Overlapping Domain Decomposition: Convergence Proofs

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1 Introduction

During the last two decades many domain decomposition algorithms have been constructed and lot of techniques have been developed to prove the convergence of the algorithms at the continuous level. Among the techniques used to prove the convergence of classical Schwarz algorithms, the first technique is the maximum principle used by Schwarz. Adopting this technique M. Gander and H. Zhao proved a convergence result for n-dimensional linear heat equation in [4]. The second technique is that of the orthogonal projections, used by P. L. Lions in [7], and his convergence results are for linear Laplace equation and linear Stokes equation. In the same paper, P. L. Lions also proved that the Schwarz sequences for linear elliptic equations are related to classical minimization methods over product spaces and this technique was then used by L. Badea in [1] for nonlinear monotone elliptic problems. Another technique is the Fourier and Laplace transforms used in the papers [3, 5] for some 1-dimensional evolution equations, with constant coefficients. In [10, 11], S. H. Lui used the idea of upper-lower solutions methods to study the convergence problem for some PDEs, with initial guess to be an upper or lower solution of the equations and monotone iterations. For nonoverlapping optimized Schwarz methods, P. L. Lions in [8] proposed to use an energy estimate argument to study the convergence of the algorithm. The energy estimate technique was then developed in [2] for Helmholtz equation and it has then become a very powerful tool to study nonoverlapping problems. J.-H. Kim in [6] proved the convergence of an overlapping optimized Schwarz method for Poisson’s equation with Robin boundary data and S. Loisel and D. B. Szyld in [9] extended the technique of J.-H. Kim to linear symmetric elliptic equation. Another technique is to use semiclassical analysis, which works for overlapping optimized Schwarz methods with rectangle subdomains, linear advection diffusion equations on the half plane (see [12]). This paper is devoted to the study of the convergence of Schwarz methods at the continuous level. We give a sketch of the proof of the convergence of optimized Schwarz methods for semilinear parabolic equations, with multiple subdomains. Complete convergence proofs for both classical
and optimized Schwarz methods, both semilinear parabolic and elliptic equations, with multiple subdomains could be found in [13].

2 Convergence for Semilinear Parabolic Equations

Consider the following parabolic equation

\[
\begin{cases}
\frac{\partial u}{\partial t}(x,t) - \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}(x,t) \\
\quad + c(x)u(x,t) = F(x,t,u(x,t)), \quad \text{in } \Omega \times (0,\infty), \\
u(x,0) = g(x,t), \quad \text{on } \partial \Omega, \\
u(x,0) = g(x,0), \quad \text{on } \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded and smooth enough domain in \( \mathbb{R}^n \). The following conditions are imposed on 1).

(A1) For all \( i,j \in \{ 1, \ldots, I \} \), \( a_{i,j}(x) = a_{j,i}(x) \). There exist strictly positive numbers \( \lambda, \Lambda \) such that \( A = (a_{i,j}(x)) \geq \lambda I \) in the sense of symmetric positive definite matrices and \( a_{i,j}(x) < \Lambda \) in \( \Omega \).

(A2) The functions \( a_{i,j}, b_i, c \) are in \( C^\infty(\mathbb{R}^n) \) and \( g \) is in \( C^\infty(\mathbb{R}^{n+1}) \).

(A3) There exists \( C > 0 \), such that \( \forall \, t \in \mathbb{R}, \forall \, x \in \mathbb{R}^n, |F(x,t,z) - F(x,t,z')| \leq C|z - z'|, \forall \, z, z' \in \mathbb{R} \). We now describe the way that we decompose the domain \( \Omega \):

The domain \( \Omega \) is divided into \( I \) smooth overlapping subdomains \( \{ \Omega_i \}_{i \in \{1,I\}} \):

\[
(\partial \Omega_i \setminus \partial \Omega) \cap (\partial \Omega_{i'} \setminus \partial \Omega) = , \quad \forall \, l, l' \in \{1, \ldots, I\}, \quad l \neq l';
\]

\[
\forall l \in \{1, \ldots, I\}, \forall l', l'' \in J_l, l'' \neq l', \quad \Omega_{l'} \cap \Omega_{l''} = ,
\]

where

\[
J_l = \{ l' | \Omega_{l'} \cap \Omega_l \neq \} ;
\]

\[
\bigcup_{l=1}^{n} \Omega_l = \Omega.
\]

This decomposition means that we do not consider cross-points in this paper.

Denote by \( \Gamma_{l,l'} \), for \( l' \in J_l \), the set \( (\partial \Omega_i \setminus \partial \Omega) \cap \Omega_l \). The transmission operator \( \mathcal{B}_{l,l'} \) is of Robin type \( \mathcal{B}_{l,l'}v = \sum_{i,j=1}^{n} a_{i,j} \frac{\partial v}{\partial x_i} n_{i,l,l'} + p_{l,l'}v \) and \( n_{i,l,l'} \) is the \( j \)-th component of the outward unit normal vector of \( \Gamma_{l,l'} \); \( p_{l,l'} \) is positive and belongs to \( L^\infty(\Gamma_{l,l'}) \).

The iterate \#k in the \( l \)-th domain, denoted by \( u^k_l \) of the Schwarz waveform relaxation algorithm is defined by:

\[
\begin{cases}
\frac{\partial u^k_l}{\partial t} - \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2 u^k_l}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u^k_l}{\partial x_i} + cu^k_l = F(t,x,u^k_l), \quad \text{in } \Omega_l \times (0,\infty), \\
\mathcal{B}_{l,l'}u^k_l = \mathcal{B}_{l,l'}u^k_{l'-1}, \quad \text{on } \Gamma_{l,l'} \times (0,\infty), \forall l' \in J_l,
\end{cases}
\]
\( u_i^k(x,t) = g(x,t) \) on \( (\partial \Omega_i \cap \partial \Omega) \times (0, \infty) \), \( u_i^k(x,0) = g(x,0) \) in \( \Omega_i \).

The initial guess \( u^0 \) is bounded in \( C^\infty(\Omega \times (0, \infty)) \); and at step 0, the Eq. (2) is solved with boundary data

\[
\mathcal{B}_{i,n} u^0_i(x,t) = u^0(x,t) \quad \text{on} \quad \Gamma_{i,n} \times (0, \infty), \forall n' \in J_i.
\]

A compatibility condition on \( u^0(x,t) \) is also assumed

\[
\mathcal{B}_{i,n} g(x,0) = u^0(x,0) \quad \text{on} \quad \Gamma_{i,n}, \forall n' \in J_i.
\]

By an induction argument, the algorithm is well-posed. Let \( e_i^k \) be the following

\[
\left\{ \begin{array}{l}
\frac{\partial e_i^k}{\partial t} - \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial^2 e_i^k}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial e_i^k}{\partial x_i} \\
\quad + c(x) e_i^k = F(t,x,u_i^k) - F(t,x,u), \quad \text{in} \quad \Omega_i \times (0, \infty),
\end{array} \right.
\]

\[
\mathcal{B}_{i,n} e_i^k(x,t) = \mathcal{B}_{i,n} e_i^{k-1}(x,t), \quad \text{on} \quad \Gamma_{i,n} \times (0, \infty), \forall n' \in J_i.
\]

Moreover,

\[
e_i^k(x,t) = 0 \text{ on } (\partial \Omega_i \cap \partial \Omega) \times (0, \infty), \quad e_i^k(x,0) = 0 \text{ in } \Omega_i.
\]

For any function \( f \) in \( L^2(0, \infty) \), define

\[
\int_0^\infty f(x) \exp(-yx)dx.
\]

For any fixed positive number \( \alpha \), define

\[
|f|_\alpha = \sup_{\alpha' > \alpha} \left[ \int_{\alpha'}^{\alpha'+1} \left( \int_0^\infty f(x) \exp(-yx)dx \right)^2 dy \right]^{\frac{1}{2}},
\]

and

\[
\mathbb{L}_\alpha^2 (0, \infty) = \{ f : f \in L^2(0, \infty), |f|_\alpha < \infty \}.
\]

Thus, \( \mathbb{L}_\alpha^2 (0, \infty), |.|_\alpha \) is a normed subspace of \( L^2(0, \infty) \).

**Theorem 1.** Consider the Schwarz algorithm with Robin transmission conditions and the initial guess \( u^0 \) in \( C^\infty_c(\Omega \times (0, \infty)) \). There exists a constant \( \alpha \) large enough such that

\[
\lim_{k \to \infty} \sum_{i=1}^{N} \int_{\Omega_i} |e_i^k|_\alpha^2 dx = 0.
\]

**Proof.** Let \( g_i \) be a function bounded and greater than 1 in \( C^\infty(\mathbb{R}^n, \mathbb{R}) \), \( \alpha \) be a positive constant, we define

\[
\Phi_i^k(x) := \left( \int_0^\infty e_i^k \exp(-\alpha t) dt \right) g_i(x),
\]
then \( \Phi^k_l(x) \) belongs to \( H^1(\Omega_l) \). Let \( B'_l \) and \( C' \) be functions in \( L^\infty(\mathbb{R}^n) \) defined by

\[
B'_l := b_l + \sum_{j=1}^n \left( a_{i,j} \frac{\partial g_l}{g_l} \right),
\]

\[
C' = \left[ \frac{\alpha}{2} + \sum_{i,j=1}^n \left( -a_{i,j} \frac{2\partial_i g_l \partial_j g_l}{(g_l)^2} - \partial_j a_{i,j} \frac{\partial_i g_l}{g_l} + a_{i,j} \frac{\partial_i g_l}{g_l} \right) - \sum_{i=1}^n b_i \frac{\partial_i g_l}{g_l} \right].
\]

Define

\[
\mathcal{L}_{lR}(\Phi^k_l) = -\sum_{i,j=1}^n \partial_j(a_{i,j} \partial_i \Phi^k_l) + \sum_{i=1}^n B'_i \partial_i \Phi^k_l + C' \Phi^k_l
\]

\[
+ \left\{ \int_0^\infty \left[ \left( \frac{\alpha}{2} + c \right) e_i^k - F(u_i^k) + F(u) \right] \exp(-\alpha t) dt \right\} g_l.
\]

It is possible to suppose \( \alpha \) to be large such that \( C' \) belongs to \( (\frac{\alpha}{4}, \alpha) \).

**Lemma 1.** Choose \( g_l, g_i' \) such that \( \nabla g_l = \nabla g_i' = 0 \) on \( \Gamma_{l,l'} \) and \( \frac{g_i'}{g_l} > 1 \) on \( \Gamma_{l,l'} \), for all \( l' \in J_l \). \( \Phi^{k+1}_l \) is then a solution of the following equation

\[
\begin{align*}
\mathcal{L}_{lR}(\Phi^{k+1}_l) &= 0, & \text{in } \Omega_l \times (0, \infty), \\
\beta_l \mathcal{B}_{l,l'}(\Phi_l^k) &= \mathcal{B}_{l,l'}(\Phi_l^{k-1}) & \text{on } \Gamma_{l,l'} \times (0, \infty), \forall l' \in J_l.
\end{align*}
\]

(4)

where \( \beta_l = \frac{g_i'}{g_l} \) on \( \Gamma_{l,l'} \), for all \( l' \) in \( J_l \).

For all \( l \) in \( \{1, l \} \), denote by \( \tilde{\Omega}_l \) the open set \( \Omega_l \setminus \bigcup_{l' \in J_l} \Omega_{l'} \). For all \( l \) in \( I \) such that \( \phi^{k+1}_l = \phi^k_l \) on \( \Gamma_{l,l'} \) for all \( l' \) in \( J_l \), let \( \phi^k_l \) and \( \phi^{k+1}_l \) be functions in \( H^1(\tilde{\Omega}_l) \) and \( H^1(\Omega_l) \).

Use the test functions \( \phi^{k+1}_l \) and \( \phi^k_l \), and take the sum (with respect to \( l \) in \( \{1, l \} \)) of

\[
\int_{\Omega_l} \mathcal{L}_{lR}(\phi^{k+1}_l) \phi^{k+1}_l dx + \int_{\Omega_l} \mathcal{L}_{lR}(\phi^k_l) \phi^k_l dx
\]

\[
= \sum_{l=1}^I \left\{ \int_{\Omega_l} C' \Phi^k_l \phi^k_l dx + \right.
\]

\[
+ \int_{\Omega_l} \sum_{i,j=1}^n a_{i,j} \partial_i \Phi^k_l \partial_j \phi^k_l dx + \sum_{i=1}^n \int_{\Omega_l} B'_i \partial_i \Phi^k_l \phi^k_l dx - \sum_{l' \in J_l} \int_{\Gamma_{l,l'}} p_{l,l'} \Phi^k_l \phi^k_l d\sigma
\]

\[
+ \left. \int_{\Omega_l} \left\{ \int_0^\infty \left[ \left( \frac{\alpha}{2} + c \right) e_i^k - F(u_i^k) + F(u) \right] \exp(-\alpha t) dt \right\} g_l \phi^k_l dx \right\}
\]

\[
= \sum_{l=1}^I \left\{ \int_{\Omega_l} C' \Phi^{k+1}_l \phi^{k+1}_l dx + \right.
\]

\[
+ \int_{\Omega_l} \sum_{i,j=1}^n a_{i,j} \partial_i \Phi^{k+1}_l \partial_j \phi^{k+1}_l dx + \sum_{l' \in J_l} \int_{\Gamma_{l,l'}} p_{l,l'} \Phi^{k+1}_l \phi^{k+1}_l d\sigma
\]

\[
+ \left. \int_{\Omega_l} \sum_{i=1}^n B'_i \partial_i \Phi^{k+1}_l \phi^{k+1}_l dx + \right.
\]

\[
+ \int_{\Omega_l} \left\{ \int_0^\infty \left[ \left( \frac{\alpha}{2} + c \right) e_i^{k+1} - F(u_i^{k+1}) + F(u) \right] \exp(-\alpha t) dt \right\} g_l \phi^{k+1}_l dx \right\}.
\]
In (5), choose $\phi_{I}^{k+1}$ to be $\Phi_{I}^{k+1}$, then there exists $\phi_{I}^{k}$, such that for all $l'$ in $J_{I}$ $\phi_{I}^{k} = \phi_{I}^{k+1}$ on $T_{l,l'}$ and

$$\|\phi_{I}^{k}\|_{H^{1}(\Omega_{I})} \leq C \sum_{l' \in J_{I}} \|\phi_{I}^{k+1}\|_{H^{1}(\Omega_{I'})}$$

and

$$\|\phi_{I}^{k}\|_{L^{2}(\Omega_{I})} \leq C \sum_{l' \in J_{I}} \|\phi_{I}^{k+1}\|_{L^{2}(\Omega_{I'})},$$

where $C$ is a positive constant.

The right hand side of (5) is then greater than or equal to

$$\sum_{l=1}^{I} \beta_{l} \left\{ \int_{\Omega_{l}} \lambda_{l} |\nabla \phi_{I}^{k+1}|^{2} dx - \sum_{l=1}^{n} \int_{\Omega_{l}} \|B_{l}^{i}\|_{L^{\infty}(\Omega_{l})} \left| \partial_{i} \phi_{I}^{k+1} \right| \phi_{I}^{k} dx \right\}.$$

Similarly, the left hand side of (5) is less than or equal to

$$\sum_{l=1}^{I} \left\{ \int_{\Omega_{l}} \Lambda |\nabla \phi_{I}^{k}| ||\nabla \phi_{I}^{k}| dx + \sum_{l=1}^{n} \int_{\Omega_{l}} \|B_{l}^{i}\|_{L^{\infty}(\Omega_{l})} \left| \partial_{i} \phi_{I}^{k} \right| \phi_{I}^{k} dx \right\}$$

$$+ \sum_{l'=1}^{I} \left( \|p_{l'}^{i}\|_{L^{\infty}(\Gamma_{l'j})} \left( \|\phi_{I}^{k}\|_{H^{1}(\Omega_{l})}^{2} + \|\phi_{I}^{k}\|_{L^{2}(\Omega_{l})}^{2} \right) \right)$$

$$\leq \sum_{l=1}^{I} \left\{ \frac{1}{2} \left( ||\nabla \phi_{I}^{k}||_{L^{2}(\Omega_{l})}^{2} + \left( \max_{i \in \{1, I\}} \|B_{l}^{i}\|_{L^{\infty}(\Omega_{l})} \right)^{2} \|\phi_{I}^{k}\|_{L^{2}(\Omega_{l})}^{2} \right) \right\}$$

$$+ \int_{\Omega_{l'}} 2 \alpha |\phi_{I}^{k}| \phi_{I}^{k} dx + \sum_{l'=1}^{I} \int_{\Gamma_{l'j}} p_{l'}^{i} \phi_{I}^{k} \phi_{I}^{k} d\sigma$$

$$+ \Lambda \left( \|\nabla \phi_{I}^{k}\|_{L^{2}(\Omega_{l})}^{2} + \|\nabla \phi_{I}^{k}\|_{L^{2}(\Omega_{l})}^{2} \right) + \alpha \|\phi_{I}^{k}\|_{L^{2}(\Omega_{l})}^{2} + \alpha \|\phi_{I}^{k}\|_{L^{2}(\Omega_{l})}^{2},$$

where $M_{l}$ depends only on $\{\Omega_{l}\}_{l \in \{1, I\}}$ and the Eq. (3). Choose $\alpha$ such that $\alpha > (\max_{i \in \{1, I\}} \|B_{l}^{i}\|_{L^{\infty}(\Omega_{l})}^{2}$, there exists $M_{2}$ positive, depending only on $\{\Omega_{l}\}_{l \in \{1, I\}}$ and (3), such that the right hand side of (7) is dominated by

$$\sum_{l=1}^{I} M_{2} \left\{ \int_{\Omega_{l}} \left( \frac{\lambda_{l}}{2} |\nabla \phi_{I}^{k}|^{2} dx + \frac{\alpha}{8} |\phi_{I}^{k}|^{2} + \frac{\lambda_{l}}{2} |\nabla \phi_{I}^{k+1}|^{2} + \frac{\alpha}{8} |\phi_{I}^{k+1}|^{2} \right) \right\}$$

$$\leq \sum_{l=1}^{I} M_{2} \left( \frac{\lambda_{l}}{2} \|\nabla \phi_{I}^{k}\|_{L^{2}(\Omega_{l})}^{2} + \frac{\alpha}{8} \|\phi_{I}^{k}\|_{L^{2}(\Omega_{l})}^{2} + \frac{\lambda_{l}}{2} \|\nabla \phi_{I}^{k+1}\|_{L^{2}(\Omega_{l})}^{2} + \frac{\alpha}{8} \|\phi_{I}^{k+1}\|_{L^{2}(\Omega_{l})}^{2} \right).$$

Define

$$E_{k} := \sum_{l=1}^{I} \left( \frac{\lambda_{l}}{2} \|\nabla \phi_{I}^{k}\|_{L^{2}(\Omega_{l})}^{2} + \frac{\alpha}{8} \|\phi_{I}^{k}\|_{L^{2}(\Omega_{l})}^{2} \right),$$

then (6), (7), and (8) imply
$$\beta - M_2 E_{k+1} \leq M_2 E_k,$$  \hfill (10)

where $\beta = \min \{\beta_1, \ldots, \beta_I\}$.  \hfill (105)

Since $M_2$ depends only on $\{\Omega\}_{l \in \{1, I\}}$ and (3), $\beta$ can be chosen such that

$$M_3 := \frac{M_2}{\beta - M_2} < 1.$$  \hfill (106)

We get

$$E_k \leq M^k E_0$$

$$\leq M^k \sum_{l=1}^I \left[ \frac{\lambda}{2} \left\| \nabla \Phi^0_l \right\|_{L^2(\Omega_l)}^2 + \frac{\alpha}{8} \left\| \Phi^0_l \right\|_{L^2(\Omega_l)}^2 \right],$$

That deduces

$$\left\| \Phi^k_l \right\|_{L^2(\Omega_l)}^2 \leq M^k \sum_{l=1}^I \left( \frac{4\lambda}{\alpha} \left\| \nabla \Phi^0_l \right\|_{L^2(\Omega_l)}^2 + \left\| \Phi^0_l \right\|_{L^2(\Omega_l)}^2 \right).$$  \hfill (11)

Since (11) still holds if $M_3$ and $\lambda$ are fixed, and $\alpha$ is replaced by $\gamma > \alpha$, then

$$\sum_{l=1}^I \int_{\Omega_l} \left( \int_0^\infty e^t \exp(-y t) dt \right) g_l^2 dx$$

$$\leq M^k \left[ \frac{4\lambda}{\gamma} \sum_{l=1}^I \int_{\Omega_l} \left( \int_0^\infty |\nabla e^t_l| \exp(-y t) dt \right)^2 g_l^2 dx + \frac{4\lambda}{\gamma} \sum_{l=1}^I \int_{\Omega_l} \left( \int_0^\infty e^t_l \exp(-y t) dt \right)^2 g_l^2 dx \right].$$

Let $\alpha'$ be a constant larger than or equal to $\alpha$, (12) implies

$$\sum_{l=1}^I \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \left( \int_0^\infty e^t \exp(-y t) dt \right)^2 g_l^2 dy dx$$

$$\leq M^k \left[ \sum_{l=1}^I \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \frac{4\lambda}{y} \left( \int_0^\infty |\nabla e^t_l| \exp(-y t) dt \right)^2 g_l^2 dy dx + \frac{4\lambda}{y} \sum_{l=1}^I \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \left( \int_0^\infty e^t_l \exp(-y t) dt \right)^2 g_l^2 dy dx \right].$$

Since $u^0$ belongs to $C_c^\infty(\overline{\Omega} \times (0, \infty))$, the right hand side of (13) is bounded by a constant $M^k_3 M_4(\alpha)$. The fact that $g_l$ is greater than 1 implies
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\[ \sum_{l=1}^{I} \int_{\Omega_l} \left( \int_{-\infty}^{\infty} e_{l}^{k} \exp(-yt)dt \right)^{2} dydx \leq M_{3}^{k} M_{4}(\alpha). \]  

(14)

Inequality (14) deduces

\[ \lim_{k \to \infty} \sum_{l=1}^{I} \int_{\Omega_l} |e_{l}^{k}|^{2} dx = 0. \]  

(15)

Acknowledgments The author would like to express his gratitude to his thesis advisor, Professor Laurence Halpern for her very kind help and support. He is also indebted to the editor for his kind help. The author has been partially supported by the ERC Advanced Grant FP7-246775 NUMERIWAVES.

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