# A Neumann-Dirichlet Preconditioner for FETI-DP Method for Mortar Discretization of a Fourth Order Problems in 2D

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## 1 Introduction

This study focuses on a construction of a parallel preconditioner for a FETI-DP (dual primal Finite Element Tearing and Interconnecting) method for a mortar Hsieh-Clough-Tocher (HCT) discretization of a model fourth order problem with discontinuous coefficients.

FETI-DP methods were introduced in [8]. They form a class of fast and efficient iterative solvers for algebraic systems of equations arising from the finite element discretizations of elliptic partial differential equations of second and fourth order, cf. [8, 10, 11, 16] and references therein. In a one-level FETI-DP method one has to solve a linear system for a set of dual variables formulated by eliminating all primal unknowns. The FETI-DP system contains in itself a coarse problem, while the preconditioner is usually fully parallel and constructed only from local problems.

There are many works investigating iterative solvers for mortar method for second order problem, e.g. cf. [1–3] and references therein. There have also been a few FETI-DP type algorithms developed for mortar discretization of second order problems, cf. e.g. [6, 7, 9]. But there is only a small number of studies focused on fast solvers for mortar discretizations of fourth order elliptic problems, cf. [12, 15, 17]. In this study we follow the approach of [9] which considers the case of a FETI-DP method for mortar discretization of a second order problem.

In this paper we first present the construction of mortar discretization of a fourth order elliptic problem which locally utilizes Hsieh-Clough-Tocher finite elements in the subdomains. Next we introduce a FETI-DP problem and then a Neumann-Dirichlet parallel preconditioner for a FETI-DP problem is proposed. Finally, we present the almost optimal bounds of the condition number, namely, a bound which grows like $C(1+\log(H/\underline{h}))^2$, where $H$ is the maximal diameter of subdomains and $\underline{h}$ is a fine mesh parameter.

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## 2 Discrete Problem

In this section we focus on a mortar Hsieh-Clough-Tocher (HCT) finite element discretization for a model fourth order elliptic problem with discontinuous coefficients.

Let $\Omega$ be a polygonal domain in the plane. We assume that there exists a partition of $\Omega$ into disjoint polygonal subdomains $\Omega_k$ such that $\overline{\Omega} = \bigcup_{k=1}^{N} \overline{\Omega}_k$ with $\overline{\Omega}_k \cap \overline{\Omega}_l$ being an empty set, an edge or a vertex (crosspoint). We also assume that these subdomains form a coarse triangulation of the domain which is shape regular in the sense of [5]. We introduce a global interface $\Gamma = \bigcup_i \overline{\partial\Omega_i \setminus \partial\Omega}$ which plays an important role in our study.

Our model differential problem is to find $u^* \in H_0^2(\Omega)$ such that

$$a(u^*, v) = \int_\Omega f v \, dx \quad \forall v \in H_0^2(\Omega), \tag{1}$$

where $f \in L^2(\Omega)$, $H_0^2(\Omega) = \{u \in H^2(\Omega) : u = \partial_n u = 0 \quad \text{on} \quad \partial\Omega\}$ and $a(u,v) = \sum_{k=1}^{N} \int_{\Omega_k} \rho_k [u_{x_1x_1} v_{x_1x_1} + 2\, u_{x_1x_2} v_{x_1x_2} + u_{x_2x_2} v_{x_2x_2}]\, dx$. The coefficients $\rho_k$ are positive and constant. Here $u_{x_kx_l} := \frac{\partial^2 u}{\partial x_k \partial x_l}$ for $k,l = 1,2$ and $\partial_n u$ is a unit normal derivative of $u$.

In each subdomain $\Omega_k$ we introduce a quasiuniform triangulation $T_h(\Omega_k)$ made of triangles with the parameter $h_k = \max_{\tau \in T_h(\Omega_k)} \operatorname{diam}(\tau)$, cf. e.g. [4]. We can now

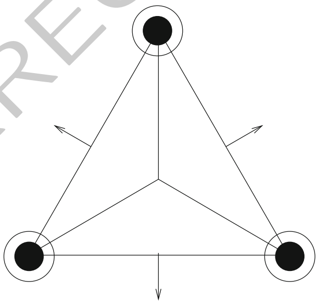

**Fig. 1.** Degrees of freedom of HCT element

introduce local finite element spaces. Let $X_h(\Omega_k)$ be the Hsieh-Clough-Tocher (HCT) macro finite element space defined as follows:

$$\begin{aligned} X_h(\Omega_k) = \{u \in C^1(\Omega_k) : u \in P_3(\tau_i),\ \tau_i \in T_h(\Omega_k),\ &\text{for the subtriangles } \tau_i, \\ &i = 1,2,3, \text{ formed by connecting the vertices of} \\ &\text{any } \tau \in T_h(\Omega_k) \text{ to its centroid, and} \\ &u = \partial_n u = 0 \text{ on } \partial\Omega_k \cap \partial\Omega\}, \end{aligned}$$

where $P_3(\tau_i)$ is the function space of cubic polynomials defined over $\tau_i$. The degrees of freedom of a function $u \in X_h(\Omega_k)$ over $\tau \in T_h(\Omega_k)$ are defined as: $\{u(p_k), \nabla u(p_k), \partial_n u(m_j)\}_{k,j=1,2,3}$, where $p_k$ is a vertex and $m_j$ is a midpoint of an edge of $\tau$, cf. Fig. 1.

Next a global space $X_h(\Omega)$ is defined as $X_h(\Omega) = \prod_{i=1}^{N} X_h(\Omega_k)$. We also introduce $\widetilde{X}_h(\Omega)$ – a subspace of $X_h(\Omega)$ formed by all functions in $X_h(\Omega)$, which has all degrees of freedom continuous at the crosspoints, i.e. the common vertices of substructures.

Let $\Gamma_{kl}$ denote the interface between two subdomains $\Omega_k$ and $\Omega_l$ i.e. the open edge that is common to these subdomains. Note that each interface $\Gamma_{kl}$ inherits two one dimensional triangulations made of segments that are edges of elements of $T_h(\Omega_k)$ and $T_h(\Omega_l)$, respectively. Thus there are two independent 1D triangulations on $\Gamma_{kl}$: $T_{h,k}(\Gamma_{kl})$ related to $\Omega_k$ and another one associated with $\Omega_l$ - $T_{h,l}(\Gamma_{lk})$, cf. Fig. 2. Let $\gamma_{kl}$ be a mortar, i.e. the side corresponding to $\Omega_k$ if $\rho_k \geq \rho_l$ and then let $\delta_{lk}$ be the other side of $\Gamma_{lk}$ associated to $\Omega_l$ called a slave (nonmortar).

For each interface $\Gamma_{kl}$ we introduce two test spaces associated with its slave triangulation $T_{h,l}(\delta_{lk})$ (cf. [13, 14]): let $M_t^h(\delta_{lk})$ be the space formed by $C^1$ smooth piecewise cubic functions on the slave triangulation of $\delta_{lk}$, which are piecewise linear in the two end elements, and let $M_n^h(\delta_{lk})$ be the space of continuous piecewise quadratic functions on the elements of this triangulation, which are piecewise linear in the two end elements.

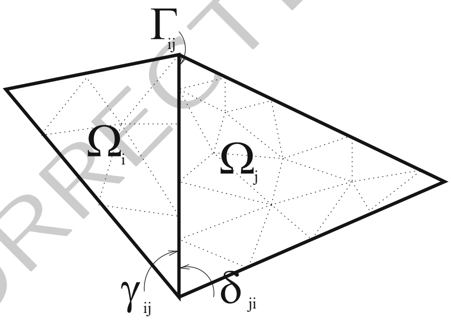


**Fig. 2.** Independent meshes on an interface $\Gamma_{ij}$

We also define a space $M = \prod_{\delta_{lk} \subset \Gamma} M_{lk}$ with $M_{lk} = M_t^l(\delta_{lk}) \times M_n^l(\delta_{lk})$ and a bilinear form $b(u, \psi)$: let $u = (u_k)_{k=1}^N \in \widetilde{X}_h(\Omega)$ and $\psi = (\psi_{lk})_{\delta_{lk}} = (\psi_{lk,t}, \psi_{lk,n})_{\delta_{lk}} \in M$, then $b(u, \psi) = \sum_{\delta_{lk} \subset \Gamma} \sum_{s \in \{t,n\}} b_{lk,s}(u, \psi_{lk,s})$ with

$$b_{lk,t}(u, \psi_{lk,t}) = \int_{\delta_{lk}} (u_k - u_l)\psi_{lk,t}\, ds,$$

$$b_{lk,n}(u, \psi_{lk,n}) = \int_{\delta_{lk}} (\partial_n u_k - \partial_n u_l)\psi_{lk,n}\, ds.$$

Further we will use the same notation for a function and for the vector with the values of degrees of freedom of this function.

We introduce discrete problem as the saddle point problem: find a pair $(u_h^*, \lambda^*) \in \widetilde{X}_h(\Omega) \times M$ such that

$$a(u_h^*, v) + b(v, \lambda^*) = f(v) \quad \forall v \in \widetilde{X}_h(\Omega), \tag{2}$$

$$b(u_h^*, \phi) = 0 \qquad \forall \phi \in M, \tag{3}$$

where $a_h(u,v) = \sum_{k=1}^{N} a_k(u,v)$ for

$$a_k(u,v) = \int_{\Omega_k} \rho_k [u_{x_1x_1} v_{x_1x_1} + 2\, u_{x_1x_2} v_{x_1x_2} + u_{x_2x_2} v_{x_2x_2}]\, dx.$$

This problem has a unique solution and error bounds are established, e.g. cf. [14].

## 3 Matrix Form of Mortar Conditions

Note that (3) is equivalent to two mortar conditions on each $\delta_{lk} = \gamma_{kl} = \Gamma_{kl}$:

$$\int_{\delta_{lk}} (u_k - u_l)\phi \, ds = 0 \quad \forall \phi \in M_t^l(\delta_{lk}), \tag{4}$$

$$\int_{\delta_{lk}} (\partial_n u_k - \partial_n u_l)\psi \, ds = 0 \quad \forall \psi \in M_n^l(\delta_{lk}). \tag{5}$$

We introduce the following splitting of two vectors representing the tangential and normal traces $u_{\delta_{lk}}$ and $\partial_n u_{\delta_{lk}}$: $u_{\delta_{lk}} = u^{(r)}_{\delta_{lk}} + u^{(c)}_{\delta_{lk}}$ and $\partial_n u_{\delta_{lk}} = \partial_n u^{(r)}_{\delta_{lk}} + \partial_n u^{(c)}_{\delta_{lk}}$ on a slave $\delta_{lk} \subset \partial\Omega_l$, where superscript $(c)$ refers to degrees of freedom related to cross-points (ends of this edge) and superscript $(r)$ refers to degrees of freedom related to remaining nodes (vertices and midpoints) on this edge. We can now rewrite (4) and (5) in a matrix form on each interface $\Gamma_{kl} \subset \Gamma$:

$$\begin{aligned} B^{(c)}_{t,\delta_{lk}} u^{(c)}_{\delta_{lk}} + B^{(r)}_{t,\delta_{lk}} u^{(r)}_{\delta_{lk}} &= B^{(c)}_{t,\gamma_{kl}} u^{(c)}_{\gamma_{kl}} + B^{(r)}_{t,\gamma_{kl}} u^{(r)}_{\gamma_{kl}}, \\ B^{(c)}_{n,\delta_{lk}} \partial_n u^{(c)}_{\delta_{lk}} + B^{(r)}_{n,\delta_{lk}} \partial_n u^{(r)}_{\delta_{lk}} &= B^{(c)}_{n,\gamma_{kl}} \partial_n u^{(c)}_{\gamma_{kl}} + B^{(r)}_{n,\gamma_{kl}} \partial_n u^{(r)}_{\gamma_{kl}}, \end{aligned} \tag{6}$$

where the matrices $B_{t,\delta_{lk}} = [B^{(c)}_{t,\delta_{lk}},\ B^{(r)}_{t,\delta_{lk}}]$ and $B_{n,\delta_{lk}} = [B^{(c)}_{n,\delta_{lk}}, B^{(r)}_{n,\delta_{lk}}]$ are mass matrices obtained by substituting the traces of standard nodal basis functions of $X_h(\Omega_l)$ and nodal basis functions of $M_t^l(\delta_{lk}), M_n^l(\delta_{lk})$, respectively, into (4). The matrices $B_{t,\gamma_{kl}} = [B^{(c)}_{t,\gamma_{kl}},\ B^{(r)}_{t,\gamma_{kl}}]$ and $B_{n,\gamma_{kl}} = [B^{(c)}_{n,\gamma_{kl}}, B^{(r)}_{n,\gamma_{kl}}]$ are constructed analogously but utilizing traces onto $\gamma_{kl}$ of standard nodal basis functions of $X_h(\Omega_k)$. Note that $B^{(r)}_{t,\delta_{lk}}, B^{(r)}_{n,\delta_{lk}}$ are positive definite square matrices, but that all other matrices in (6) are rectangular in general.

## 4 FETI-DP Problem

Let $K_l$ be a matrix of $a_l(\cdot,\cdot)$ in the standard basis of $X_h(\Omega_l)$. Then let $\tilde{K}$ be the matrix obtained from a block diagonal matrix $K := \mathrm{diag}(K_l)_{l=1}^N$ by taking into account the continuity of the degrees of freedom at crosspoints. We can partition $\tilde{K}$ into

$$\tilde{K} = \begin{pmatrix} K_{ii} & K_{ic} & K_{ir} \\ K_{ci} & K_{cc} & K_{cr} \\ K_{ri} & K_{rc} & K_{rr} \end{pmatrix},$$

where the superscript $(i)$ refer to the degrees of freedom associated with nodal points interior to subdomain, $(c)$ to the degrees of freedom related to crosspoints, and $(r)$ to the degrees of freedom associated the remaining nodes on masters and slaves. Then the matrix formulation of (2) and (3) is the following:

$$\begin{pmatrix} K_{ii} & K_{ic} & K_{ir} & 0 \\ K_{ci} & K_{cc} & K_{cr} & (B^{(c)})^T \\ K_{ri} & K_{rc} & K_{rr} & (B^{(r)})^T \\ 0 & B^{(c)} & B^{(r)} & 0 \end{pmatrix} \begin{pmatrix} u^{(i)} \\ u^{(c)} \\ u^{(r)} \\ \lambda^* \end{pmatrix} = \begin{pmatrix} f_i \\ f_c \\ f_r \\ 0 \end{pmatrix}. \tag{7}$$

Here $B^{(c)}$ is the matrix built from $B^{(c)}_{t,\delta_{lk}}, B^{(c)}_{n,\delta_{lk}}, B^{(c)}_{t,\gamma_{kl}}, B^{(c)}_{n,\gamma_{kl}}$ for all $\Gamma_{kl} = \gamma_{kl} = \delta_{lk} \subset \Gamma$ and $B^{(r)} := \mathrm{diag}([-B^{(r)}_{\gamma_{kl}}, B^{(r)}_{\delta_{lk}}])_{\Gamma_{kl} \subset \Gamma}$ is the block diagonal matrix with

$$B^{(r)}_{\gamma_{kl}} := \begin{pmatrix} B^{(r)}_{t,\gamma_{kl}} & 0 \\ 0 & B^{(r)}_{n,\gamma_{kl}} \end{pmatrix}, \qquad B^{(r)}_{\delta_{lk}} := \begin{pmatrix} B^{(r)}_{t,\delta_{lk}} & 0 \\ 0 & B^{(r)}_{n,\delta_{lk}} \end{pmatrix}. \tag{8}$$

Next we eliminate the unknowns related to the interior nodes and crosspoints i.e. $u^{(i)}$, $u^{(c)}$ in (7) and we get

$$\begin{aligned} \tilde{S}u^{(r)} + \tilde{B}^T \lambda^* &= \tilde{f}_r, \\ \tilde{B}u^{(r)} + \tilde{S}_{cc}\lambda^* &= \tilde{f}_c, \end{aligned} \tag{9}$$

where the respective matrices are defined as follows:

$$\tilde{S} := K_{rr} - (K_{ri}\, K_{rc})(\tilde{K}^{(ic)})^{-1} \begin{pmatrix} K_{ir} \\ K_{cr} \end{pmatrix},$$

$$\tilde{B} := B^{(r)} - (0\, B^{(c)})(\tilde{K}^{(ic)})^{-1} \begin{pmatrix} K_{ir} \\ K_{cr} \end{pmatrix},$$

and $\tilde{S}_{cc} := -(0\ B^{(c)})(\tilde{K}^{(ic)})^{-1} \begin{pmatrix} 0 \\ (B^{(c)})^T \end{pmatrix}$ with the nonsingular matrix $\tilde{K}^{(ic)} := \begin{pmatrix} K_{ii} & K_{ic} \\ K_{ci} & K_{cc} \end{pmatrix}$.

Eliminating $u^{(r)}$ we obtain the following FETI-DP problem: find $\lambda^* \in M$ such that

$$F(\lambda^*) = d, \tag{10}$$

where $d := \tilde{f}_c - \tilde{B}\tilde{S}^{-1}\tilde{f}_r$ and $F := \tilde{S}_{cc} - \tilde{B}\tilde{S}^{-1}\tilde{B}^T$.

## 5 Parallel Preconditioner

Let $W_r = \{w^{(r)} : w \in \widetilde{X}_h(\Omega)\}$ i.e. $W_r$ is the space of vectors representing all degrees of freedom of functions from $\widetilde{X}_h(\Omega)$ associated with nodes (vertices and midpoints) on $\Gamma$ but are *not* associated with crosspoints.

We can decompose any vector $w^{(r)} \in W_r$ into vectors related to masters and slaves:

$$w^{(r)} = \left(w_{\Gamma}^{(r)}, w_{\Delta}^{(r)}\right)^T,$$

where $w_{\Gamma}^{(r)}$ is the vector with the values of degrees of freedom which are associated with the nodes on the masters and $w_{\Delta}^{(r)}$ is the vector with the values of degrees of freedom which are related to the nodes on the slaves. We then introduce $W_\Delta = \{w_{\Delta}^{(r)} : w^{(r)} \in W_r\}$ i.e. the space formed by vectors in $W_r$ which have only entries related to the degrees of freedom which are associated with the nodes on the slaves. It is very important to note that

$$\dim M = \dim W_\Delta.$$

Let $S_\Delta$ be the matrix obtained by restricting $\tilde{S} : W_r \to W_r$ to $W_\Delta$.

Note that this matrix is can be represented as a block diagonal matrix with non-singular diagonal blocks $S_{k,\Delta}$, i.e.

$$S_\Delta := \operatorname{diag}(S_{k,\Delta})_k,$$

where the subscript $k$ runs over all subdomains that have at least one edge on $\Gamma$ as a slave. Naturally, we could also partitioned this matrix with respect to the slaves.

Define nonsingular block diagonal matrix $B_\Delta : W_\Delta \to W_\Delta$:

$$B_\Delta := \operatorname{diag}(B_{\delta_{lk}}^{(r)})_{\delta_{lk} \subset \Gamma},$$

where $B_{\delta_{lk}}^{(r)}$ are block diagonal matrices (with two nonsingular blocks) defined in (8).

Then we introduce our parallel preconditioner:

$$\mathscr{M}_{DN}^{-1} := B_\Delta^{-T} S_\Delta B_\Delta^{-1},$$

which is nonsingular, or equivalently its inverse: $\mathscr{M}_{DN} := B_\Delta S_\Delta^{-1} B_\Delta^T$. Note that $S_\Delta$ and thus $\mathscr{M}_{DN}$ are dependent on the discontinuous coefficients $\rho_k$.



# 6 Condition Number Bounds

The main result of this paper is the following theorem which yields the bound of the condition number of preconditioned problem:

**Theorem 1.** *It holds that*

$$\langle \mathscr{M}_{DN}\lambda, \lambda\rangle \le \langle F\lambda, \lambda\rangle \le C\left(1 + \log\left(\frac{H}{\underline{h}}\right)\right)^2 \langle \mathscr{M}_{DN}\lambda, \lambda\rangle \qquad \forall \lambda \in M,$$

*where $H = \max_k h_k$, $\underline{h} = \min_k h_k$, and $C$ a positive constant independent of the coefficients, or the parameters $H_k$ and $h_k$. Here $\langle \cdot, \cdot \rangle$ is the standard $l_2$ inner product.*

As a direct consequence of this theorem we see that the condition number of $\mathscr{M}_{DN}^{-1}F$ is bounded by $C\left(1+\log\left(\frac{H}{\underline{h}}\right)\right)^2$.

The lower bound in the theorem is obtained by purely algebraic arguments. And we get the upper bound by using several technical results of which the most important one is the estimate of special trace norms of jumps of tangential and normal traces over an interface $\Gamma_{kl} \subset \Gamma$.

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