A Robust FEM-BEM Solver for Time-Harmonic Eddy Current Problems

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Summary. This paper is devoted to the construction and analysis of robust solution techniques for time-harmonic eddy current problems in unbounded domains. We discretize the time-harmonic eddy current equation by means of a symmetrically coupled finite and boundary element method, taking care of the different physical behavior in conducting and non-conducting subdomains, respectively. We construct and analyse a block-diagonal preconditioner for the system of coupled finite and boundary element equations that is robust with respect to the space discretization parameter as well as all involved "bad" parameters like the frequency, the conductivity and the reluctivity. Block-diagonal preconditioners can be used for accelerating iterative solution methods such like the Minimal Residual Method.

1 Introduction

In many practical applications, the excitation is time-harmonic. Switching from the time domain to the frequency domain allows us to replace expensive time-integration procedures by the solution of a system of partial differential equations for the amplitudes belonging to the sine- and to the cosine-excitation. Following this strategy, [7, 13] and [4, 5] applied harmonic and multiharmonic approaches to parabolic initial-boundary value problems and the eddy current problem, respectively. Indeed, in [13], a preconditioned MinRes solver for the solution of the eddy current problem in bounded domains was constructed that is robust with respect to both the discretization parameter h and the frequency ω. The key point of this parameter-robust solver is the construction of a block-diagonal preconditioner, where standard $\mathbf{H}(\mathbf{curl})$ FEM magneto-static problems have to be solved or preconditioned. The aim of this contribution is to generalize these ideas to the case of unbounded domains in terms of a coupled Finite Element (FEM) – Boundary Element (BEM) Method. In this case we are also able to construct a block-diagonal preconditioner, where now standard coupled FEM-BEM $\mathbf{H}(\mathbf{curl})$ problems, as arising in the magneto-static case, have to be solved or preconditioned. We mention, that this preconditioning technique fits into the framework of operator preconditioning, see, e.g. [1, 11, 16, 19].

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The paper is now organized as follows. We introduce the frequency domain equations in Sect. 2. In the same section, we provide the symmetrically coupled FEM-BEM discretization of these equations. In Sect. 3, we construct and analyse our parameter-robust block-diagonal preconditioner used in a MinRes setting for solving the resulting system of linear algebraic equations. Finally, we discuss the practical realization of our preconditioner.

2 Frequency Domain FEM-BEM

As a model problem, we consider the following eddy current problem:

$$\begin{cases} \sigma \frac{\partial \mathbf{u}}{\partial t} + \mathbf{curl}\,(\nu_1\,\mathbf{curl}\,\mathbf{u}) = \mathbf{f} & \text{in } \Omega_1 \times (0,T), \\ \mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}) = \mathbf{0} & \text{in } \Omega_2 \times (0,T), \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega_2 \times (0,T), \\ \mathbf{u} = \mathscr{O}(|\mathbf{x}|^{-1}) & \text{for } |\mathbf{x}| \to \infty, \\ \mathbf{curl}\,\mathbf{u} = \mathscr{O}(|\mathbf{x}|^{-1}) & \text{for } |\mathbf{x}| \to \infty, \\ \mathbf{u} = \mathbf{u_0} & \text{on } \Omega_1 \times \{0\}, \\ \mathbf{u_1} \times \mathbf{n} = \mathbf{u_2} \times \mathbf{n} & \text{on } \Gamma \times (0,T), \\ \nu_1 \mathbf{curl}\,\mathbf{u_1} \times \mathbf{n} = \mathbf{curl}\,\mathbf{u_2} \times \mathbf{n} & \text{on } \Gamma \times (0,T), \end{cases} \tag{1}$$

where the computational domain $\Omega = \mathbb{R}^3$ is split into the two non-overlapping subdomains Ω_1 and Ω_2. The conducting subdomain Ω_1 is assumed to be a simply connected Lipschitz polyhedron, whereas the non-conducting subdomain Ω_2 is the complement of Ω_1 in $\mathbb{R}^3$, i.e $\mathbb{R}^3 \backslash \overline{\Omega}_1$. Furthermore, we denote by Γ the interface between the two subdomains, i.e. $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$. The exterior unit normal vector of Ω_1 on Γ is denoted by $\mathbf{n}$, i.e. $\mathbf{n}$ points from Ω_1 to Ω_2. The reluctivity ν_1 is supposed to be independent of $|\mathbf{curl}\,\mathbf{u}|$, i.e. we assume the eddy current problem (1) to be linear. The conductivity σ is zero in Ω_2, and piecewise constant and uniformly positive in Ω_1.

We assume, that the source $\mathbf{f}$ is given by a time-harmonic excitation with the frequency $\omega > 0$ and amplitudes $\mathbf{f^c}$ and $\mathbf{f^s}$ in the conducting domain Ω_1. Therefore, the solution $\mathbf{u}$ is time-harmonic as well, with the same base frequency ω, i.e.

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u^c}(\mathbf{x}) \cos(\omega t) + \mathbf{u^s}(\mathbf{x}) \sin(\omega t). \tag{2}$$

In fact, (2) is the real reformulation of a complex time-harmonic approach $\mathbf{u}(\mathbf{x},t) = \hat{\mathbf{u}}(\mathbf{x})\mathrm{e}^{\mathrm{i}\omega t}$ with the complex-valued amplitude $\hat{\mathbf{u}} = \mathbf{u^c} - \mathrm{i}\mathbf{u^s}$. Using the time-harmonic representation (2) of the solution, we can state the eddy current problem (1) in the frequency domain as follows:

$$\text{Find } \mathbf{u} = (\mathbf{u^c}, \mathbf{u^s}) : \quad \begin{cases} \omega\,\sigma\,\mathbf{u^s} + \mathbf{curl}\,(\nu_1\,\mathbf{curl}\,\mathbf{u^c}) = \mathbf{f^c} & \text{in } \Omega_1, \\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{u^c} = \mathbf{0} & \text{in } \Omega_2, \\ -\omega\,\sigma\,\mathbf{u^c} + \mathbf{curl}\,(\nu_1\,\mathbf{curl}\,\mathbf{u^s}) = \mathbf{f^s} & \text{in } \Omega_1, \\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{u^s} = \mathbf{0} & \text{in } \Omega_2, \end{cases} \tag{3}$$

with the corresponding decay and interface conditions from (1).

Remark 1. In practice, the reluctivity ν_1 depends on the inductivity $|\mathbf{curl}\,\mathbf{u}|$ in a nonlinear way in ferromagnetic materials. Having in mind applications to problems with nonlinear reluctivity, we prefer to use the real reformulation (3) instead of a complex approach. For overcoming the nonlinearity the preferable way is to apply Newton's method due to its fast convergence. It turns out, that Newton's method cannot be applied to the nonlinear complex-valued system (see [4]), but it can be applied to the reformulated real-valued system. Anyhow, the analysis of the linear problem also helps to construct efficient solvers for the nonlinear problem.

Deriving the variational formulation and integrating by parts once more in the exterior domain yields: Find $(\mathbf{u^c},\mathbf{u^s}) \in \mathbf{H}(\mathbf{curl},\Omega_1)^2$ such that

$$\begin{cases} \omega(\sigma\mathbf{u^s},\mathbf{v^c})_{L_2(\Omega_1)} + (\nu_1\mathbf{curl}\,\mathbf{u^c},\mathbf{curl}\,\mathbf{v^c})_{L_2(\Omega_1)} - \langle\gamma_N\mathbf{u^c},\gamma_D\mathbf{v^c}\rangle_\tau = \langle\mathbf{f^c},\mathbf{v^c}\rangle, \\ -\omega(\sigma\mathbf{u^c},\mathbf{v^s})_{L_2(\Omega_1)} + (\nu_1\mathbf{curl}\,\mathbf{u^s},\mathbf{curl}\,\mathbf{v^s})_{L_2(\Omega_1)} - \langle\gamma_N\mathbf{u^s},\gamma_D\mathbf{v^s}\rangle_\tau = \langle\mathbf{f^s},\mathbf{v^s}\rangle, \end{cases}$$

for all $(\mathbf{v^c},\mathbf{v^s}) \in \mathbf{H}(\mathbf{curl},\Omega_1)^2$. Here γ_D and γ_N denote the Dirichlet trace $\gamma_D := \mathbf{n}\times(\mathbf{u}\times\mathbf{n})$ and the Neumann trace $\gamma_N := \mathbf{curl}\,\mathbf{u}\times\mathbf{n}$ on the interface Γ. $\langle\cdot,\cdot\rangle_\tau$ denotes the $L_2(\Gamma)$-based duality product. In order to deal with the expression on the interface Γ, we use the framework of the symmetric FEM-BEM coupling for eddy current problems (see [10]). So, using the boundary integral operators $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{N}$, as defined in [10], we end up with the weak formulation of the time-harmonic eddy current problem: Find $(\mathbf{u^c},\mathbf{u^s}) \in \mathbf{H}(\mathbf{curl},\Omega_1)^2$ and $(\lambda^\mathbf{c},\lambda^\mathbf{s}) \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0,\Gamma)^2$ such that

$$\begin{cases} \omega(\sigma\mathbf{u^s},\mathbf{v^c})_{L_2(\Omega_1)} + (\nu_1\mathbf{curl}\,\mathbf{u^c},\mathbf{curl}\,\mathbf{v^c})_{L_2(\Omega_1)}, \\ \qquad -\langle\mathbf{N}(\gamma_D\mathbf{u^c}),\gamma_D\mathbf{v^c}\rangle_\tau + \langle\mathbf{B}(\lambda^\mathbf{c}),\gamma_D\mathbf{v^c}\rangle_\tau = \langle\mathbf{f^c},\mathbf{v^c}\rangle, \\ \qquad \langle\mu^\mathbf{c},(\mathbf{C}-\mathbf{Id})(\gamma_D\mathbf{u^c})\rangle_\tau - \langle\mu^\mathbf{c},\mathbf{A}(\lambda^\mathbf{c})\rangle_\tau = 0, \\ -\omega(\sigma\mathbf{u^c},\mathbf{v^s})_{L_2(\Omega_1)} + (\nu_1\mathbf{curl}\,\mathbf{u^s},\mathbf{curl}\,\mathbf{v^s})_{L_2(\Omega_1)}, \\ \qquad -\langle\mathbf{N}(\gamma_D\mathbf{u^s}),\gamma_D\mathbf{v^s}\rangle_\tau + \langle\mathbf{B}(\lambda^\mathbf{s}),\gamma_D\mathbf{v^s}\rangle_\tau = \langle\mathbf{f^s},\mathbf{v^s}\rangle, \\ \qquad \langle\mu^\mathbf{s},(\mathbf{C}-\mathbf{Id})(\gamma_D\mathbf{u^s})\rangle_\tau - \langle\mu^\mathbf{s},\mathbf{A}(\lambda^\mathbf{s})\rangle_\tau = 0, \end{cases} \tag{4}$$

for all $(\mathbf{v^c},\mathbf{v^s}) \in \mathbf{H}(\mathbf{curl},\Omega_1)^2$ and $(\mu^\mathbf{c},\mu^\mathbf{s}) \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0,\Gamma)^2$. This variational form is the starting point of the discretization in space. Therefore, we use a regular triangulation $\mathscr{T}_h$, with mesh size $h > 0$, of the domain Ω_1 with tetrahedral elements. $\mathscr{T}_h$ induces a mesh $\mathscr{K}_h$ of triangles on the boundary Γ. On these meshes, we consider Nédélec basis functions of order p yielding the conforming finite element subspace $\mathscr{N}\mathscr{D}_p(\mathscr{T}_h)$ of $\mathbf{H}(\mathbf{curl},\Omega_1)$, see [17]. Further, we use the space of divergence free Raviart-Thomas basis functions $\mathscr{R}\mathscr{T}_p^0(\mathscr{K}_h) := \{\lambda_h \in \mathscr{R}\mathscr{T}_p(\mathscr{K}_h), \mathrm{div}_\Gamma \lambda_h = 0\}$ being a conforming finite element subspace of $\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0,\Gamma)$. Let $\{\varphi_i\}$ denote the basis of $\mathscr{N}\mathscr{D}_p(\mathscr{T}_h)$, and let $\{\psi_i\}$ denote the basis of $\mathscr{R}\mathscr{T}_p^0(\mathscr{K}_h)$. Then the matrix entries corresponding to the operators in (4) are given by the formulas

$$
\begin{aligned}
(\mathbf{K})_{ij} &:= (\nu\, \mathbf{curl}\, \varphi_\mathbf{i}, \mathbf{curl}\, \varphi_\mathbf{j})_{\mathbf{L_2}(\Omega_1)} - \langle \mathbf{N}(\gamma_D \varphi_\mathbf{i}), \gamma_D \varphi_\mathbf{j} \rangle_\tau, \\
(\mathbf{M})_{ij} &:= \omega(\sigma \varphi_\mathbf{i}, \varphi_\mathbf{j})_{\mathbf{L_2}(\Omega_1)}, \\
(\mathbf{A})_{ij} &:= \left\langle \psi_\mathbf{i}, \mathbf{A}(\psi_\mathbf{j}) \right\rangle_\tau, \\
(\mathbf{B})_{ij} &:= \langle \psi_\mathbf{i}, (\mathbf{C} - \mathbf{Id})(\gamma_D \varphi_\mathbf{j}) \rangle_\tau.
\end{aligned}
$$

The entries of the right-hand side vector are given by the formulas $(\mathbf{f^c})_i := (\mathbf{f^c}, \varphi_\mathbf{i})_{\mathbf{L_2}(\Omega_1)}$ and $(\mathbf{f^s})_i := (\mathbf{f^s}, \varphi_\mathbf{i})_{\mathbf{L_2}(\Omega_1)}$. The resulting system $\mathscr{A}\,\mathbf{x} = \mathbf{f}$ of the coupled finite and boundary element equations has now the following structure:

$$
\begin{pmatrix}
\mathbf{M} & 0 & \mathbf{K} & \mathbf{B}^T \\
0 & 0 & \mathbf{B} & -\mathbf{A} \\
\mathbf{K} & \mathbf{B}^T & -\mathbf{M} & 0 \\
\mathbf{B} & -\mathbf{A} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{u^s} \\ \lambda^s \\ \mathbf{u^c} \\ \lambda^c
\end{pmatrix}
=
\begin{pmatrix}
\mathbf{f^c} \\ 0 \\ \mathbf{f^s} \\ 0
\end{pmatrix}. \tag{5}
$$

In fact, the system matrix $\mathscr{A}$ is symmetric and indefinite and obtains a double saddle-point structure. Since $\mathscr{A}$ is symmetric, the system can be solved by a MinRes method, see, e.g., [18]. Anyhow, the convergence rate of any iterative method deteriorates with respect to the meshsize h and the "bad" parameters ω, ν and σ, if applied to the unpreconditioned system (5). Therefore, preconditioning is a challenging topic.

3 A Parameter-Robust Preconditioning Technique

In this section, we investigate a preconditioning technique for double saddle-point equations with the block-structure (5). Due to the symmetry and coercivity properties of the underlying operators, the blocks fulfill the following properties: $\mathbf{K} = \mathbf{K}^T \geq 0$, $\mathbf{M} = \mathbf{M}^T > 0$ and $\mathbf{A} = \mathbf{A}^T > 0$.

In [19] a parameter-robust block-diagonal preconditioner for the distributed optimal control of the Stokes equations is constructed. The structural similarities to that preconditioner gives us a hint how to choose the block-diagonal preconditioner in our case. Therefore, we propose the following preconditioner

$$
\mathscr{C} = \mathrm{diag}\ (\mathscr{I}_{FEM}, \mathscr{I}_{BEM}, \mathscr{I}_{FEM}, \mathscr{I}_{BEM}),
$$

where the diagonal blocks are given by $\mathscr{I}_{FEM} = \mathbf{M} + \mathbf{K}$ and $\mathscr{I}_{BEM} = \mathbf{A} + \mathbf{B}\mathscr{I}_{FEM}^{-1}\mathbf{B}^T$. Being aware that $\mathscr{I}_{FEM}$ and $\mathscr{I}_{BEM}$ are symmetric and positive definite, we conclude that $\mathscr{C}$ is also symmetric and positive definite. Therefore, $\mathscr{C}$ induces the energy norm $\|\mathbf{u}\|_{\mathscr{C}} = \sqrt{\mathbf{u}^T \mathscr{C} \mathbf{u}}$. Using this special norm, we can apply the Theorem of Babuška-Aziz [3] to the variational problem:

$$
\text{Find } \mathbf{x} \in \mathbb{R}^N: \quad \mathbf{w}^T \mathscr{A} \mathbf{x} = \mathbf{w}^T \mathbf{f}, \quad \forall \mathbf{w} \in \mathbb{R}^N.
$$

The main result is now summarized in the following lemma.

Lemma 1. *The matrix $\mathscr{A}$ satisfies the following norm equivalence inequalities:*

$$\frac{1}{\sqrt{7}}\|\mathbf{x}\|_{\mathscr{C}} \leq \sup_{\mathbf{w}\neq 0}\frac{\mathbf{w}^T\mathscr{A}\mathbf{x}}{\|\mathbf{w}\|_{\mathscr{C}}} \leq 2\,\|\mathbf{x}\|_{\mathscr{C}} \quad \forall \mathbf{x}\in\mathbb{R}^N.$$

Proof. Throughout the proof, we use the following notation: $\mathbf{x} = (\mathbf{x_1},\mathbf{x_2},\mathbf{x_3},\mathbf{x_4})^T$ and $\mathbf{y} = (\mathbf{y_1},\mathbf{y_2},\mathbf{y_3},\mathbf{y_4})^T$. The upper bound follows by reapplication of Cauchy's inequality several time. The expressions corresponding to the Schur complement can be derived in the following way:

$$\mathbf{y_1}^T\mathbf{B}^T\mathbf{x_4} = \mathbf{y_1}\mathscr{I}_{FEM}^{1/2}\mathscr{I}_{FEM}^{-1/2}\mathbf{B}^T\mathbf{x_4} \leq \|\mathscr{I}_{FEM}^{1/2}\mathbf{y_1}\|_{l_2}\|\mathscr{I}_{FEM}^{-1/2}\mathbf{B}^T\mathbf{x_4}\|_{l_2}.$$

Therefore, we end up with an upper bound with constant 2.

In order to compute the lower bound, we use a linear combination of special test vectors. For the choice $\mathbf{w_1} = (\mathbf{x_1},\mathbf{x_2},-\mathbf{x_3},-\mathbf{x_4})^T$, we obtain

$$\mathbf{w_1}^T\mathscr{A}\mathbf{x} = \mathbf{x_1}^T\mathbf{M}\mathbf{x_1} + \mathbf{x_3}^T\mathbf{M}\mathbf{x_3};$$

for $\mathbf{w_2} = (\mathbf{x_3},-\mathbf{x_4},\mathbf{x_1},-\mathbf{x_2})^T$, we get

$$\mathbf{w_2}^T\mathscr{A}\mathbf{x} = \mathbf{x_1}^T\mathbf{K}\mathbf{x_1} + \mathbf{x_3}^T\mathbf{K}\mathbf{x_3} + \mathbf{x_2}^T\mathbf{A}\mathbf{x_2} + \mathbf{x_4}^T\mathbf{A}\mathbf{x_4};$$

for $\mathbf{w_3} = ((\mathbf{x_4}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1})^T,\mathbf{0},(\mathbf{x_2}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1})^T,\mathbf{0})^T$, we have

$$\begin{aligned}\mathbf{w_3}^T\mathscr{A}\mathbf{x} = {} & \mathbf{x_4}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T\mathbf{x_4} + \mathbf{x_2}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T\mathbf{x_2}\\ & + \mathbf{x_4}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1} + \mathbf{x_4}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3}\\ & + \mathbf{x_2}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1} - \mathbf{x_2}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3};\end{aligned}$$

for $\mathbf{w_4} = (-(\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1})^T,\mathbf{0},-(\mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1})^T,\mathbf{0})^T$, we get

$$\begin{aligned}\mathbf{w_4}^T\mathscr{A}\mathbf{x} = {} & -\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1} - \mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3}\\ & - \mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T\mathbf{x_4} - \mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1}\\ & - \mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T\mathbf{x_2} + \mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3};\end{aligned}$$

and, finally, for the choice $\mathbf{w_5} = (-(\mathbf{x_1}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1})^T,\mathbf{0},(\mathbf{x_3}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1})^T,\mathbf{0})^T$, we obtain

$$\begin{aligned}\mathbf{w_5}^T\mathscr{A}\mathbf{x} = {} & -\mathbf{x_1}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1} - \mathbf{x_1}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3}\\ & - \mathbf{x_1}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T\mathbf{x_4} + \mathbf{x_3}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1}\\ & + \mathbf{x_3}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T\mathbf{x_2} - \mathbf{x_3}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3}.\end{aligned}$$

Therefore, we end up with the following expression

$$\begin{aligned}(\mathbf{w_1}+\mathbf{w_2}+\mathbf{w_3}+\mathbf{w_4}+\mathbf{w_5})^T\mathscr{A}\mathbf{x} = {} & \mathbf{x_1}^T\mathbf{M}\mathbf{x_1}+\mathbf{x_3}^T\mathbf{M}\mathbf{x_3}\\ & +\mathbf{x_1}^T\mathbf{K}\mathbf{x_1}+\mathbf{x_3}^T\mathbf{K}\mathbf{x_3}+\mathbf{x_2}^T\mathbf{A}\mathbf{x_2}+\mathbf{x_4}^T\mathbf{A}\mathbf{x_4}\\ & +\mathbf{x_4}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T\mathbf{x_4}+\mathbf{x_2}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T\mathbf{x_2}\\ & -\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3}-\mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1}\\ & -\mathbf{x_3}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3}-\mathbf{x_1}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1}\\ & -2\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1}+2\mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3}.\end{aligned}$$

For estimating the non-symmetric terms, we use the following result:

$$\begin{aligned}-2\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1} & \geq -2\|(\mathbf{K}+\mathbf{M})^{-1/2}\mathbf{K}\mathbf{x_3}\|_{l_2}\|(\mathbf{K}+\mathbf{M})^{-1/2}\mathbf{M}\mathbf{x_1}\|_{l_2}\\ & \geq -\|(\mathbf{K}+\mathbf{M})^{-1/2}\mathbf{K}\mathbf{x_3}\|_{l_2}^2-\|(\mathbf{K}+\mathbf{M})^{-1/2}\mathbf{M}\mathbf{x_1}\|_{l_2}^2\\ & = -\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3}-\mathbf{x_1}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1}.\end{aligned}$$

Analogously, we obtain

$$2\mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3} \geq -\mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1}-\mathbf{x_3}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3}.$$

Hence, putting all terms together, we have

$$\begin{aligned}(\mathbf{w_1}+\mathbf{w_2}+\mathbf{w_3}+\mathbf{w_4}+\mathbf{w_5})^T\mathscr{A}\mathbf{x} = {} & \mathbf{x}^T\mathscr{C}\mathbf{x}\\ & -2\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3}-2\mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1}\\ & -2\mathbf{x_3}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3}-2\mathbf{x_1}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1}.\end{aligned}$$

In order to get rid of the four remaining terms, we use, for $i=1,3$,

$$\mathbf{x_i}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_i} \leq \mathbf{x_i}^T\mathbf{K}\mathbf{x_i} \quad \text{and} \quad \mathbf{x_i}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_i} \leq \mathbf{x_i}^T\mathbf{M}\mathbf{x_i}.$$

Hence by adding $\mathbf{w_1}$ and $\mathbf{w_2}$ twice more, we end up with the desired result

$$\underbrace{(3\mathbf{w_1}+3\mathbf{w_2}+\mathbf{w_3}+\mathbf{w_4}+\mathbf{w_5})^T}_{:=\mathbf{w}^T}\mathscr{A}\mathbf{x} \geq \mathbf{x}^T\mathscr{C}\mathbf{x}+\mathbf{x_2}^T\mathbf{A}\mathbf{x_2}+\mathbf{x_4}^T\mathbf{A}\mathbf{x_4} \geq \mathbf{x}^T\mathscr{C}\mathbf{x}.$$

The next step is to compute (and estimate) the $\mathscr{C}$ norm of the special test vector. Straightforward estimations yield

$$\|\mathbf{w}\|_{\mathscr{C}}^2 = \|3\mathbf{w_1}+3\mathbf{w_2}+\mathbf{w_3}+\mathbf{w_4}+\mathbf{w_5}\|_{\mathscr{C}}^2 \leq 7\,\|\mathbf{x}\|_{\mathscr{C}}^2.$$

This completes the proof.

Now, from Lemma 1, we obtain that the condition number of the preconditioned system can be estimated by the constant $c=2\sqrt{7}$ that is obviously independent of the meshsize h and all involved parameters ω, ν and σ, i.e.

$$\kappa_{\mathscr{C}}(\mathscr{C}^{-1}\mathscr{A}) := \|\mathscr{C}^{-1}\mathscr{A}\|_{\mathscr{C}}\|\mathscr{A}^{-1}\mathscr{C}\|_{\mathscr{C}} \leq 2\sqrt{7}. \tag{6}$$

The condition number defines the convergence behaviour of the MinRes method applied to the preconditioned system (see e.g. [9]), as stated in the following theorem:

Theorem 1 (Robust solver). *The MinRes method applied to the preconditioned system* $\mathscr{C}^{-1}\mathscr{A}\mathbf{u} = \mathscr{C}^{-1}\mathbf{f}$ *converges. At the 2m-th iteration, the preconditioned residual* $\mathbf{r^m} = \mathscr{C}^{-1}\mathbf{f} - \mathscr{C}^{-1}\mathscr{A}\mathbf{u^m}$ *is bounded as*

$$\left\|\mathbf{r^{2m}}\right\|_{\mathscr{C}} \leq \frac{2q^m}{1+q^{2m}} \left\|\mathbf{r^0}\right\|_{\mathscr{C}}, \quad where \quad q = \frac{2\sqrt{7}-1}{2\sqrt{7}+1}. \tag{7}$$

4 Conclusion, Outlook and Acknowledgments

The method developed in this work shows great potential for solving time-harmonic eddy current problems in an unbounded domain in a robust way. The solution of a fully coupled 4×4 block-system can be reduced to the solution of a block-diagonal matrix, where each block corresponds to standard problems. We mention, that by analogous procedure, we can state another robust block-diagonal preconditioner $\tilde{\mathscr{C}} = \text{diag}\,(\tilde{\mathscr{I}}_{FEM}, \tilde{\mathscr{I}}_{BEM}, \tilde{\mathscr{I}}_{FEM}, \tilde{\mathscr{I}}_{BEM})$, with $\tilde{\mathscr{I}}_{FEM} = \mathbf{M} + \mathbf{K} + \mathbf{B}^T \tilde{\mathscr{I}}_{BEM}^{-1} \mathbf{B}$ and $\tilde{\mathscr{I}}_{BEM} = \mathbf{A}$, leading to a condition number bound of 4, see e.g. [15].

Of course this block-diagonal preconditioner is only a theoretical one, since the exact solution of the diagonal blocks corresponding to a standard FEM discretized stationary problem and the Schur-complement of a standard FEM-BEM discretized stationary problem are still prohibitively expensive. Nevertheless, as for the FEM discretized version in [13], this theoretical preconditioner allows us replace the solution of a time-dependent problem by the solution of a sequence of time-independent problems in a robust way, i.e. independent of the space and time discretization parameters h and ω and all additional "bad" parameters. Therefore, the issue of finding robust solvers for the fully coupled time-harmonic system matrix $\mathscr{A}$ can be reduced to finding robust solvers for the blocks $\mathscr{I}_{FEM}$ and $\mathscr{I}_{BEM}$, or $\tilde{\mathscr{I}}_{FEM}$ and $\tilde{\mathscr{I}}_{BEM}$. By replacing these diagonal blocks by standard preconditioners, it is straight-forward to derive mesh-independent convergence rates, see, e.g., [8]. Unfortunately, the construction of fully robust preconditioners for the diagonal blocks is not straight forward and has to be studied. Candidates are $\mathscr{H}$ matrix, multigrid multigrid and domain decomposition preconditioners, see, e.g. [2, 6] and [12], respectively.

The preconditioned MinRes solver presented in this paper can also be generalized to eddy current optimal control problems studied in [14] for the pure FEM case in bounded domains.

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