Domain Decomposition Methods of Stochastic PDEs

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1 Introduction

AQ1 In conjunction with modern high performance computing systems, domain decomposition algorithms permit simulation of PDEs with extremely high resolution numerical models. Such computational models substantially reduce discretization errors. In realistic simulation of certain physical systems, it is however necessary to consider the heterogeneities of the model parameters. Whenever sufficient statistical information is available, such heterogeneities can be modeled by stochastic processes (e.g. [2]). For uncertainty propagation, the traditional Monte Carlo simulation may be impractical for these high resolution models. As an alternative, a domain decomposition algorithm for stochastic PDEs (SPDEs) is proposed [4] using the spectral stochastic finite element method (SSFEM). The SSFEM discretization leads to a linear system with a block sparsity structure, and the size of the resulting system grows rapidly with the spatial mesh resolution and the order of the stochastic dimension [2]. The solution of this large-scale system constitutes a computationally challenging task and therefore efficient solvers are required. Extending the formulation in [4], the iterative substructuring based non-overlapping domain decomposition methods are proposed to solve the large-scale linear system arising in the SSFEM. The methodology is based on domain decomposition in the geometric space and a functional decomposition in the stochastic space [4]. Firstly, we describe a primal version of iterative substructuring methods of SPDEs. The method offers a straightforward approach to formulate a two-level scalable preconditioner. In the proposed preconditioner, the continuity of the solution field is strictly enforced on the corner nodes of the interface boundary, but weakly satisfied over the remaining interface nodes. This approach naturally leads to a coarse grid connecting the subdomains globally and provides a mechanism to propagate information across the subdomains which makes the algorithm scalable. The proposed preconditioner may be viewed as an extension of BDDC [3] for SPDEs. Secondly, a dual-primal iterative substructuring method is introduced for SPDEs. In this approach, the continuity condition on the corner nodes is strictly satisfied and Lagrange multipliers are used to weakly enforce the continu-

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ity on the remaining nodes of the interface boundary. This method may be construed to be an extension of FETI-DP [1] for SPDEs.

2 Uncertainty Representation by Stochastic Processes

We briefly review the theories of stochastic processes, relevant to subsequent theoretical developments, by closely following [2, 4–6]. Assuming the input data (containing sufficient statistical information) permits a representation of the model parameters as stochastic processes that span the Hilbert space $\mathscr{H}_G$. Using Karhunen-Loeve expansion (KLE), a set of basis functions $\{\xi_i(\theta)\}$ for the Hilbert space $\mathscr{H}_G$ is identified. The KLE of a stochastic process $\alpha(\mathbf{x},\theta)$ is based on the spectral expansion of its covariance function $C_{\alpha\alpha}(\mathbf{x},\mathbf{y})$, and takes the following form [2]

$$\alpha(\mathbf{x},\theta) = \bar{\alpha}(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\xi_i(\theta)\phi_i(\mathbf{x}), \tag{1}$$

where $\bar{\alpha}(\mathbf{x})$ is the mean of the stochastic process, $\{\xi_i(\theta)\}$ is a set of uncorrelated random variables and $\{\lambda_i,\phi_i(\mathbf{x})\}$ are the eigenpairs of the covariance function, obtained from the following integral equation

$$\int_{\Omega} C_{\alpha\alpha}(\mathbf{x},\mathbf{y})\phi_i(\mathbf{y})d\mathbf{y} = \lambda_i\phi_i(\mathbf{x}). \tag{2}$$

For a smooth stochastic process, only a finite number of KLE basis is sufficient to represent the stochastic process. Given the covariance function of the solution is not known a priori, the KLE cannot be used to represent solution process. Assuming the solution process $u(\mathbf{x},\theta)$ belong to the Hilbert space $\mathscr{H}_L$, a generic basis of this space can be identified using the Polynomial Chaos (PC) [2]. Consequently, the solution process can be approximated as

$$u(\mathbf{x},\theta) = \sum_{j=0}^{N} \Psi_j(\theta)u_j(\mathbf{x}), \tag{3}$$

where the polynomials $\Psi_j(\theta)$ are orthogonal in the statistical sense, meaning $\langle\Psi_j,\Psi_k\rangle = \langle\Psi_j^2\rangle\delta_{jk}$ where $\langle\cdot\rangle$ denotes the expectation operator and δ_{jk} is the Kronecker delta, and $u_j(\mathbf{x})$ are the PC coefficients to be determined by Galerkin projection.

3 Review of Schur Complement Based Domain Decomposition Method of SPDEs

A review of the domain decomposition method for SPDEs based on [4–6] is provided in this section. For an elliptic SPDE defined on a domain Ω with a prescribed boundary condition on $\partial\Omega$, the finite element discretization leads to the following linear system

$$\mathbf{A}(\theta)\mathbf{u}(\theta) = \mathbf{f}, \tag{4}$$

where $\mathbf{A}(\theta)$ is the random stiffness matrix, $\mathbf{u}(\theta)$ is the stochastic response and $\mathbf{f}$ is the applied force. The physical domain Ω is split into n_s non-overlapping subdomains $\{\Omega_s\}_{s=1}^{n_s}$. For a typical subdomain Ω_s the nodal vector $\mathbf{u}^s(\theta)$ is partitioned into interior $\mathbf{u}_I^s(\theta)$ and interface $\mathbf{u}_\Gamma^s(\theta)$ unknowns. This decomposition leads to the following subdomain equilibrium equation

$$\begin{bmatrix} \mathbf{A}_{II}^s(\theta) & \mathbf{A}_{I\Gamma}^s(\theta) \\ \mathbf{A}_{\Gamma I}^s(\theta) & \mathbf{A}_{\Gamma\Gamma}^s(\theta) \end{bmatrix} \begin{Bmatrix} \mathbf{u}_I^s(\theta) \\ \mathbf{u}_\Gamma^s(\theta) \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_I^s \\ \mathbf{f}_\Gamma^s \end{Bmatrix}. \tag{5}$$

Enforcing the transmission conditions and expanding the solution vector by the PCE (as in Eq. (3)) and then performing Galerkin projection, we obtain the following block linear systems of equations [4–6]:

$$\langle \sum_{i=0}^{L} \Psi_i(\theta) \begin{bmatrix} \mathbf{A}_{II,i}^1 & \dots & 0 & \mathbf{A}_{I\Gamma,i}^1\mathbf{R}_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \mathbf{A}_{II,i}^{n_s} & \mathbf{A}_{I\Gamma,i}^{n_s}\mathbf{R}_{n_s} \\ \mathbf{R}_1^T\mathbf{A}_{\Gamma I,i}^1 & \dots & \mathbf{R}_{n_s}^T\mathbf{A}_{\Gamma I,i}^{n_s} & \sum_{s=1}^{n_s}\mathbf{R}_s^T\mathbf{A}_{\Gamma\Gamma,i}^s\mathbf{R}_s \end{bmatrix} \sum_{j=0}^{N} \Psi_j(\theta) \begin{Bmatrix} \mathbf{u}_{I,j}^1 \\ \vdots \\ \mathbf{u}_{I,j}^{n_s} \\ \mathbf{u}_{\Gamma,j} \end{Bmatrix} \Psi_k(\theta)\rangle$$

$$= \langle \begin{Bmatrix} \mathbf{f}_I^1 \\ \vdots \\ \mathbf{f}_I^{n_s} \\ \sum_{s=1}^{n_s}\mathbf{R}_s^T\mathbf{f}_\Gamma^s \end{Bmatrix} \Psi_k(\theta)\rangle, \quad k = 0, \dots, N. \tag{6}$$

where the restriction operator $\mathbf{R}_s$ maps the global interface vector $\mathbf{u}_\Gamma(\theta)$ to the local interface unknown $\mathbf{u}_\Gamma^s(\theta)$ as $\mathbf{u}_\Gamma^s(\theta) = \mathbf{R}_s\mathbf{u}_\Gamma(\theta)$. Compactly, Eq. (6) can be expressed as

$$\begin{bmatrix} \mathscr{A}_{II}^1 & \dots & 0 & \mathscr{A}_{I\Gamma}^1\mathscr{R}_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \mathscr{A}_{II}^{n_s} & \mathscr{A}_{I\Gamma}^{n_s}\mathscr{R}_{n_s} \\ \mathscr{R}_1^T\mathscr{A}_{\Gamma I}^1 & \dots & \mathscr{R}_{n_s}^T\mathscr{A}_{\Gamma I}^{n_s} & \sum_{s=1}^{n_s}\mathscr{R}_s^T\mathscr{A}_{\Gamma\Gamma}^s\mathscr{R}_s \end{bmatrix} \begin{Bmatrix} \mathscr{U}_I^1 \\ \vdots \\ \mathscr{U}_I^{n_s} \\ \mathscr{U}_\Gamma \end{Bmatrix} = \begin{Bmatrix} \mathscr{F}_I^1 \\ \vdots \\ \mathscr{F}_I^{n_s} \\ \sum_{s=1}^{n_s}\mathscr{R}_s^T\mathscr{F}_\Gamma^s \end{Bmatrix}, \tag{7}$$

where $[\mathscr{A}_{\alpha\beta}^s]_{jk} = \sum_{i=0}^{L}\langle\Psi_i\Psi_j\Psi_k\rangle\mathbf{A}_{\alpha\beta,i}^s$, $\mathscr{F}_{\alpha,k}^s = \langle\Psi_k\mathbf{f}_\alpha^s\rangle$, $\mathscr{U}_I^m = (\mathbf{u}_{I,0}^m, \dots, \mathbf{u}_{I,N}^m)^T$ and $\mathscr{R}_s = blockdiag(\mathbf{R}_s^0, \dots, \mathbf{R}_s^N)$. The subscripts α and β represent the index I and Γ. Performing Gaussian elimination in Eq. (7), we obtain the global *extended* Schur complement system as

$$\mathscr{S}\,\mathscr{U}_\Gamma = \mathscr{G}_\Gamma, \tag{8}$$

where $\mathscr{S} = \sum_{s=1}^{n_s}\mathscr{R}_s^T[\mathscr{A}_{\Gamma\Gamma}^s - \mathscr{A}_{\Gamma I}^s(\mathscr{A}_{II}^s)^{-1}\mathscr{A}_{I\Gamma}^s]\mathscr{R}_s$, $\mathscr{G}_\Gamma = \sum_{s=1}^{n_s}\mathscr{R}_s^T[\mathscr{F}_\Gamma^s - \mathscr{A}_{\Gamma:I}^s(\mathscr{A}_{II}^s)^{-1}\mathscr{F}_I^s]$.

4 Primal Iterative Substructuring Method of SPDEs

In this section, a two-level domain decomposition method is formulated in the context of SPDEs. The subdomain nodal vector, namely the primal variable, is partitioned into interior, remaining interface and corner nodes as schematically shown in Fig. 1 [3]. Using PCE to represent the random coefficients of the system parameters and performing Galerkin projection, lead to the following coupled deterministic system

$$\begin{bmatrix} \mathcal{A}_{ii}^s & \mathcal{A}_{ir}^s & \mathcal{A}_{ic}^s \\ \mathcal{A}_{ri}^s & \mathcal{A}_{rr}^s & \mathcal{A}_{rc}^s \\ \mathcal{A}_{ci}^s & \mathcal{A}_{cr}^s & \mathcal{A}_{cc}^s \end{bmatrix} \begin{Bmatrix} \mathcal{U}_i^s \\ \mathcal{U}_r^s \\ \mathcal{U}_c^s \end{Bmatrix} = \begin{Bmatrix} \mathcal{F}_i^s \\ \mathcal{F}_r^s \\ \mathcal{F}_c^s \end{Bmatrix}. \tag{9}$$

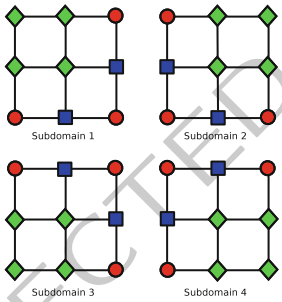


Fig. 1. Partitioning domain nodes into: interior (♦), remaining (■) and corner (●)

Enforcing the transmission conditions along the boundary interfaces, the subdomain equilibrium equation can be written as

$$\begin{bmatrix} \mathcal{A}_{ii}^s & \mathcal{A}_{ir}^s\mathcal{B}_r^s & \mathcal{A}_{ic}^s\mathcal{B}_c^s \\ \sum_{s=1}^{n_s}\mathcal{B}_r^{sT}\mathcal{A}_{ri}^s & \sum_{s=1}^{n_s}\mathcal{B}_r^{sT}\mathcal{A}_{rr}^s\mathcal{B}_r^s & \sum_{s=1}^{n_s}\mathcal{B}_r^{sT}\mathcal{A}_{rc}^s\mathcal{B}_c^s \\ \sum_{s=1}^{n_s}\mathcal{B}_c^{sT}\mathcal{A}_{ci}^s & \sum_{s=1}^{n_s}\mathcal{B}_c^{sT}\mathcal{A}_{cr}^s\mathcal{B}_r^s & \sum_{s=1}^{n_s}\mathcal{B}_c^{sT}\mathcal{A}_{cc}^s\mathcal{B}_c^s \end{bmatrix} \begin{Bmatrix} \mathcal{U}_i^s \\ \mathcal{U}_r \\ \mathcal{U}_c \end{Bmatrix} = \begin{Bmatrix} \mathcal{F}_i^s \\ \sum_{s=1}^{n_s}\mathcal{B}_r^{sT}\mathcal{F}_r^s \\ \sum_{s=1}^{n_s}\mathcal{B}_c^{sT}\mathcal{F}_c^s \end{Bmatrix}, \tag{10}$$

where $\mathcal{B}_r^s$ and $\mathcal{B}_c^s$ are Boolean rectangular matrices that extract the subdomain remaining interface and corner degrees of freedom from the corresponding global vectors $\mathcal{U}_r$ and $\mathcal{U}_c$ as $\mathcal{U}_r^s = \mathcal{B}_r^s\mathcal{U}_r$ and $\mathcal{U}_c^s = \mathcal{B}_c^s\mathcal{U}_c$. Eliminating $\mathcal{U}_i^s$ from Eq. (10), we obtain

$$\begin{bmatrix} \sum_{s=1}^{n_s}\mathcal{B}_r^{sT}\mathcal{S}_{rr}^s\mathcal{B}_r^s & \sum_{s=1}^{n_s}\mathcal{B}_r^{sT}\mathcal{S}_{rc}^s\mathcal{B}_c^s \\ \sum_{s=1}^{n_s}\mathcal{B}_c^{sT}\mathcal{S}_{cr}^s\mathcal{B}_r^s & \sum_{s=1}^{n_s}\mathcal{B}_c^{sT}\mathcal{S}_{cc}^s\mathcal{B}_c^s \end{bmatrix} \begin{Bmatrix} \mathcal{U}_r \\ \mathcal{U}_c \end{Bmatrix} = \begin{Bmatrix} \sum_{s=1}^{n_s}\mathcal{B}_r^{sT}\mathcal{G}_r^s \\ \sum_{s=1}^{n_s}\mathcal{B}_c^{sT}\mathcal{G}_c^s \end{Bmatrix}, \tag{11}$$

where $\mathcal{S}^s_{\alpha\beta} = \mathcal{A}^s_{\alpha\beta} - \mathcal{A}^s_{\alpha i}[\mathcal{A}^s_{ii}]^{-1}\mathcal{A}^s_{i\beta}$ and $\mathcal{G}^s_\alpha = \mathcal{F}^s_\alpha - \mathcal{A}^s_{\alpha i}[\mathcal{A}^s_{ii}]^{-1}\mathcal{F}^s_i$. Eliminating $\mathcal{U}_c$ from Eq. (11) leads to the following symmetric positive definite *reduced interface problem*

$$(F_{rr} - F_{rc}[F_{cc}]^{-1}F_{cr})\mathcal{U}_r = d_r - F_{rc}[F_{cc}]^{-1}d_c, \tag{12}$$

where $F_{\alpha\beta} = \sum_{s=1}^{n_s} \mathcal{B}^{s\,T}_\alpha \mathcal{S}^s_{\alpha\beta}\mathcal{B}^s_\beta$ and $d_\alpha = \sum_{s=1}^{n_s} \mathcal{B}^{s\,T}_\alpha \mathcal{G}^s_\alpha$.

4.1 Two-Level Primal Preconditioner

The Preconditioned Conjugate Gradient Method (PCGM) can be used to solve the reduced interface problem in Eq. (12). At each iteration of the PCGM, the continuity of the solution field is enforced strictly on the corner nodes, but weakly satisfied on the remaining interface nodes. Consequently we obtain the following partially assembled Schur complement system:

$$\begin{bmatrix} \mathcal{S}^s_{rr} & \mathcal{S}^s_{rc}\mathcal{B}^s_c \\ \sum_{s=1}^{n_s} \mathcal{B}^{s\,T}_c \mathcal{S}^s_{cr}\mathcal{B}^s_r & \sum_{s=1}^{n_s} \mathcal{B}^{s\,T}_c \mathcal{S}^s_{cc}\mathcal{B}^s_c \end{bmatrix} \begin{Bmatrix} \mathcal{U}^s_r \\ \mathcal{U}_c \end{Bmatrix} = \begin{Bmatrix} \mathcal{F}^s_r \\ 0 \end{Bmatrix}, \tag{13}$$

where $\mathcal{F}^s_r = \mathcal{D}^s_r\mathcal{B}^s_r\mathbf{r}_j$, and $\mathbf{r}_j$ is the residual of the jth iteration of PCGM, and $\mathcal{D}^s_r$ is a block diagonal weighting matrix which satisfies $\sum_{s=1}^{n_s} \mathcal{B}^{s\,T}_r \mathcal{D}^s_r \mathcal{B}^s_r = \mathbf{I}$. Next, $\mathcal{U}^s_r$ can be eliminated from Eq. (13) leading to the following coarse problem

$$\widetilde{F}_{cc}\mathcal{U}_c = \widetilde{d}_c, \tag{14}$$

where $\widetilde{F}_{cc} = \sum_{s=1}^{n_s} \mathcal{B}^{s\,T}_c(\mathcal{S}^s_{cc} - \mathcal{S}^s_{cr}[\mathcal{S}^s_{rr}]^{-1}\mathcal{S}^s_{rc})\mathcal{B}^s_c$ and $\widetilde{d}_c = -\sum_{s=1}^{n_s} \mathcal{B}^{s\,T}_c \mathcal{S}^s_{cr}[\mathcal{S}^s_{rr}]^{-1}\mathcal{F}^s_r$.
The two-level preconditioner can be expressed as

$$\mathcal{M}^{-1} = \sum_{s=1}^{n_s} \mathcal{B}^{s\,T}_r \mathcal{D}^s_r[\mathcal{S}^s_{rr}]^{-1}\mathcal{D}^s_r\mathcal{B}^s_r + R_0^T[\widetilde{F}_{cc}]^{-1}R_0, \tag{15}$$

where $R_0 = \sum_{s=1}^{n_s} \mathcal{B}^{s\,T}_c \mathcal{S}^s_{cr}[\mathcal{S}^s_{rr}]^{-1}\mathcal{D}^s_r\mathcal{B}^s_r$.

5 Dual-Primal Iterative Substructuring of SPDEs

In the dual-primal method [1], the continuity condition on the corner nodes is enforced strictly while Lagrange multipliers are used to weakly enforce the continuity on the remaining interface. Partial assembly of the corner node unknowns leads to the following system

$$\begin{bmatrix} \mathscr{A}_{ii}^s & \mathscr{A}_{ir}^s & \mathscr{A}_{ic}^s\mathscr{B}_c^s & 0 \\ \mathscr{A}_{ri}^s & \mathscr{A}_{rr}^s & \mathscr{A}_{rc}^s\mathscr{B}_c^s & \mathscr{B}_r^{sT} \\ \sum_{s=1}^{n_s}\mathscr{B}_c^{sT}\mathscr{A}_{ci}^s & \sum_{s=1}^{n_s}\mathscr{B}_c^{sT}\mathscr{A}_{cr}^s & \sum_{s=1}^{n_s}\mathscr{B}_c^{sT}\mathscr{A}_{cc}^s\mathscr{B}_c^s & 0 \\ 0 & \sum_{s=1}^{n_s}\mathscr{B}_r^s & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathscr{U}_i^s \\ \mathscr{U}_r^s \\ \mathscr{U}_c \\ \Lambda \end{Bmatrix} = \begin{Bmatrix} \mathscr{F}_i^s \\ \mathscr{F}_r^s \\ \sum_{s=1}^{n_s}\mathscr{B}_c^{sT}\mathscr{F}_c^s \\ 0 \end{Bmatrix}, \tag{16}$$

where $\sum_{s=1}^{n_s}\mathscr{B}_r^s\mathscr{U}_r^s = 0$ and $\Lambda^T = \{\boldsymbol{\lambda}_0, \cdots, \boldsymbol{\lambda}_N\}$. The matrix $\mathscr{B}_r^s$ is a block diagonal signed Boolean continuity operator and $\boldsymbol{\lambda}_j$ is the nodal force vector required to satisfy continuity on the remaining interface nodes. Eliminating $\mathscr{U}_i^s$ and $\mathscr{U}_r^s$ from Eq. (16) leads to the following interface problem

$$\begin{bmatrix} \bar{F}_{cc} & -\bar{F}_{cr} \\ \bar{F}_{rc} & \bar{F}_{rr} \end{bmatrix} \begin{Bmatrix} \mathscr{U}_c \\ \Lambda \end{Bmatrix} = \begin{Bmatrix} \bar{d}_c \\ \bar{d}_r \end{Bmatrix}, \tag{17}$$

where

$$\bar{F}_{cc} = \sum_{s=1}^{n_s}\mathscr{B}_c^{sT}(\mathscr{S}_{cc}^s - \mathscr{S}_{cr}^s[\mathscr{S}_{rr}^s]^{-1}\mathscr{S}_{rc}^s)\mathscr{B}_c^s, \quad \bar{F}_{cr} = \sum_{s=1}^{n_s}\mathscr{B}_c^{sT}\mathscr{S}_{cr}^s[\mathscr{S}_{rr}^s]^{-1}\mathscr{B}_r^{sT}$$

$$\bar{F}_{rc} = \sum_{s=1}^{n_s}\mathscr{B}_r^s[\mathscr{S}_{rr}^s]^{-1}\mathscr{S}_{rc}^s\mathscr{B}_c^s, \quad \bar{F}_{rr} = \sum_{s=1}^{n_s}\mathscr{B}_r^s[\mathscr{S}_{rr}^s]^{-1}\mathscr{B}_r^{sT}$$

$$\bar{d}_c = \sum_{s=1}^{n_s}\mathscr{B}_c^{sT}(\mathscr{G}_c^s - \mathscr{S}_{cr}^s[\mathscr{S}_{rr}^s]^{-1}\mathscr{G}_r^s), \quad \bar{d}_r = \sum_{s=1}^{n_s}\mathscr{B}_r^s[\mathscr{S}_{rr}^s]^{-1}\mathscr{G}_r^s$$

Solving for $\mathscr{U}_c$ from Eq. (17) gives the following coarse problem

$$\bar{F}_{cc}\mathscr{U}_c = (\bar{d}_c + \bar{F}_{cr}\Lambda) \tag{18}$$

Substituting $\mathscr{U}_c$ into Eq. (17) leads to the following symmetric positive definite Lagrange multiplier system

$$(\bar{F}_{rr} + \bar{F}_{rc}[\bar{F}_{cc}]^{-1}\bar{F}_{cr})\Lambda = \bar{d}_r - \bar{F}_{rc}[\bar{F}_{cc}]^{-1}\bar{d}_c. \tag{19}$$

The Lagrange multiplier system in Eq. (19) is solved using PCGM equipped with a Dirichlet precondtioner defined as $\bar{\mathscr{M}} = \sum_{s=1}^{n_s}\mathscr{B}_r^s\mathscr{D}_r^s\mathscr{S}_{rr}^s\mathscr{D}_r^s\mathscr{B}_r^{sT}$.

6 Numerical Results

For numerical illustrations, we consider the following elliptic SPDE

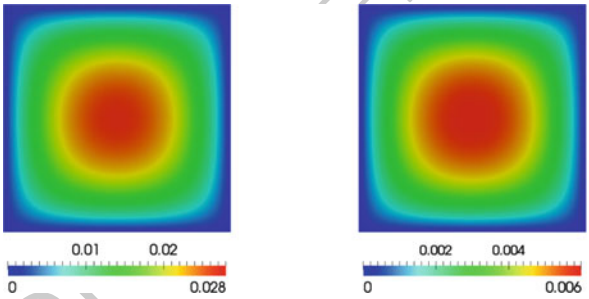
$$\nabla\cdot(\kappa(\mathbf{x},\theta)\nabla u(\mathbf{x},\theta)) = f(\mathbf{x}), \quad \mathbf{x}\in\Omega, \tag{20}$$

$$u(\mathbf{x},\theta) = 0, \quad \mathbf{x}\in\partial\Omega. \tag{21}$$

The coefficient $\kappa(\mathbf{x}, \theta)$ is modeled as a lognormal stochastic process, obtained from the underlying Gaussian process with an exponential covariance function given as

$$C_{\alpha\alpha}(\mathbf{x}, \mathbf{y}) = \sigma^2 \exp\left(-\frac{|x_1 - y_1|}{b_1} - \frac{|x_2 - y_2|}{b_2}\right). \tag{22}$$

The lognormal process is approximated using four-dimensional second order PC expansion ($L = 15$). Finite element discretization results in 375,444 elements and 186,925 nodes. The response is expressed using third order PCE ($N = 34$) leading to a linear system of order 6,542,375. The mean and standard deviation of the solution process are shown in Fig. 2. The PCGM iteration counts for the primal and dual-primal methods for fixed problem size in the spatial domain is reported in Table 1 for 1st, 2nd and 3rd order of PCE. The results suggest that the methods are numerically scalable with respect to number of subdomains. Table 2 shows the iteration counts of the methods when we fix spatial problem size per subdomain and increase the overall problem size by adding more subdomains. Again these results suggest that both the methods are numerically scalable with respect to fixed problem size per subdomain.



(a) (b)

Fig. 2. The mean and standard deviation of the solution field. **(a)** Mean. **(b)** Standard deviation

Table 1. Iteration counts for fixed problem size in geometric space

Subdomain	PP-DDM			DP-DDM		
	1*st*	2*nd*	3*rd*	1*st*	2*nd*	3*rd*
8	11	12	12	9	9	9
16	12	13	13	10	10	10
32	14	14	14	11	11	11
64	13	14	14	10	10	10
128	14	14	14	10	10	10
256	14	14	14	10	10	10

Table 2. Iteration counts for fixed problem size per subdomain in geometric space

Subdomain	PP-DDM			DP-DDM		
	1*st*	2*nd*	3*rd*	1*st*	2*nd*	3*rd*
8	9	9	9	8	8	8
16	12	12	12	10	10	10
32	12	13	13	10	10	10
64	13	14	14	10	10	10
128	14	14	14	10	10	10
256	15	15	15	11	11	11

7 Conclusion

Primal and dual-primal domain decomposition methods are proposed to solve the large-scale linear system arising from the finite element discretization of SPDEs. The proposed techniques exploit a coarse grid in the geometric space which makes the methods numerically scalable with respect to fixed geometric problem size, fixed geometric size per subdomain and the order of PCE.

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