On Block Preconditioners for Generalized Saddle Point Problems

Piotr Krzyżanowski

University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland;
piotr.krzyzanowski@mimuw.edu.pl

1 Introduction

We consider a symmetric system of linear equations with a block structure,

$$\mathscr{M}\begin{pmatrix} u \\ p \end{pmatrix} \equiv \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}. \tag{1}$$

We assume that A is $n \times n$ and C is an $m \times m$ matrix. Many such systems arise from the discretization of (systems of) partial differential equations. For example, Stokes equations discretized with stable finite elements or a mixed finite element method for second order elliptic PDEs lead to a positive definite matrix A and to $C = 0$, so that (1) has a genuine saddle point structure. Certain other PDE problems may result in an indefinite matrix A, or a semidefinite matrix A with a large kernel, which gives (1) the structure of a so called generalized saddle point problem. Linear elasticity equations modelling nearly incompressible materials discretized with mixed finite elements result in both matrices A and C being positive definite, having thus a nature of a penalized saddle point problem. All systems mentioned above have a common feature that the matrix of (1) is indefinite.

The specific structure of (1) makes it possible to design efficient solution methods which intensively exploit the properties of the system, see the recent survey of [4] on the state-of-the-art in this field. Systems derived from the discretization of PDEs are usually very large and sparse, and typically are solved by some iterative method. Unfortunately, these systems are ill-conditioned with respect to the mesh size h, so preconditioning is necessary in order to keep the number of iterations within a reasonable limit. Applying a left preconditioner $\mathscr{P}$, one then solves a problem with a preconditioned matrix $\mathscr{P}^{-1}\mathscr{M}$. We shall consider preconditioners of the form

$$\mathscr{P}_{\mathrm{d}} = \begin{pmatrix} I & \\ cBA_0^{-1} & I \end{pmatrix}\begin{pmatrix} A_0 & \\ & S_0 \end{pmatrix}\begin{pmatrix} I & dA_0^{-1}B^T \\ & I \end{pmatrix} \tag{2}$$

or

$$\mathscr{P}_{\mathrm{p}} = \begin{pmatrix} I & dB^TS_0^{-1} \\ & I \end{pmatrix}\begin{pmatrix} A_0 & \\ & S_0 \end{pmatrix}\begin{pmatrix} I & \\ cS_0^{-1}B & I \end{pmatrix}, \tag{3}$$

R. Bank et al. (eds.), *Domain Decomposition Methods in Science and Engineering XX*,
Lecture Notes in Computational Science and Engineering 91,
DOI 10.1007/978-3-642-35275-1_70, © Springer-Verlag Berlin Heidelberg 2013

where A_0 and S_0 are symmetric, positive (or negative) definite matrices whose inverses are *easy to apply* and $c,d \in \{-1,+1\}$. In accordance with [8], we will refer to $\mathscr{P}_{\mathrm{d}}$ as the family of dual block preconditioners and to $\mathscr{P}_{\mathrm{p}}$ as the family of primal block preconditioners.

Many popular block preconditioners can be formed by choosing appropriate values of c and d in the formulas above. For example, a block diagonal preconditioner, cf. e.g. [2, 6, 9, 13, 19, 21] corresponds to $c = d = 0$ above. Block triangular preconditioners considered e.g. in [7, 14, 22] and the Bramble–Pasciak preconditioner as well, see [5], are obtained with either c or d equal to zero. The choice $c = d = 1$ in (2) produces a symmetric indefinite preconditioner, see [3, 20, 24, 25], while the same choice in (3) leads to a primal based penalty preconditioner, [1, 8].

It is straightforward that solving a system with $\mathscr{P}_{\mathrm{d}}$ requires one solve with S_0 and at most two solves with A_0, while applying $\mathscr{P}_{\mathrm{p}}$ to a vector takes one solve with A_0 and at most two solves with S_0. When $cd = 0$, both types of preconditioners require only one solve with A_0 and one with S_0.

Let us stress that when (1) arises from finite element discretization of PDEs, there is a possibility to use other than block preconditioning approaches. On the other hand, for many types of discretizations and problems, specialized methods based on direct construction of a multigrid or domain decomposition preconditioner—although usually outperforming block preconditioners, [15]—may take a considerable effort to develop, implement and analyse. Since the block preconditioning approach as discussed here turns out to be based on preconditioners for symmetric positive definite matrices, this property makes it a viable and robust alternative to custom methods, as in this case one can efficiently reuse existing theory and software to solve more complex problems. This feature has been recognized in the software package PETSc, see [23], where a family of so called field-splitting preconditioners has recently been implemented.

2 Eigenvalue Estimates of the Preconditioned System

Eigenvalue clustering is vital for the convergence of a Krylov method, so it is important to bound the spectrum of $\mathscr{P}^{-1}\mathscr{M}$, where $\mathscr{P}$ stands for either $\mathscr{P}_{\mathrm{d}}$ or $\mathscr{P}_{\mathrm{p}}$. Inspired by the block nature of the problem, which imposes a decomposition of the unknowns into two parts $(u,p) \in R^n \times R^m$, let us define a block diagonal, symmetric, positive definite matrix

$$\mathscr{J} = \begin{pmatrix} \tilde{A}_0 & \\ & \tilde{S}_0 \end{pmatrix},$$

where $\tilde{A}_0$ is either A_0, if A_0 is positive definite, or $(-A_0)$, if A_0 is negative definite; $\tilde{S}_0$ is defined in the same way. We assume there exist positive constants m_0 and m_1 such that

$$m_0 ||x||_{\mathscr{J}} \le ||\mathscr{M}x||_{\mathscr{J}^{-1}} \le m_1 ||x||_{\mathscr{J}} \qquad \forall x \in R^n \times R^m,$$

where

$$|| \begin{pmatrix} u \\ p \end{pmatrix} ||^2_{\mathscr{J}} = ||u||^2_{\tilde{A}_0} + ||p||^2_{\tilde{S}_0},$$

This is nothing but a stability and continuity assumption in an appropriate norm, see also [18]. At the same time we suppose there exists a constant b_0 such that for any $u \in R^n$ and $p \in R^m$,

$$|p^T Bu| \leq b_0 ||u||_{\tilde{A}_0} ||p||_{\tilde{S}_0}.$$

Finally, we assume that for some $\delta \in \{-1, +1\}$, the matrix $\mathscr{H}$ is positive definite, where $\mathscr{H}$ is equal to either $\mathscr{H}_\mathrm{d}$ or $\mathscr{H}_\mathrm{p}$ (depending on whether we are addressing $\mathscr{P}_\mathrm{d}$ or $\mathscr{P}_\mathrm{p}$), with

$$\mathscr{H}_\mathrm{d} = \delta \begin{pmatrix} A_0 - cA & \\ & S_0 + cdBA_0^{-1}B^T + dC \end{pmatrix},$$

$$\mathscr{H}_\mathrm{p} = \delta \begin{pmatrix} A_0 + cdB^T S_0^{-1} B - cA & \\ & S_0 + dC \end{pmatrix}.$$

It turns out that then both $\mathscr{H}_\mathrm{d} {\mathscr{P}_\mathrm{d}}^{-1} \mathscr{M}$ and $\mathscr{H}_\mathrm{p} {\mathscr{P}_\mathrm{p}}^{-1} \mathscr{M}$ are symmetric and the eigenvalues of the preconditioned matrix are bounded as stated in the following theorem, whose proof appeared in [16]:

Theorem 1. *Suppose the above assumptions are fulfilled. If λ is an eigenvalue of ${\mathscr{P}_\mathrm{d}}^{-1}\mathscr{M}$ or of ${\mathscr{P}_\mathrm{p}}^{-1}\mathscr{M}$, then it is real and satisfies*

$$\frac{m_0}{2(1+b_0^2)} \leq |\lambda| \leq 2m_1(1+b_0^2).$$

Let us mention that earlier Klawonn [12] proved a similar result for block diagonal preconditioning matrices.

2.1 Example Application: Stabilized Stokes Equations

Theorem 1 relies on the stability of (1) and therefore indicates that block preconditioners can be used also in the case when the inf-sup condition is not satisfied and one uses a so called stabilized method. As a model example let us consider a stabilized $Q_1 - Q_1$ discretization of Stokes equations

$$\begin{aligned} -\Delta u + \nabla p &= f, \\ \nabla \cdot u &= 0. \end{aligned}$$

Let $\mathscr{T}_h$ denote a shape-regular, quasi-uniform triangulation of a polygonal $\Omega \subset R^2$ into quadrilaterals. Define the finite dimensional spaces of bilinear finite elements:

$$V_h = \{v \in [H_0^1(\Omega)]^2 : v_{|_\kappa} \in [Q_1(\kappa)]^2 \quad \forall \kappa \in \mathscr{T}_h\}$$

and

$$W_h = \{q \in L_0^2(\Omega) \cap C(\Omega) : q_{|_\kappa} \in Q_1(\kappa) \quad \forall \kappa \in \mathscr{T}_h\},$$

where $Q_1(\kappa)$ denotes the space of bilinear functions on κ. Since V_h and W_h do not satisfy the inf-sup condition the following stabilized discretization has been introduced in [11]:

$$\begin{cases} (\nabla u_h, \nabla v_h)_{L^2(\Omega)} - (\mathrm{div} v_h, p_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} & \forall v_h \in V_h, \\ -(\mathrm{div}\, u_h, q_h)_{L^2(\Omega)} - c(p_h, q_h) = -\tau \sum_{\kappa \in \mathscr{T}_h} h_\kappa^2 (f, \nabla q_h)_{L^2(\kappa)} & \forall q_h \in W_h, \end{cases} \tag{4}$$

where

$$c(p_h, q_h) = \tau \sum_{\kappa \in \mathscr{T}_h} h_\kappa^2 (\nabla p_h, \nabla q_h)_{L^2(\kappa)}$$

and $\tau > 0$ is some prescribed parameter, independent of h. As the above system is stable and continuous in the norm $\left(||u||^2_{H^1_0} + ||p||^2_{L^2} \right)^{1/2}$, one concludes that an optimal preconditioner (with respect to the mesh size h) can be obtained with either $\mathscr{P}_\mathrm{d}$ or $\mathscr{P}_\mathrm{p}$, where $\tilde{A}_0$ is spectrally equivalent to the discrete Lapacian operator and $\tilde{S}_0$ is spectrally equivalent to the pressure mass matrix. These operators may require some pre-scaling in order to make either $\mathscr{H}_\mathrm{d}$ or $\mathscr{H}_\mathrm{d}$ positive definite.

Numerical Experiments

We confirm the above findings running experiments for a stabilized $Q_1 - Q_1$ discretization of the Stokes system on a unit square, obtained under MATLAB with the software package IFISS 2.2, see [10].

We investigated the number of iterations of the preconditioned conjugate residual method required to reduce the residual norm by a factor of 10^6. We experimented with $\mathscr{P}_\mathrm{d}$ having one of the following forms: block diagonal ($c = 1$, $d = 0$), upper triangular ($c = 0$, $d = 1$) and lower triangular ($c = d = 0$) (see [17] for implementation details) for varying mesh size h. The results for the case when $A_0 = A$ and $S_0 = M$ (as suggested by the above analysis) are provided below, confirming a convergence rate independent of h:

$n+m$	243	867	3,267	12,675	49,923
Lower triangular	17	21	21	22	23
Upper triangular	16	16	16	16	16
Diagonal	32	35	37	39	39

In order to show a more realistic choice of A_0, we used A_0^{-1} defined by means of the incomplete Cholesky factorization of A, with drop tolerance 10^{-3}. Since for our model problem the quality of the incomplete Cholesky factorization degrades slowly with increasing size of the system, this is also reflected in an increase of the iteration counts:

$n+m$	243	867	3,267	12,675	49,923
Lower triangular	18	20	24	35	113
Upper triangular	17	17	20	33	—
Diagonal	33	38	48	74	132

It has been observed that (at least in our implementation) the best solution times were obtained mostly for triangular preconditioners.

3 Conclusions

We have presented two classes of block preconditioners for symmetric saddle point problems and provided eigenvalue estimates of the preconditioned system $\mathscr{P}^{-1}\mathscr{M}$ under a quite general assumption of the stability and continuity of the problem being solved. In the context of PDEs, based upon this result, an iterative method, optimal with respect to the mesh size h, can be designed, which may reuse existing state-of-the-art preconditioners or fast solvers for certain elliptic problems.

Acknowledgments The research has been partially supported in part by Polish Ministry of Science and Higher Education grant N N201 0069 33.

Bibliography

[1] O. Axelsson. Preconditioning of indefinite problems by regularization. *SIAM J. Numer. Anal.*, 16(1):58–69, 1979.

[2] Owe Axelsson and Maya Neytcheva. Eigenvalue estimates for preconditioned saddle point matrices. *Numer. Linear Algebra Appl.*, 13(4):339–360, 2006.

[3] Randolph E. Bank, Bruno D. Welfert, and Harry Yserentant. A class of iterative methods for solving saddle point problems. *Numer. Math.*, 56(7):645–666, 1990.

[4] Michele Benzi, Gene H. Golub, and Jörg Liesen. Numerical solution of saddle point problems. *Acta Numer.*, 14:1–137, 2005.

[5] James H. Bramble and Joseph E. Pasciak. A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. *Math. Comp.*, 50(181):1–17, 1988.

[6] J.H. Bramble and J.E. Pasciak. Iterative techniques for time dependent Stokes problems. *Comput. Math. Appl.*, 33(1–2):13–30, 1997.

[7] Zhi-Hao Cao. Positive stable block triangular preconditioners for symmetric saddle point problems. *Appl. Numer. Math.*, 57(8):899–910, 2007.

[8] C. R. Dohrmann and R. B. Lehoucq. A primal-based penalty preconditioner for elliptic saddle point systems. *SIAM J. Numer. Anal.*, 44(1):270–282 (electronic), 2006.

[9] E. G. D'yakonov. Iterative methods with saddle operators. *Dokl. Akad. Nauk SSSR*, 292(5):1037–1041, 1987.

[10] Howard C. Elman, Alison Ramage, and David J. Silvester. Algorithm 886: IFISS, a Matlab toolbox for modelling incompressible flow. *ACM Trans. Math. Software*, 33(2):Art. 14, 18, 2007. software available at http://www.cs.umd.edu/~elman/ifiss.html.

[11] L.P. Franca, T.J.R. Hughes, and R. Stenberg. Stabilised finite element methods for the Stokes problem. In R.A. Nicolaides and M.D. Gunzburger, editors, *Incompressible Computational Fluid Dynamics – Trends and Advances*, pages 87–107, London, 1993. Cambridge University Press.

[12] Axel Klawonn. *Preconditioners for Indefinite Problems*. PhD thesis, Universität Münster, Germany, 1996.

[13] Axel Klawonn. An optimal preconditioner for a class of saddle point problems with a penalty term. *SIAM J. Sci. Comput.*, 19(2):540–552 (electronic), 1998.

[14] Axel Klawonn. Block-triangular preconditioners for saddle point problems with a penalty term. *SIAM J. Sci. Comput.*, 19(1):172–184 (electronic), 1998. Special issue on iterative methods (Copper Mountain, CO, 1996).

[15] Axel Klawonn and Luca F. Pavarino. A comparison of overlapping Schwarz methods and block preconditioners for saddle point problems. *Numer. Linear Algebra Appl.*, 7(1):1–25, 2000.

[16] Piotr Krzyżanowski. Block preconditioners for saddle point problems resulting from discretizations of partial differential equations. In Owe Axelsson and Janos Karatson, editors, *Efficient preconditioned solution methods for elliptic partial differential equations*. Bentham Science Publishers, 2011. URL `www.benthamscience.com/ebooks/9781608052912`. E-book available online.

[17] Piotr Krzyżanowski. On block preconditioners for saddle point problems with singular or indefinite (1, 1) block. *Numer. Linear Algebra Appl.*, 18(1):123–140, 2011.

[18] Kent-Andre Mardal and Ragnar Winther. Uniform preconditioners for the time dependent Stokes problem. *Numer. Math.*, 98(2):305–327, 2004.

[19] Torgeir Rusten and Ragnar Winther. A preconditioned iterative method for saddlepoint problems. *SIAM J. Matrix Anal. Appl.*, 13(3):887–904, 1992. Iterative methods in numerical linear algebra (Copper Mountain, CO, 1990).

[20] Joachim Schöberl and Walter Zulehner. Symmetric indefinite preconditioners for saddle point problems with applications to PDE-constrained optimization problems. *SIAM J. Matrix Anal. Appl.*, 29(3):752–773 (electronic), 2007.

[21] David Silvester and Andrew Wathen. Fast iterative solution of stabilised Stokes systems. II. Using general block preconditioners. *SIAM J. Numer. Anal.*, 31(5):1352–1367, 1994.

[22] V. Simoncini. Block triangular preconditioners for symmetric saddle-point problems. *Appl. Numer. Math.*, 49(1):63–80, 2004.

[23] B.F. Smith, W.D. Gropp, and L.C. McInnes. PETSc 2.0 users manual. Technical Report ANL-95/11, Argonne National Laboratory, 1997. Also available via `ftp://www.mcs.anl/pub/petsc/manual.ps`.

[24] Panayot S. Vassilevski and Raytcho D. Lazarov. Preconditioning mixed finite element saddle-point elliptic problems. *Numer. Linear Algebra Appl.*, 3(1):1–20, 1996.

[25] Walter Zulehner. Analysis of iterative methods for saddle point problems: a unified approach. *Math. Comp.*, 71(238):479–505 (electronic), 2002.