Domain Decomposition Methods for Auxiliary Linear Problems of an Elliptic Variational Inequality

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Summary. Elliptic variational inequalities with multiple bodies are considered. It is assumed that an active set method is used to handle the nonlinearity of the inequality constraint, which results in auxiliary linear problems. We describe two domain decomposition methods for solving such linear problems, namely, the FETI-FETI (finite element tearing and interconnecting) and hybrid methods, which are combinations of already existing domain decomposition methods.

Estimates of the condition numbers of both methods are provided. The FETI-FETI method has a condition number which depends linearly on the number of subdomains across each body and polylogarithmically on the number of element across each subdomain. The hybrid method is a scalable alternative to the FETI-FETI method, and has a condition number with two polylogarithmic factors depending on the number of elements across each subdomain and across each body. We present numerical results confirming these theoretical findings.

1 Introduction

Consider the following inequality constrained minimization problem,

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} \left(\frac{1}{2} \int_{\Omega_i} \rho(x) |\nabla u^i(x)|^2 dx - \int_{\Omega_i} f(x) u^i(x) dx \right), \\ \text{where} \quad & u^i \in H^1(\Omega_i),\ u^i = 0 \quad \text{on} \quad \Gamma_u^i,\ i = 1, \cdots, N, \\ & u^i - u^j \leq 0 \quad \text{on} \quad \partial\Omega_i \cap \partial\Omega_j, \forall i < j, \end{aligned} \tag{1}$$

with variable coefficients and multiple bodies $\Omega_i \subset \mathbb{R}^2$ with their boundaries and the Dirichlet boundaries denoted by $\partial\Omega_i$ and Γ_u^i, respectively, for $i = 1, \cdots, N$. The bodies are decomposed into subdomains,

$$\Omega_i = \bigcup_{j=1}^{N_i} \Omega_{i,j}, \quad i = 1, \cdots, N.$$

Here, *bodies* mean separate physical entities; for instance, two rubber balls in contact with each other are considered two bodies. *Subdomains*, on the other hand, is artificially introduced for convenience; a rubber ball can consist of as many subdomains

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as the modeler wants. We assume that the coefficient ρ varies moderately within each body, $\Omega_i, i = 1, \cdots, N$. The diameters of Ω_i and $\Omega_{i,j}$ are denoted by H_i and $H_{i,j}$, respectively. The smallest diameters of any element in Ω_i and $\Omega_{i,j}$ are denoted by h_i and $h_{i,j}$, respectively. Also, $H_b := \max_i H_i$, $H_s := \max_{i,j} H_{i,j}$, $\frac{H_b}{h} := \max_i \frac{H_i}{h_i}$, $\frac{H_s}{h} := \max_{i,j} \frac{H_{i,j}}{h_{i,j}}$. We introduce the following:

$$\begin{aligned} \Gamma_{gl} &:= \bigcup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j, \text{ potential contact surface between bodies,,} \\ \Gamma_{loc}^{(i)} &:= \bigcup_{j \neq k} (\partial\Omega_{i,j} \cap \partial\Omega_{i,k}), \text{interface between subdomains, } i = 1, \cdots, N. \end{aligned} \tag{2}$$

Here, the subscripts *gl* and *loc* stand for global and local, respectively, referring to nature of the interfaces. For each body, $\Omega_i, i = 1, \cdots, N$, two kinds of finite element spaces are introduced: $\widehat{W}^{(i)}$ is a standard finite element space of continuous, piecewise linear functions and, as such, is continuous across $\Gamma_{loc}^{(i)}$; $\widetilde{W}^{(i)}$ is a more general space, consisting of finite element functions required to be continuous only at the *primal* nodes (i.e., the vertex nodes of $\Gamma_{loc}^{(i)}$ in this two-dimensional case; more sophisticated continuity couplings, i.e., primal constraints, are required in $\widetilde{W}^{(i)}$ for three-dimensional problems; see [9, 10]), as in the FETI-DP (dual-primal FETI) method. The trace spaces of $\widetilde{W}^{(i)}$ and $\widehat{W}^{(i)}$ on $\Gamma_{loc}^{(i)} \cup (\partial\Omega_i \cap \Gamma_{gl})$ are denoted by $\widetilde{V}^{(i)}$ and $\widehat{V}^{(i)}$, respectively. The trace space of $\widehat{W}^{(i)}$ on $\partial\Omega_i \cap \Gamma_{gl}$ is denoted by $V_{OL}^{(i)}$, where OL stands for "one level." The Schur complements of the stiffness matrices for $\widetilde{W}^{(i)}$ and $\widehat{W}^{(i)}$, obtained by eliminating unknowns corresponding to the *subdomain interiors*, that is, those *not* associated with $\Gamma_{loc}^{(i)} \cup (\partial\Omega_i \cap \Gamma_{gl})$, are denoted by $\widetilde{S}_{\Gamma}^{(i)}$ and $\widehat{S}_{\Gamma}^{(i)}$, respectively. The Schur complement $S_{OL}^{(i)}$ of the stiffness matrix for $\widehat{W}^{(i)}$, on the other hand, is obtained by eliminating unknowns corresponding to the *body interior*, i.e., those *not* associated with $\partial\Omega_i \cap \Gamma_{gl}$. Therefore $\widetilde{S}_{\Gamma}^{(i)}, \widehat{S}_{\Gamma}^{(i)}$, and $S_{OL}^{(i)}$ can be viewed as operators on $\widetilde{V}^{(i)}, \widehat{V}^{(i)}$, and $V_{OL}^{(i)}$, respectively. We note that applying $S_{OL}^{(i)}$ requires solving a Dirichlet problem on Ω_i.

Let $\widetilde{V} := \prod_{i=1}^{N} \widetilde{V}^{(i)}, \widehat{V} := \prod_{i=1}^{N} \widehat{V}^{(i)}, V_{OL} := \prod_{i=1}^{N} V_{OL}^{(i)}, \widetilde{S} = \text{diag}_{i=1}^{N} \widetilde{S}_{\Gamma}^{(i)}, \widehat{S} = \text{diag}_{i=1}^{N} \widehat{S}_{\Gamma}^{(i)}$, and $S_{OL} := \text{diag}_{i=1}^{N} S_{OL}^{(i)}$. We also introduce matrices $\widetilde{B}, \widehat{B}$, and B_{OL}, with elements of $\{0, -1, 1\}$: $\widetilde{B}u \Leftrightarrow u \in \widetilde{V}$ is continuous across $\Gamma_{loc}^{(i)}, \forall i$, as well as Γ_{gl}; $\widehat{B}v \Leftrightarrow v \in \widehat{V}$ is continuous across Γ_{gl}; $B_{OL}w \Leftrightarrow w \in V_{OL}$ is continuous across Γ_{gl}.

2 Algorithms

With the matrices defined in Sect. 1, we can consider the following algorithm for solving (1):

Algorithm: Active set method + Krylov subspace method

1. Initialize u^0. Set $k = 0$. Set $\mathscr{A}_k$, a subset of the index set $\{1, \cdots, \#(\text{rows}(\widetilde{B}))\}$ (resp. $\#(\text{rows}(\widehat{B}))$), according to the active set method of choice.
2. Solve

$$\min_{u \in \widetilde{V}} \frac{1}{2} u^T \widetilde{S} u - \widetilde{g}^T u, \quad \text{with} \quad Z^k \widetilde{B} u = 0 \tag{3}$$

$$\left(\text{resp.} \min_{u \in \widehat{V}} \frac{1}{2} u^T \widehat{S} u - \widehat{g}^T u, \quad \text{with} \quad \widehat{Z}^k \widehat{B} u = 0 \right) \tag{4}$$

approximately to a given precision, using a Krylov subspace method. Set u^{k+1} to the resulting approximate solution. Find $\mathscr{A}_{k+1}$ accordingly.
3. Set $k = k+1$. Stop if $\mathscr{A}_{k-1} = \mathscr{A}_k$; return to Step 2 otherwise.

Note that the linear problem in the kth iteration of the active set method is formulated as a minimization problem in terms of the interface variables in $\widetilde{V}$ or $\widehat{V}$. Here, $\widetilde{g} \in \widetilde{V}$ and $\widehat{g} \in \widehat{V}$ are appropriate load vectors. The square, diagonal matrix Z^k, with all elements equal to 0 or 1, is chosen such that $Z^k \widetilde{B} = \widetilde{B}_{\mathscr{A}_k}$, where $\widetilde{B}_{\mathscr{A}_k}$ is obtained by replacing the ith row of $\widetilde{B}$ with zeros for $\forall i \notin \mathscr{A}_k$. The matrix $\widehat{Z}^k$ is defined analogously. The minimization problems (3) and (4) are equivalent to the following saddle point problems,

$$\begin{bmatrix} \widetilde{S} & (Z^k \widetilde{B})^T \\ Z^k \widetilde{B} & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} \widetilde{g} \\ 0 \end{bmatrix}, \tag{5}$$

and

$$\begin{bmatrix} \widehat{S} & (\widehat{Z}^k \widehat{B})^T \\ \widehat{Z}^k \widehat{B} & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} \widehat{g} \\ 0 \end{bmatrix}, \tag{6}$$

respectively. We now consider the preconditioning of (5) and (6).
The **FETI-FETI** method is a combination of the one-level FETI method with a Dirichlet preconditioner [4] and the FETI-DP method [5], and was used in [1, 2] to solve frictionless contact problems. For (6), it is natural to follow the approach in the one-level and FETI-DP methods and form a Schur complement equation

$$\underbrace{Z^k \widetilde{B} \widetilde{S}^{\dagger} \widetilde{B}^T Z^k}_{:=F} \lambda = Z^k \widetilde{B} \widetilde{S}^{\dagger} \widetilde{g} + Z^k \widetilde{B} R \alpha, \tag{7}$$

where $\widetilde{S}^{\dagger}$ is a pseudoinverse of $\widetilde{S}$, $\text{range}(R) = \text{null}(\widetilde{S})$, and the vector α is to be determined. We solve (7) with the preconditioned conjugate gradient (PCG) method, using the following preconditioner:

$$P_F^{-1} := Z^k \widetilde{B}_D \widetilde{S} \widetilde{B}_D^T Z^k. \tag{8}$$

If $\widetilde{S}$ is singular, then the PCG method needs to be confined to the following subspace:

$$V^k := \{\lambda : Z^k \widetilde{B} \lambda \in \text{range}(\widetilde{S})\}. \tag{9}$$

Most of the computational work in each iteration of the PCG method goes into the applications of $\widetilde{S}^{\dagger}$ and $\widetilde{S}$, in the applications of F and P_F^{-1}, respectively. The application of $\widetilde{S}$ involves solving a Dirichlet problem on each subdomain, $\Omega_{i,j}, i=1,\cdots,N, j=1,\cdots,N_i$. The application of $\widetilde{S}^{\dagger}$ involves solving a Dirichlet problem in each subdomain, with the Dirichlet boundary condition imposed only at subdomain vertices, plus solving a coarse problem on each body, associated with the set of vertices of $\Gamma_{loc}^{(i)}, i=1,\cdots,N$; for details, see, e.g., [13],[14, Chap. 6].

The **hybrid** method is a combination of the one-level FETI method with a Dirichlet preconditioner and the BDDC (balancing domain decomposition by constraints) method [3]. For (6), forming a Schur complement equation similar to (7) is much more expensive because of the dense structure of $\widehat{S}$. Hence we keep the saddle point formulation (6) as is and solve it with the preconditioned conjugate residual (PCR) method. As in the FETI-FETI method, the PCR method needs to be confined to the following subspace:

$$\widehat{V}^k := \{\lambda : \widehat{Z}^k \widehat{B}\lambda \in \text{range}(\widehat{S})\}.$$

Letting P^k denote an orthogonal projection onto V^k, we rewrite (6) as

$$\underbrace{\begin{bmatrix} \widehat{S} & (P^k \widehat{Z}^k \widehat{B})^T \\ P^k \widehat{Z}^k \widehat{B} & 0 \end{bmatrix}}_{:=\mathscr{A}} \begin{bmatrix} u \\ \mu \end{bmatrix} = \begin{bmatrix} \widehat{g} - \widehat{B}^T \lambda_0 \\ 0 \end{bmatrix}, \tag{10}$$

with λ_0 satisfying $(\widehat{Z}^k \widehat{B}^T)\lambda_0 \in \text{range}(\widehat{S})$. For details on how to recover a solution of (6) from a solution of (10), see [8]. Letting P_R denote an orthogonal projection onto range$(\widehat{S})$, we introduce the preconditioner $\mathscr{B}$, where

$$\mathscr{B}^{-1} = \begin{bmatrix} P_R M_{BDDC}^{-1} P_R & 0 \\ 0 & P^k M_D^{-1} P^k \end{bmatrix}. \tag{11}$$

Here, M_{BDDC} is a block diagonal matrix consisting of the *BDDC* preconditioners [3] for the bodies:

$$M_{BDDC}^{-1} = \text{diag}_{i=1}^{N} M_{BDDC}^{(i)^{-1}} = \text{diag}_{i=1}^{N} \widetilde{R}_{D,\Gamma}^{(i)^T} \widetilde{S}_{\Gamma}^{(i)^{\dagger}} \widetilde{R}_{D,\Gamma}^{(i)},$$

where $\widetilde{R}_{D,\Gamma}^{(i)^T}, i=1,\cdots,N$, is a scaled restriction from $\widetilde{V}^{(i)}$ to $\widehat{V}^{(i)}$, with the scaling factors determined by the material coefficients; similarly, $B_{OL,D}$ is a scaled version of B_{OL}. For details on the definition of these matrices, see, for instance, [11, 13]. Then M_D can be viewed as a Dirichlet preconditioner of the one-level FETI method, obtained by viewing each body, Ω_i, as a subdomain:

$$M_D^{-1} = \widehat{Z}^k B_{OL,D} S_{OL} B_{OL,D}^T \widehat{Z}^{k^T}.$$

Most of the computational work in each iteration of the PCR method goes into the application of $\widehat{S}$, in the application of $\mathscr{A}$, and the application of $\widetilde{S}_{\Gamma}^{(i)^{\dagger}}, i=1,\cdots,N$ and S_{OL}, in the application of $\mathscr{B}^{-1}$. The application of $\widehat{S}$ requires solving a Dirichlet problem on each subdomain, $\Omega_{i,j}, i=1,\cdots,N, j=1,\cdots,N_i$. The application of

$\widetilde{S}_{\Gamma}^{(i)^{\dagger}}, i = 1, \cdots, N$, which is carried out in the FETI-FETI method as well, requires solving a Dirichlet problem on $\Omega_{i,j}, j = 1, \cdots, N_i$ with the Dirichlet boundary condition imposed only at the vertices, plus solving a coarse problem on Ω_i associated with the vertices of $\Gamma_{loc}^{(i)}$. The application of S_{OL}, however, requires solving a Dirichlet problem on each body, which is expensive; therefore in practice such a Dirichlet problem needs only to be solved inexactly, for instance with a Krylov subspace method. A preconditioner for solving such a Dirichlet problem is proposed and tested in [11].

3 Theory

We now present condition number estimates for the FETI-FETI and hybrid methods. Because of space limitations, details and proofs are given elsewhere; see [11, 12].

Theorem 1. *Let F, P_F, and V^k be defined as in (7) and (9), respectively. For any $\lambda \in V^k$, we have*

$$\langle P_F \lambda, \lambda \rangle \leq \langle F\lambda, \lambda \rangle \leq C(H_b/H_s)(1 + \log(H_s/h))^2 \langle P_F \lambda, \lambda \rangle,$$

where $C > 0$ is a constant independent of the sizes of the bodies, subdomains, and elements.

Convergence of the PCR method for the hybrid method is determined by

$$\mathscr{K}(\mathscr{B}^{-1}\mathscr{A}) := \frac{\mu_{max}}{\mu_{min}} = \frac{\max\{|\lambda| : \lambda \in \sigma(\mathscr{B}^{-1}\mathscr{A})\}}{\min\{|\lambda| : \lambda \in \sigma(\mathscr{B}^{-1}\mathscr{A})\}}, \qquad (12)$$

where $\sigma(\mathscr{B}^{-1}\mathscr{A})$ is the spectrum of $\mathscr{B}^{-1}\mathscr{A}$ on range$(P_R) \times \widehat{V}^k$.

Theorem 2. *Let $\mathscr{B}^{-1}, \mathscr{A}$, and $\mathscr{K}(\mathscr{B}^{-1}\mathscr{A})$ be defined as in (11)–(12), respectively. We then have the following bound:*

$$\mathscr{K}(\mathscr{B}^{-1}\mathscr{A}) \leq C(1 + \log(H_b/h))^2 (1 + \log(H_s/h))^2,$$

where $C > 0$ is a constant independent of the sizes of the bodies, subdomains, and elements.

4 Numerical Results: Auxiliary Linear Problems

We solve the following equality-constrained minimization problem:

$$\min \sum_{i=1}^{N_b \times N_b} \left(\frac{1}{2} \int_{\Omega_i} |\nabla u^i|^2 dx - \int_{\Omega_i} f u^i dx \right), \quad \text{with equality constraints to be specified,} \qquad (13)$$

Table 1. Results of FETI-FETI and hybrid.

			FETI-FETI				Hybrid	
			I		II		I	II
$1/H_b$	H_b/H_s	H_s/h	cond	iter	cond	iter	iter	iter
2	fixed	fixed	2.89	7	2.31	7	10	10
4	at 2	at 2	4.41	12	2.85	10	11	8
6			4.51	13	2.91	10	11	9
8			4.55	14	2.93	10	11	8
10			4.56	14	2.94	10	11	8
12			4.57	13	2.95	10	11	7
14			4.58	14	2.96	10	11	7
16			4.58	14	2.96	10	11	7
fixed	4	fixed	7.68	10	5.02	9	10	10
at 2	6	at 2	12.70	12	7.46	10	10	10
	8		17.80	13	8.12	10	10	10
	10		22.93	15	10.96	11	10	8
	12		28.08	16	13.43	12	10	8
	14		33.25	17	14.01	12	9	8
	16		38.41	17	16.90	12	8	7
fixed	fixed	4	4.71	9	4.73	9	12	11
at 2	at 2	6	5.90	10	6.37	10	13	13
		8	6.90	10	7.08	10	13	13
		10	7.79	11	8.27	11	14	14
		12	8.55	11	9.25	11	14	14
		14	9.23	12	9.71	12	14	14
		16	9.83	12	10.52	12	14	14

where $\Omega_i \subset \mathbb{R}^2, i = 1, \cdots, N_b \times N_b$ are square bodies with side length $H_b := 1/N_b$, which collectively form the domain $\bar{\Omega} = \bigcup_{i=1}^{N_b \times N_b} \bar{\Omega}_i = [0,1] \times [0,1]$. We require $u^i \in H^1(\Omega_i), u^i|_{\partial\Omega_i \cap \partial\Omega} = 0$. Each Ω_i is decomposed into $N_s \times N_s$ square subdomains, each of which is discretized by square bilinear elements of side length h. Also, $\Gamma := \cup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j$ denotes the interface between the bodies.

We supplement (13) with two different equality constraints, associated with different *contact areas* between the bodies. In the first problem, the entire Γ is considered as the contact area, that is, we require the continuity of the displacement vector across the entire Γ. This case has already been considered by Klawonn and Rheinbach [6] and Klawonn and Rheinbach [7]. In the second problem, continuity is imposed only on the middle third of the faces between the bodies. We solve these problems with both the FETI-FETI and hybrid methods. The PCG and PCR iterations are stopped when the norm of the residual has been reduced by a factor of 10^{-6}.

The results are shown in Table 1. We have three parameters to vary: the number of bodies across Ω ($N_b = 1/H_b$), the number of subdomains across each body

($N_s = H_b/H_s$), and the number of elements across each subdomain (H_s/h). We vary one parameter while keeping the other two fixed. The results for the first set of experiments, with the entire Γ as the contact surface, are shown in column I; those for the second set of experiments with a reduced contact area are shown in column II.

Note the linear dependence of the condition number on the number of subdomains across each body, H_b/H_s, for the FETI-FETI method, which confirms our theoretical finding. Note also that the iteration counts of the hybrid method do not increase as the number of subdomains is increased. Similar numerical results for the FETI-FETI method have been obtained independently by Klawonn and Rheinbach [6] and Klawonn and Rheinbach [7].

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