 0 \text{ for each } E\}.$$

We shall use $\kappa^{\perp}(B^{-1}A)$ to denote the *induced condition number* of the preconditioned system $B^{-1}A$ associated with the subspace $V_h^{\perp}(\Omega)$, namely the condition number of $B^{-1}A$ restricted on the subspace $V_h^{\perp}(\Omega)$ (cf. [17]). At this moment we are able to establish only the following estimate of the induced condition number. As the estimate is quite lengthy and technical, we cannot include it here due to the page limitation.

Theorem 1. *Under the assumptions (2), the preconditioner B given in (8) is nearly optimal in the sense that*

$$\kappa^{\perp}(B^{-1}A) \leq C[1 + \log(d/h)]^2[1 + \log(1/h)]^2 \tag{9}$$

where the constant C is independent of h, d and the jumps of the coefficients.

As we see from the above theorem that the induced condition number grows logarithmically with the degrees of freedom in each subdomain, but also with the

degrees of freedom of the entire fine mesh. We believe this is mainly due to the restriction of our current analysis technique, namely the estimate must be done for the induced condition number in the subspace $V_h^{\perp}(\Omega)$ associated with the coarse triangulation formed by the material subdomains D_r. We expect the estimate should be finally carried out directly in the entire edge element space $V_h(\Omega)$, that will remove the logarithmic factor of $1/h$ in the estimate (9). This expectation has already been confirmed by our three-dimensional numerical experiments; see the next section.

4 Numerical Experiments

In this section we shall conduct some numerical experiments to check the convergence of the newly proposed preconditioner, and find out whether they are consistent with the prediction of the convergence theory developed in the previous sections.

In our experiments, we take the domain to be the unit cube $\Omega = (0,1)^3$, while the right-hand side $\mathbf{f}$ of the system (1) is selected such that the exact solution $\mathbf{u} = (u_1, u_2, u_3)^T$ is given by

$$\begin{aligned} u_1 &= xyz(x-1)(y-1)(z-1), \\ u_2 &= \sin(\pi x)\sin(\pi y)\sin(\pi z), \\ u_3 &= (1-e^x)(1-e^{x-1})(1-e^y)(1-e^{y-1})(1-e^z)(1-e^{z-1}), \end{aligned}$$

when the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are both constant 1. This right-hand side $\mathbf{f}$ is then fixed in all our experiments, but the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ may be taken differently.

We then need to triangulate the domain Ω into subdomains $\{\Omega_k\}$. For this, we first partition the three edges of Ω on x-, y- and z-axis into n equal subintervals from which one can naturally generate n^3 equal smaller cubes of size $d = 1/n$. This yields the desired subdomain decomposition in our experiments.

Next, we further triangulate each subdomain Ω_k to get a fine triangulation $\mathcal{T}_h$ of size h over the domain Ω. To generate $\mathcal{T}_h$, we divide each subdomain into m^3 equal smaller cubes of size $h = 1/(mn)$, in the same manner as done in the previous subdomain generation. Then $\mathcal{T}_h$ is obtained by triangulating each cube into six tetrahedra. For easy identification, we may denote the triangulation $\mathcal{T}_h$ as $m^3(n^3)$ below.

The edge finite element space of the lowest order is used for the discretization of (3). The resulting system (5) is solved by PCG method with the newly proposed preconditioners B defined in Sect. 4. We shall choose the balancing parameter ω in front of the E-*basket* local solvers B_{E} in (8) as $\omega = 1$ or $\omega = 2.5$.

We consider various distributions of the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ and report the corresponding numbers of PCG iterations, and the condition numbers of $B^{-1}A$ for some representative cases. The PCG iteration is terminated in our experiments when the relative residual is less than 10^{-6}.

Case (i): coefficients $\alpha(\mathbf{x}) = \beta(\mathbf{x}) = 1$, with no jumps. The PCG iterations and the condition numbers (in brackets) for $\omega = 2.5$ are listed in Table 1.

	$\omega = 1.0$				$\omega = 2.5$			
$m \backslash n$	4	6	8	10	4	6	8	10
4	34	33	32	32	31 (34.24)	31 (36.31)	31 (36.94)	30 (37.40)
8	41	40	39	38	39 (52.15)	38 (53.78)	37 (54.21)	37 (54.61)
12	48	47	44	42	43 (64.29)	43 (65.91)	41 (66.19)	41 (66.62)
16	51	50	49	45	47 (74.40)	46 (75.69)	44 (75.82)	44 (76.39)

Table 1. Iterations (and condition numbers) with smooth coefficients

We observe from the above table that the number of PCG iterations grows slowly when $m = d/h$ increases but $n = 1/d$ is fixed, and that these numbers vary stably when m is fixed but n increases. This justifies our early expection that the condition number of the preconditioned system $B^{-1}A$ should grow logarithmically with d/h only, not with $1/h$.

One important issue we like to draw the readers' attention to is the large-scale of the discrete system we are solving. For instance, when $m = 16$ and $n = 10$, the total number of degrees of freedom for the fine edge element system is about 28,672,000.

Case (ii): coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ have large jumps:

$$\alpha(\mathbf{x}) = \beta(\mathbf{x}) = \alpha_0 \quad \text{in} \quad D; \quad \alpha(\mathbf{x}) = \beta(\mathbf{x}) = 1 \quad \text{in} \quad \Omega \backslash D.$$

where $D \subset \Omega$ is a union of several subdomains Ω_k. We choose $\alpha_0 = 10^{-5}$ or $\alpha_0 = 10^5$, and consider two choices of D, where one does not have *cross-points*, while the other has one *cross-point*.

Example 1:

$$D = [\frac{1}{4}, \frac{1}{2}]^3.$$

Example 2:

$$D = [\frac{1}{4}, \frac{1}{2}]^3 \bigcup [\frac{1}{2}, \frac{3}{4}]^3.$$

The numerical results are given in Tables 2 and 3, from which we can make some similar observations about the PCG convergence in terms of the mesh and subdomain quantities d/h and d as we did for Case (i).

	Example 1				Example 2			
	$\omega = 1.0$		$\omega = 2.5$		$\omega = 1.0$		$\omega = 2.5$	
$m \backslash n$	4	8	4	8	4	8	4	8
4	29	31	26 (32.00)	29 (35.97)	28	30	26 (35.51)	30 (35.97)
8	35	38	32 (44.88)	37 (52.97)	35	38	32 (45.88)	37 (52.59)
12	38	45	36 (56.02)	42 (64.96)	37	45	35 (55.66)	41 (63.81)
16	40	49	37 (64.65)	45 (74.68)	40	49	37 (65.65)	45 (74.31)

Table 2. Iterations (and condition numbers) with $\alpha_0 = 10^{-5}$

	Example 1				Example 2			
	$\omega = 1.0$		$\omega = 2.5$		$\omega = 1.0$		$\omega = 2.5$	
$m \backslash n$	4	8	4	8	4	8	4	8
4	42	42	36 (40.47)	36 (42.71)	42	44	38 (40.55)	37 (42.72)
8	49	48	45 (61.08)	44 (62.89)	52	51	46 (60.20)	45 (62.89)
12	55	54	50 (74.04)	49 (76.28)	56	56	50 (76.24)	51 (76.28)
16	59	57	54 (91.51)	52 (86.45)	59	59	53 (83.35)	54 (86.45)

Table 3. Iterations (and condition numbers) with $\alpha_0 = 10^5$

Case (iii): coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ have large jumps:

$$\alpha(\mathbf{x}) = \begin{cases} \alpha_0, & \text{in } D \\ 1, & \text{in } \Omega \backslash D, \end{cases} \qquad \beta(\mathbf{x}) = \begin{cases} \beta_0, & \text{in } D \\ 1, & \text{in } \Omega \backslash D, \end{cases}$$

where $D \subset \Omega$ is a union of several subdomains Ω_k. We choose $\alpha_0 = 10^{-5}$ or $\alpha_0 = 10^5$, but $\beta_0 \neq \alpha_0$. We still consider two different regions D from Examples 1 and 2 in the previous Case (ii), but choose the balancing parameter ω in front of the E-*basket* local solvers B_{E} in (8) as $\omega = 2.5$.

The numerical results are given in Tables 4 and 5. Again, we can make similar observations about the PCG convergence in terms of the mesh and subdomain quantities d/h and d as we did for Case (i).

	Example 1				Example 2			
	$\beta_0 = \alpha_0 \times 10^2$		$\beta_0 = \alpha_0 \times 10^{-2}$		$\beta_0 = \alpha_0 \times 10^2$		$\beta_0 = \alpha_0 \times 10^{-2}$	
$m \backslash n$	4	8	4	8	4	8	4	8
4	30	36	46	47	30	36	45	47
8	39	43	56	56	39	45	56	56
16	49	52	65	65	49	52	63	65

Table 4. Iterations with $\alpha_0 = 10^{-5}$

	Example 1				Example 2			
	$\beta_0 = \alpha_0 \times 10^2$		$\beta_0 = \alpha_0 \times 10^{-2}$		$\beta_0 = \alpha_0 \times 10^2$		$\beta_0 = \alpha_0 \times 10^{-2}$	
$m \backslash n$	4	8	4	8	4	8	4	8
4	31	37	38	41	31	37	39	46
8	37	47	46	49	37	47	53	58
16	48	56	55	57	48	56	66	73

Table 5. Iterations with $\alpha_0 = 10^5$

We may also observe from the previous numerical experiments that appropriate choices of the parameter ω can significantly improve the efficiency of the preconditioner B. It is important to see that the choices of ω seem independent of the fine

and coarse meshsizes h and d, so we may determine ω by solving some small scale systems, e.g., a system with $m = n = 4$.

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