A DG Space–Time Domain Decomposition Method

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Summary. In this paper we present a hybrid domain decomposition approach for the parallel solution of linear systems arising from a discontinuous Galerkin (DG) finite element approximation of initial boundary value problems. This approach allows a general decomposition of the space–time cylinder into finite elements, and is therefore applicable for adaptive refinements in space and time.

1 A Space–Time DG Finite Element Method

As a model problem we consider the transient heat equation

$$\partial_t u(x,t) - \Delta u(x,t) = f(x,t) \quad \text{for } (x,t) \in Q := \Omega \times (0,T), \tag{1}$$

$$u(x,t) = 0 \quad \text{for } (x,t) \in \Sigma := \partial\Omega \times (0,T), \tag{2}$$

$$u(x,0) = u_0(x) \quad \text{for } (x,t) \in \Omega \times \{0\} \tag{3}$$

where $\Omega \subset \mathbb{R}^n, n = 1,2,3$, is a bounded Lipschitz domain, and $T > 0$. Let $\mathscr{T}_N$ be a decomposition of the space–time cylinder $Q = \Omega \times (0,T) \subset \mathbb{R}^{n+1}$ into simplices τ_k of mesh size h. For simplicity we assume that the space time cylinder Q has a polygonal ($n = 1$), a polyhedral ($n = 2$), or a polychoral ($n = 3$) boundary ∂Q. With $\mathscr{I}_N$ we denote the set of all interfaces (interior facets) e between two neighboring elements τ_k and τ_ℓ. For an admissible decomposition the interior facets are edges ($n = 1$), triangles ($n = 2$), or tetrahedrons ($n = 3$).

With respect to an interior facet $e \in \mathscr{I}_N$ we define for a function v the jump

$$[v]_e(x,t) := v_{|\tau_k}(x,t) - v_{|\tau_\ell}(x,t) \quad \text{for all } (x,t) \in e,$$

the average

$$\langle v \rangle_e(x,t) := \frac{1}{2}\left[v_{|\tau_k}(x,t) + v_{|\tau_\ell}(x,t)\right] \quad \text{for all } (x,t) \in e,$$

and the upwind in time direction by

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$$\{v\}_e^{\mathrm{up}}(x,t) := \begin{cases} v_{|\tau_k}(x,t) & \text{for } n_t \geq 0, \\ v_{|\tau_\ell}(x,t) & \text{for } n_t < 0 \end{cases} \quad \text{for all } (x,t) \in e,$$

where $\mathbf{n} = (\mathbf{n}_x, n_t)$ is the normal vector of the interior facet e.

For a decomposition $\mathscr{T}_N$ of the space–time cylinder Q we introduce the discrete function space of piecewise polynomials of order p

$$S_{h,0}^p(\mathscr{T}_N) := \left\{ v : v_{|\tau_k} \in \mathbb{P}_p(\tau_k) \text{ for all } \tau_k \in \mathscr{T}_N \text{ and } v_{|\Sigma} = 0 \right\}.$$

The proposed space–time approach is based on the use of an interior penalty Galerkin approximation of the Laplace operator and an upwind scheme for the approximation of the time derivative, see, e.g., [3, 5]. Hence we have to find $u_h \in S_{h,0}^p(\mathscr{T}_N)$ such that

$$\begin{aligned} a_{\mathrm{DG}}(u_h, v_h) :=& -\sum_{k=1}^N \int_{\tau_k} u_h \, \partial_t v_h \, dxdt + \int_{\Sigma_T} u_h \, v_h \, dx \\ &+ \sum_{e \in \mathscr{I}_N} \int_e n_t \, \{u_h\}_e^{\mathrm{up}} [v_h]_e \, ds_{(x,t)} + \sum_{k=1}^N \int_{\tau_k} \nabla_x u_h \cdot \nabla_x v_h \, dxdt \\ &- \sum_{e \in \mathscr{I}_N} \int_e \left[\langle \mathbf{n}_x \cdot \nabla_x u_h \rangle_e \, [v_h]_e - \varepsilon \, [u_h]_e \langle \mathbf{n}_x \cdot \nabla_x v_h \rangle_e \right] ds_{(x,t)} \\ &+ \frac{\sigma}{h} \sum_{e \in \mathscr{I}_N} \int_e |\mathbf{n}_x|^2 \, [u_h]_e \, [v_h]_e \, ds_{(x,t)} \\ =& \int_Q f \, v_h \, dxdt + \int_{\Sigma_0} u_0 \, v_h \, dx =: F(v_h) \end{aligned} \tag{4}$$

is satisfied for all $v_h \in S_{h,0}^p(\mathscr{T}_N)$. The parameters σ and ε have to be chosen appropriately. For $v_h \in S_{h,0}^p(\mathscr{T}_N)$ and $\sigma > 0$ the related energy norm is given by

$$\|v_h\|_{\mathrm{DG}}^2 := \|v_h\|_A^2 + \|v_h\|_B^2,$$

where

$$\|v_h\|_A^2 := \sum_{k=1}^N \|\nabla_x v_h\|_{\tau_k}^2 + \frac{\sigma}{h} \sum_{e \in \mathscr{I}_N} \| \, |\mathbf{n}_x| \, [v_h]_e \|_{L_2(e)}^2,$$

$$\|v_h\|_B^2 := h \sum_{k=1}^N \|\partial_t v_h\|_{\tau_k}^2 + \frac{1}{2} \|v_h\|_{L_2(\Sigma_0 \cup \Sigma_T)}^2 + \frac{1}{2} \sum_{e \in \mathscr{I}_N} \| \sqrt{|n_t|} \, [v_h]_e \|_{L_2(e)}^2.$$

The unique solvability of the variational formulation (4) is based on the following stability result.

Lemma 1. *Let $\varepsilon \in \{-1,0,1\}$ and $\sigma > 0$. For $\varepsilon \in \{-1,0\}$ let σ be sufficient large. Then the stability estimate*

$$\sup_{0 \neq v_h \in S_{h,0}^p(\mathscr{T}_N)} \frac{a_{\mathrm{DG}}(u_h, v_h)}{\|v_h\|_{\mathrm{DG}}} \geq c_1^A \|u_h\|_{\mathrm{DG}} \quad \textit{for all } u_h \in S_{h,0}^p(\mathscr{T}_N)$$

is satisfied where the constant c_1^A depends on the shape of the finite elements, and on the stabilization parameter σ. However, for a sufficient large choice of σ we can ensure $c_1^A = \frac{1}{2}$.

Proof. The proof follows as in [5], by using the technique as in [2]; see also [3]. □

By using standard arguments we can then conclude the energy error estimate

$$\|u - u_h\|_{\mathrm{DG}} \leq c h^{\min\{s,p+1\}-1} |u|_{H^s(Q)}$$

when assuming $u \in H^s(Q)$ for some $s \leq p+1$, and, by applying the Aubin–Nitsche trick, for $\varepsilon = -1$,

$$||u - u_h||_{L_2(\Omega)} \leq c h^{\min\{s,p+1\}} |u|_{H^s(Q)} \,. \tag{5}$$

To illustrate the proposed DG finite element method in space and time as well as the given error estimates we consider a first numerical example for the initial boundary value problem (1)–(3) for $n = 1$ and $\Omega = (0,1)$, $T = 1$. This implies $Q = (0,1)^2$. The given data f and u_0 are chosen such that the solution is given as

$$u(x,t) = \sin(\pi x)(1-t)^{3/4} \in H^{1.25-\bar{\varepsilon}}(Q) \quad \text{with } \bar{\varepsilon} > 0.$$

Starting from a triangulation of $Q = (0,1)^2$ into four triangles we consider a sequence of several uniform refinement steps to analyze the convergence behavior of the presented method. Using piecewise linear basis functions, i.e. $p = 1$, $\varepsilon = -1$ and $\sigma = 10$, the numerical results are given in Table 1 which confirm the convergence rate of 1.25 as predicted by the error estimate (5).

level	elements	dof	$\|\|u - u_h\|\|_{L_2(Q)}$	eoc
0	4	8	2.2679 – 1	–
1	16	40	5.1354 – 2	2.14
2	64	176	1.3107 – 2	1.97
3	256	736	3.4813 – 3	1.91
4	1024	3008	9.7383 – 4	1.84
5	4096	12160	3.0406 – 4	1.68
6	16384	48896	1.0923 – 4	1.48
7	65536	196096	4.3315 – 5	1.33
8	262144	785408	1.7935 – 5	1.27
9	1048576	3143680	7.5278 – 6	1.25
10	4194304	12578816	3.1694 – 6	1.25
11	16777216	50323456	1.3345 – 6	1.25

Table 1. Numerical results for $p = 1$, $\varepsilon = -1$ and $\sigma = 10$.

2 A Hybrid Space-Time Domain Decomposition Method

The presented space–time method (4) results in a large linear system of algebraic equations. For its iterative solution we introduce a hybrid formulation as in [1, 2]. Therefore we subdivide the space–time domain Q into P non–overlapping subdomains Q_i, $i = 1, \ldots, P$,

$$\overline{Q} = \bigcup_{i=1}^{P} \overline{Q}_i, \quad Q_i \cap Q_j = \emptyset \quad \text{for } i \neq j.$$

By

$$\Gamma := \bigcup_{i=1}^{P} \Gamma_i \quad \text{with } \Gamma_i := \overline{\partial Q_i \setminus \partial Q}$$

we denote the interface of the space–time domain decomposition, see Fig. 1.

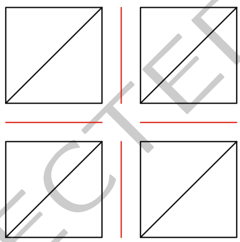

Fig. 1. Space–time decomposition of Q and the interface Γ

With respect to the interface Γ we introduce the discrete function space of piecewise polynomials of order p,

$$S_h^p(\Gamma) := \left\{ v \in L_2(\Gamma) : v_{|e} \in \mathbb{P}_p(e) \text{ for all } e \in \mathscr{I}_N \text{ with } e \subseteq \Gamma \right\}.$$

For the solution of the local partial differential equations in all subdomains Q_i we apply the space–time method as described by the variational formulation (4). For this we denote by $a_{\mathrm{DG}}^{(i)}(\cdot,\cdot)$ the restriction of the bilinear form $a_{\mathrm{DG}}(\cdot,\cdot)$ on the subdomain Q_i, $i = 1, \ldots, P$, i.e.

$$\begin{aligned} a_{\mathrm{DG}}^{(i)}(u_h, v_h) :=& -\sum_{k=1}^{N} \int_{\tau_k \cap Q_i} u_h \, \partial_t v_h \, dx dt + \int_{\Sigma_T \cap \partial Q_i} u_h \, v_h \, dx \\ &+ \sum_{e \in \mathscr{I}_N} \int_{e \cap Q_i} n_t \, \{u_h\}_e^{\mathrm{up}} \, [v_h]_e \, ds_{(x,t)} + \sum_{k=1}^{N} \int_{\tau_k \cap Q_i} \nabla_x u_h \cdot \nabla_x v_h \, dx dt \\ &- \sum_{e \in \mathscr{I}_N} \int_{e \cap Q_i} [\langle \mathbf{n}_x \cdot \nabla_x u_h \rangle_e \, [v_h]_e - \varepsilon \, [u_h]_e \, \langle \mathbf{n}_x \cdot \nabla_x v_h \rangle_e] \, ds_{(x,t)} \\ &+ \frac{\sigma}{h} \sum_{e \in \mathscr{I}_N} \int_{e \cap Q_i} |\mathbf{n}_x|^2 \, [u_h]_e [v_h]_e ds_{(x,t)}. \end{aligned}$$

Accordingly, the restriction of the linear form $F(\cdot)$ on a subdomain Q_i is given by

$$F^{(i)}(v_h) := \int_{Q_i} f\, v_h\, dx dt + \int_{\Sigma_0 \cap \partial Q_i} u_0\, v_h\, dx.$$

For the coupling of the local fields we first introduce a new unknown on the interface,

$$\lambda := \langle u \rangle_e = \frac{1}{2}\left[u_{|\tau_k} + u_{|\tau_\ell}\right] \quad \text{on } \Gamma \cap e.$$

With this we can rewrite the jump of a function as

$$[u]_e = u_{|\tau_k} - u_{|\tau_\ell} = 2\left(u_{|\tau_k} - \lambda\right) = 2\left(\lambda - u_{|\tau_\ell}\right) \quad \text{on } \Gamma \cap e.$$

Therefore we obtain for the coupling terms related to the Laplace operator

$$\sum_{e \in \mathscr{I}_N} \int_{e \cap \Gamma} \langle \mathbf{n}_x \cdot \nabla_x u \rangle_e\, [v]_e\, ds_{(x,t)} = \sum_{k=1}^{N} \int_{\partial\tau_k \cap \Gamma} \mathbf{n}_{k,x} \cdot \nabla_x u\, (v - \mu)\, ds_{(x,t)},$$

$$\sum_{e \in \mathscr{I}_N} \int_{e \cap \Gamma} [u]_e\, \langle \mathbf{n}_x \cdot \nabla_x v \rangle_e\, ds_{(x,t)} = \sum_{k=1}^{N} \int_{\partial\tau_k \cap \Gamma} (u - \lambda)\, \mathbf{n}_{k,x} \cdot \nabla_x v\, ds_{(x,t)},$$

$$\sum_{e \in \mathscr{I}_N} \int_{e \cap \Gamma} |\mathbf{n}_x|^2\, [u]_e\, [v]_e\, ds_{(x,t)} = 2 \sum_{k=1}^{N} \int_{\partial\tau_k \cap \Gamma} |\mathbf{n}_{k,x}|^2\, (u - \lambda)(v - \mu)\, ds_{(x,t)}.$$

For the classical solution u of the transient heat equation (1)–(3) there obviously holds for an interior facet $e \in \mathscr{I}_N$

$$\lambda = \langle u \rangle_e = \frac{1}{2}\left[u_{|\tau_k} + u_{|\tau_\ell}\right] = u_{|\tau_k} = u_{|\tau_\ell} \quad \text{on } e.$$

Therefore the upwind in time can be written as

$$\{u\}_e^{\mathrm{up}} = \begin{cases} u_{|\tau_k} & \text{for } n_t \ge 0, \\ u_{|\tau_\ell} & \text{for } n_t < 0 \end{cases} = \begin{cases} u_{|\tau_k} & \text{for } n_{k,t} \ge 0, \\ \lambda & \text{for } n_{k,t} < 0 \end{cases} =: \{u/\lambda\}_{\partial\tau_k}^{\mathrm{up}} \quad \text{on } \Gamma \cap e.$$

The coupling containing the upwind part can now be expressed by

$$\sum_{e \in \mathscr{I}_N} \int_{e \cap \Gamma} n_t\, \{u\}_e^{\mathrm{up}}\, [v]_e\, ds_{(x,t)} = \sum_{k=1}^{N} \int_{\partial\tau_k \cap \Gamma} n_{k,t}\, \{u/\lambda\}_{\partial\tau_k}^{\mathrm{up}} (v - \mu)\, ds_{(x,t)}.$$

With respect to each subdomain Q_i we therefore can define the bilinear form

$$\begin{aligned} c^{(i)}(u_h, \lambda_h; v_h, \mu_h) := & \sum_{\substack{k=1 \\ \tau_k \subseteq Q_i}}^{N} \int_{\partial\tau_k \cap \Gamma} n_{k,t}\, \{u_h/\lambda_h\}_{\partial\tau_k}^{\mathrm{up}} (v_h - \mu_h)\, ds_{(x,t)} \\ & - \sum_{\substack{k=1 \\ \tau_k \subseteq Q_i}}^{N} \int_{\partial\tau_k \cap \Gamma} \left[\mathbf{n}_{k,x} \cdot \nabla_x u_h\, (v_h - \mu_h) - \varepsilon (u_h - \lambda_h)\, \mathbf{n}_{k,x} \cdot \nabla_x v_h\right] ds_{(x,t)} \\ & + \frac{2\sigma}{h} \sum_{\substack{k=1 \\ \tau_k \subseteq Q_i}}^{N} \int_{\partial\tau_k \cap \Gamma} |\mathbf{n}_{k,x}|^2\, (u_h - \lambda_h)(v_h - \mu_h)\, ds_{(x,t)}. \end{aligned}$$

Hence we can write the discrete hybrid space–time variational formulation to find $u_h \in S^p_{h,0}(\mathscr{T}_N)$ and $\lambda_h \in S^p_h(\Gamma)$ satisfying

$$\sum_{i=1}^{P} \left[a^{(i)}_{\mathrm{DG}}(u_h, v_h) + c^{(i)}(u_h, \lambda_h; v_h, \mu_h) \right] = \sum_{i=1}^{P} F^{(i)}(v_h) \tag{6}$$

for all $v_h \in S^p_{h,0}(\mathscr{T}_N)$ and $\mu_h \in S^p_h(\Gamma)$. As in [2] we can prove unique solvability of the hybrid scheme (6). Moreover, related error estimates as derived for the DG scheme remain valid.

The discrete variational formulation (6) is equivalent to the solution of the linear equations

$$\begin{pmatrix} A^{(1)}_{II} & & & & A^{(1)}_{I\Gamma} \\ & A^{(2)}_{II} & & & A^{(2)}_{I\Gamma} \\ & & \ddots & & \vdots \\ & & & A^{(P)}_{II} & A^{(P)}_{I\Gamma} \\ A^{(1)}_{\Gamma I} & A^{(2)}_{\Gamma I} & \cdots & A^{(P)}_{\Gamma I} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{u}^{(1)}_I \\ \mathbf{u}^{(2)}_I \\ \vdots \\ \mathbf{u}^{(P)}_I \\ \lambda_\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{(1)}_I \\ \mathbf{f}^{(2)}_I \\ \vdots \\ \mathbf{f}^{(P)}_I \\ \mathbf{f}_\Gamma \end{pmatrix} \tag{7}$$

where the local block matrices $A^{(i)}_{II}$ correspond to the local bilinear forms $a^{(i)}_{\mathrm{DG}}(\cdot,\cdot)$ and $c^{(i)}(\cdot,0;\cdot,0)$, while the remaining block matrices describe the coupling across the interface. For an appropriate choice of the DG parameters, see Lemma 1, the local matrices $A^{(i)}_{II}$ are invertible. Hence we obtain the Schur complement system

$$\left[A_{\Gamma\Gamma} - \sum_{i=1}^{P} A^{(i)}_{\Gamma I} \left(A^{(i)}_{II} \right)^{-1} A^{(i)}_{I\Gamma} \right] \lambda_\Gamma = \mathbf{f}_\Gamma - \sum_{i=1}^{P} A^{(i)}_{\Gamma I} \left(A^{(i)}_{II} \right)^{-1} \mathbf{f}^{(i)}_I, \tag{8}$$

with

$$\mathbf{u}^{(i)}_I = \left(A^{(i)}_{II} \right)^{-1} \left[\mathbf{f}^{(i)}_I - A^{(i)}_{I\Gamma} \lambda_\Gamma \right] \quad \text{for } i = 1, \ldots, P.$$

The inversion of the local matrices $A^{(i)}_{II}$ can be done in parallel either by using some appropriate direct approach, or suitable iterative schemes. For the solution of the global Schur complement system (8) we can use, for example the GMRES method.

3 Numerical Examples

To illustrate the hybrid domain decomposition approach we consider for $n = 3$ the spatial domain $\Omega = (0,1)^3$ and $T = 1$, i.e. $Q = (0,1)^4$. As initial triangulation for the space-time domain we use 96 pentatopes of the same size, see also [4]. The initial triangulation is used as a partition of the space-time domain into $P = 96$ subdomains, which we keep fixed for all computations. As exact solution of the transient heat equation (1) we now consider the smooth function

$$u(x,t) = \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) t^2.$$

For the iterative solution of the Schur complement system (8) we use the GMRES method without preconditioning with a relative error reduction of $\varepsilon_{\text{GMRES}} = 10^{-8}$. In the Tables 2 and 3 we present the iteration numbers of the GMRES method for different levels of a uniform refinement of the space–time mesh for $p = 1$ and $p = 2$. We observe that the number of required iterations grows slightly indicating the need of using an appropriate preconditioner. The results also show the optimal convergence rates for the error in the $L_2(Q)$ norm when using linear and quadratic basis functions.

level	elements	dof $\mathbf{u}_I^{(i)}$	dof λ_Γ	iter.	$\|\|u-u_h\|\|_{L_2(Q)}$	eoc
0	96	192	768	68	$6.120-2$	–
1	1536	5376	6144	143	$3.821-2$	0.68
2	24576	104448	49152	197	$1.356-2$	1.49
3	393216	1818624	393216	294	$4.024-3$	1.75
4	6291456	30277632	3145728	475	$1.111-3$	1.86

Table 2. Numerical results with 96 subdomains for $p = 1$, $\varepsilon = -1$ and $\sigma = 10$.

level	elements	dof $\mathbf{u}_I^{(i)}$	dof λ_Γ	iter.	$\|\|u-u_h\|\|_{L_2(Q)}$	eoc
0	96	720	1920	404	$4.199-2$	–
1	1536	17280	15360	699	$7.492-3$	2.49
2	24576	322560	122880	900	$1.005-3$	2.90
3	393216	5529600	983040	1131	$1.293-4$	2.96

Table 3. Numerical results with 96 subdomains for $p = 2$, $\varepsilon = -1$ and $\sigma = 10$.

4 Conclusions

In this paper we have presented a hybrid DG domain decomposition approach for the parallel solution of initial boundary value problems. Numerical examples for one– and three–dimensional spatial domains indicate the accuracy and applicability of the proposed method. However, the numerical results also indicate the need to use an appropriate global preconditioner for the Schur complement system (8). Moreover, when solving the coupled system (7) iteratively, suitable local preconditioners are mandatory as well. A possible choice is to use space-time multigrid methods. Although we have only considered uniform refinements in this paper, the proposed approach is also applicable to non–uniform and adaptive refinements, see, for example, [4]. For this we need to use suitable a posteriori error estimators, and the solution algorithms need to be robust with respect to adaptive refinements. Although we have only considered the simple model problem of the transient heat equation, the proposed approach can be extended to more complicated problems, see, e.g., [4] for a first example for the transient Navier-Stokes system.

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Bibliography

[1] B. Cockburn, J. Gopalakrishnan, and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 47:1319–1365, 2009.

[2] H. Egger and J. Schöberl. A hybrid mixed discontinuous Galerkin finite–element method for convection–diffusion problems. *IMA J. Numer. Anal.*, 30:1206–1234, 2010.

[3] M. Neumüller. Eine Finite Element Methode für optimale Kontrollprobleme mit parabolischen Randwertaufgaben. Masterarbeit, Institut für Numerische Mathematik, Technische Universität Graz, 2010.

[4] M. Neumüller and O. Steinbach. Refinement of flexible space–time finite element meshes and discontinuous Galerkin methods. *Comput. Visual. Sci.*, accepted, 2012.

[5] B. Rivière. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations*. SIAM, Philadelphia, 2008.