A Two-Level Additive Schwarz Preconditioner for C^0 Interior Penalty Methods for Cahn-Hilliard Equations

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Summary. We study a two-level additive Schwarz preconditioner for C^0 interior penalty methods for a biharmonic problem with essential and natural boundary conditions with Cahn-Hilliard type. We show that the condition number of the preconditioned system is bounded by $C(1+(H^3/\delta^3))$, where H is the typical diameter of a subdomain, δ measures the overlap among the subdomains, and the positive constant C is independent of the mesh sizes and the number of subdomains.

1 Introduction

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$, and $\mathbb{V} = \{v \in H^2(\Omega) : \partial v/\partial n = 0$ on $\partial\Omega\}$, where $\partial/\partial n$ denotes the outward normal derivative. Consider the following model problem which is the weak form of the biharmonic problem with boundary conditions of Cahn-Hilliard type:

Find $u \in H^2(\Omega)$ such that

$$a(u,v) = (f,v) \qquad \forall v \in \mathbb{V}, \tag{1}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{2}$$

where $f \in L_2(\Omega)$, $(\cdot,\cdot)$ is the $L_2(\Omega)$ inner product, and

$$a(w,v) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}\, dx$$

is the inner product of the Hessian matrices of w and v.

Let p_* be a corner of Ω, and

$$\mathbb{V}^* = \{v \in \mathbb{V} : v(p_*) = 0\}.$$

Then by elliptic regularity [1], the unique solution $u \in V^*$ of our model problem belongs to $H^{2+\alpha}(\Omega)$, where $0 < \alpha \le 2$ is the index of elliptic regularity.

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C^0 interior penalty methods are discontinuous Galerkin methods for fourth order problems. These approaches for our model problem have recently been analyzed in [5]. Let $\mathscr{T}_h$ be a simplicial or convex quadrilateral triangulation of Ω, and V_h be a Lagrange (triangular or tensor product) finite element space associated with $\mathscr{T}_h$. Let

$$V_h^* = \{v \in V_h : v(p_*) = 0\}.$$

Then the C^0 interior penalty method for (1) and (2) is to find $u_h \in V_h^*$ such that

$$\mathscr{A}_h(u_h, v) = (f, v) \qquad \forall v \in V_h^*, \tag{3}$$

where for $w, v \in V_h^*$,

$$\begin{aligned}\mathscr{A}_h(w,v) = &\sum_{D\in\mathscr{T}_h}\sum_{i,j=1}^{2}\int_D \frac{\partial^2 w}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j}\,dx + \sum_{e\in\mathscr{E}_h}\frac{\eta}{|e|}\int_e \left[\!\left[\frac{\partial w}{\partial n}\right]\!\right]\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds\\ &+ \sum_{e\in\mathscr{E}_h}\int_e \left(\left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] + \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial w}{\partial n}\right]\!\right]\right) ds,\end{aligned} \tag{4}$$

$\mathscr{E}_h$ denotes the set of edges of the triangulation $\mathscr{T}_h$, and η is a penalty parameter. The jumps and averages are defined as follows.

For interior edges $e \in \mathscr{E}_h$ shared by two elements $D_\pm \in \mathscr{T}_h$, we take n_e to be the unit normal of e pointing from D_- into D_+, and define

$$\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] = \frac{\partial v_+}{\partial n_e} - \frac{\partial v_-}{\partial n_e} \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_+}{\partial n_e^2} + \frac{\partial^2 v_-}{\partial n_e^2}\right),$$

where $v_\pm = v\big|_{D_\pm}$. Note that the definitions of $[\![\partial v/\partial n]\!]$ and $\{\!\{\partial^2 v/\partial n^2\}\!\}$ are independent of the choice of e.

For $e \in \mathscr{E}_h$ which is on the boundary of Ω, we take n_e to be the unit normal of e pointing outside Ω and define

$$\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] = -\frac{\partial v}{\partial n_e} \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} = \frac{\partial^2 v}{\partial n_e^2}.$$

Remark 1. The discrete problem (3) resulting from the C^0 interior penalty method is consistent, and for the penalty parameter η large enough, it is also stable [3].

For fourth order problems, C^0 interior penalty methods have certain advantages over classical finite element methods. However, due to the nature of fourth order problems, the discrete system resulting from the C^0 interior penalty method is very ill-conditioned. Therefore, it is necessary to develop modern fast solvers to overcome this drawback. In this paper, we construct a two-level additive Schwarz preconditioner and extend the results in [4] for biharmonic problems with essential Dirichlet boundary conditions to the ones with the essential and natural boundary conditions.

The rest of this paper is organized as follows. We first introduce the framework of a two-level additive Schwarz preconditioner in Sect. 2, followed by the condition number estimates of the preconditioned system in Sect. 3. Section 4 demonstrates some numerical results.

2 A Two-Level Additive Schwarz Preconditioner

For simplicity, we will focus on the case where $\mathscr{T}_h$ is a rectangular mesh. The results obtained in this paper are also true for triangular and general convex quadrilateral meshes.

Let $V_h^* = \{v : v \in C(\bar{\Omega}), v(p_*) = 0, v_D = v\big|_D = \mathbb{Q}_2(D)\ \forall D \in \mathscr{T}_h\}$ be the standard quadratic Lagrange finite element space associated with $\mathscr{T}_h$, and the operator $A_h : V_h^* \longrightarrow V_h^{*'}$ can then be defined by

$$\langle A_h v, w\rangle = \mathscr{A}_h(v,w) \qquad \forall v,w \in V_h^*,$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form between a vector space and its dual.

Note that for η sufficiently large, the following relation [3] is true.

$$C_1|v|^2_{H^2(\Omega,\mathscr{T}_h)} \le \langle A_h v, v\rangle \le C_2|v|^2_{H^2(\Omega,\mathscr{T}_h)} \qquad \forall v \in V_h^*,$$

where

$$|v|^2_{H^2(\Omega,\mathscr{T}_h)} = \sum_{D\in\mathscr{T}_h} |v|^2_{H^2(D)} + \sum_{e\in\mathscr{E}_h} \frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)},$$

and the constants C_1 and C_2 depend only on the shape regularity of $\mathscr{T}_h$.

We now construct a two-level additive Schwarz preconditioner for the operator A_h which involves a coarse grid solve and subdomain solves.

First of all, let $\mathscr{T}_H$ be a coarse rectangular mesh for Ω, and $V_0 \subset H^1(\Omega)$ be the $\mathbb{Q}_1$ finite element space associated with $\mathscr{T}_H$. We define $A_0 : V_0^* \longrightarrow V_0^{*'}$ by

$$\langle A_0 v, w\rangle = \mathscr{A}_H(v,w) \qquad \forall v,w \in V_0^*,$$

where $\mathscr{A}_H$ is the analog of $\mathscr{A}_h$ for the coarse grid $\mathscr{T}_H$, and $V_0^* = \{v : v \in V_0, v(p_*) = 0\}$.

Let $\Omega_j, 1 \le j \le J$, be overlapping subdomains of Ω such that $\Omega = \cup_{j=1}^J \Omega_j$, and the boundaries of Ω_j are aligned with the edges of $\mathscr{T}_h$. We assume that there exist nonnegative $\theta_j \in C^\infty(\bar{\Omega})$ for $1 \le j \le J$ such that

$$\theta_j = 0 \qquad \text{on} \quad \Omega\backslash\Omega_j,$$

$$\sum_{j=1}^J \theta_j = 1 \qquad \text{on} \quad \bar{\Omega},$$

$$\|\nabla\theta_j\|_{L_\infty(\Omega)} \le \frac{C}{\delta}, \qquad \|\nabla^2\theta_j\|_{L_\infty(\Omega)} \le \frac{C}{\delta^2},$$

where $\nabla^2\theta_j$ is the Hessian of θ_j, $\delta > 0$ measures the overlap among the subdomains, and C is a positive constant independent of h, H and J.

Remark 2. Suppose $\mathscr{T}_h$ is a refinement of $\mathscr{T}_H$. We can construct Ω_j by enlarging the elements of $\mathscr{T}_H$ by the amount of δ so that the boundaries of $\Omega_j, 1 \le j \le J$, are aligned with the edges of $\mathscr{T}_h$ (cf. Fig. 1). The construction of $\theta_j, 1 \le j \le J$, is then standard.

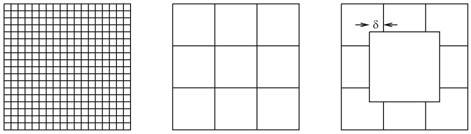

Fig. 1. $\mathscr{T}_h, \mathscr{T}_H$ and Ω_j

Moreover, we assume that the maximum number of subdomains Ω_j that share a common point is bounded by a constant N_c.

Let $V_j = \{v : v \in V_h^*, v = 0 \text{ on } \bar{\Omega}_\ell \text{ if } \ell \neq j\}$ be the $\mathbb{Q}_2$ finite element space associated with $\mathscr{T}_h$ on $\bar{\Omega}_j$. Then we define the operator $A_j : V_j \longrightarrow V_j'$ by

$$\langle A_j v, w \rangle = \mathscr{A}_j(v, w) \qquad \forall v, w \in V_j,$$

where $\mathscr{A}_j, 1 \leq j \leq J$, are the analogs of $\mathscr{A}_h$ restricted on $\bar{\Omega}_j$. Similarly, we obtain that

$$C_3 |v|^2_{H^2(\Omega_j, \mathscr{T}_h)} \leq \langle A_j v, v \rangle \leq C_4 |v|^2_{H^2(\Omega_j, \mathscr{T}_h)} \qquad \forall v \in V_j,$$

where

$$|v|^2_{H^2(\Omega_j, \mathscr{T}_h)} = \sum_{\substack{D \in \mathscr{T}_h \\ D \subset \Omega_j}} |v|^2_{H^2(D)} + \sum_{\substack{e \in \mathscr{E}_h \\ e \subset \bar{\Omega}_j}} \|[\![\partial v / \partial n]\!]\|^2_{L_2(e)},$$

and C_3, C_4 are constants independent of h, H, J, N_c and δ.

For simplicity, from now on, we will use C to denote a generic positive constant independent of h, H, δ, and J that will take different values in different occurrences.

The subdomain finite element space $V_j, 1 \leq j \leq J$, is connected to V_h^* by the natural injection operator I_j which satisfies the following inequality.

$$|I_j v|_{H^2(\Omega, \mathscr{T}_h)} \leq C |v|_{H^2(\Omega_j, \mathscr{T}_h)} \qquad \forall v \in V_j.$$

Furthermore, the coarse space V_0^* and the fine space V_h^* are connected by the operator I_0 which is defined as follows.

Let $\tilde{V}_0 \subset H^2(\Omega)$ be the $\mathbb{Q}_3$ Bogner-Fox-Schmit finite element space associated with $\mathscr{T}_H$, and $\tilde{V}_0^* = \{v : v \in \tilde{V}_0, v(p_*) = 0\}$. The $\mathbb{Q}_1$ Lagrange element and the $\mathbb{Q}_3$ Bogner-Fox-Schmit element are depicted in Fig. 2, where we use the solid dot • to denote pointwise evaluation of the shape functions, the circle ∘ and the arrow ⬀ to denote pointwise evaluation of all the first order derivatives and the mixed second order derivative of the shape functions, respectively.

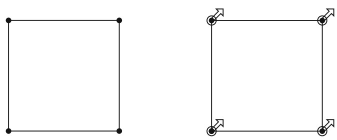

Fig. 2. $\mathbb{Q}_1$ element and $\mathbb{Q}_3$ Bogner-Fox-Schmit element

We define $E_H : V_0^* \longrightarrow \tilde{V}_0^*$ to be the operator that for all $p \in \mathscr{T}_H$,

$$
\begin{aligned}
(E_H v)(p) &= v(p), \\
\nabla(E_H v)(p) &= \begin{cases} \dfrac{1}{|\mathscr{T}_p|} \displaystyle\sum_{D \in \mathscr{T}_p} \nabla v_D(p), & \text{if } p \in \Omega, \\ 0, & \text{if } p \in \partial\Omega, \end{cases} \\
\frac{\partial^2 (E_H v)}{\partial x_1 \partial x_2}(p) &= \begin{cases} \dfrac{1}{|\mathscr{T}_p|} \displaystyle\sum_{D \in \mathscr{T}_p} \frac{\partial^2 v_D}{\partial x_1 \partial x_2}(p), & \text{if } p \in \Omega, \\ 0, & \text{if } p \in \partial\Omega, \end{cases}
\end{aligned}
$$

where $\mathscr{T}_p$ is the set of rectangles in $\mathscr{T}_H$ sharing p as a vertex, $|\mathscr{T}_p|$ is the number of elements in $\mathscr{T}_p$, and $v_D = v\big|_D$.

Then for all $v \in V_0^*$, we take $I_0 v \in V_h^*$ to be the one whose nodal values are identical with the corresponding nodal values of $E_H v$.

Remark 3. Instead of using the operator E_H, if we define the operator I_0 as the natural injection operator from V_0^* to V_h^*, then the performance of the preconditioner will be affected by the different scalings that appear in the penalty terms for $\mathscr{A}_h$ and $\mathscr{A}_H$. However, this problems can be avoided by defining I_0 as above since $E_H v \in H^2(\Omega)$.

We can now define the two-level additive Schwarz preconditioner $B : V_h^{*'} \longrightarrow V_h^*$ by

$$B = \sum_{j=0}^{J} I_j A_j^{-1} I_j^t,$$

where $I_j^t : V_h^{*'} \longrightarrow V_j'$ is the transpose of I_j, i.e.,

$$\langle I_j^t \Psi, v \rangle = \langle \Psi, I_j v \rangle \qquad \forall \Psi \in V_h^{*'}, v \in V_j.$$

From the additive Schwarz theory [2, 6], the preconditioner B is symmetric positive definite and therefore the eigenvalues of BA_h are positive. Moreover, the maximum and minimum eigenvalues of BA_h are given by the following formulas, which will be used to estimate the condition number of the preconditioned system.

$$
\lambda_{\max}(BA_h) = \max_{\substack{v \in V_h \\ v \neq 0}} \frac{\langle A_h v, v \rangle}{\displaystyle\min_{\substack{v = \sum_{j=0}^J I_j v_j \\ v_j \in V_j}} \sum_{j=0}^{J} \langle A_j v_j, v_j \rangle},
$$

$$
\lambda_{\min}(BA_h) = \min_{\substack{v \in V_h \\ v \neq 0}} \frac{\langle A_h v, v \rangle}{\displaystyle\min_{\substack{v = \sum_{j=0}^J I_j v_j \\ v_j \in V_j}} \sum_{j=0}^{J} \langle A_j v_j, v_j \rangle}.
$$

3 Condition Number Estimates

From the construction of our two-level additive Schwarz preconditioner, by the similar arguments as we did in [4], it is not difficult to derive the following results on the estimates of the eigenvalues of the preconditioned system.

Theorem 1. *The following upper bound for the eigenvalues of BA_h holds:*

$$\lambda_{\max}(BA_h) \le C,$$

where the positive constant C depends on the shape regularity of $\mathscr{T}_h$ and $\mathscr{T}_H$ but not h, H, δ nor J.

Theorem 2. *The following lower bound for the eigenvalues of BA_h holds:*

$$\lambda_{\min}(BA_h) \ge C\left(1 + \frac{H^4}{\delta^4}\right),$$

where the positive constant C depends on the shape regularity of $\mathscr{T}_h$ and $\mathscr{T}_H$ but not h, H, δ nor J.

Finally, from Theorems 1 and 2, the following estimate on the condition number of the preconditioned system can be obtained immediately.

Theorem 3. *It holds that*

$$\kappa(BA_h) = \frac{\lambda_{\max}(BA_h)}{\lambda_{\min}(BA_h)} \le C\left(1 + \frac{H^4}{\delta^4}\right),$$

where the positive constant C depends on the shape regularity of $\mathscr{T}_h$ and $\mathscr{T}_H$ but not h, H, δ nor J.

Remark 4. In the case of a small overlap, i.e. $\delta \ll H$, the estimate on the condition number of the preconditioned system can be improved to $(1 + (H/\delta)^3)$, provided with more assumptions on the subdomains Ω_j [4].

4 Numerical Results

In this section, we present some numerical results for the biharmonic problem with Cahn-Hilliard type of boundary conditions on the unit square. We choose the penalty parameter in $\mathscr{A}_h, \mathscr{A}_H$ and $\mathscr{A}_j$ to be 5, which guarantees the coerciveness of the variational form (4) on V_h^*.

First of all, for different choices of H and h, we generate a vector $v_h \in V_h^*$, compute the right-hand side vector $g = A_h v_h$, and apply the preconditioned conjugate gradient algorithm to the system $A_h z = g$ using our two-level additive Schwarz preconditioner. We compute the iteration numbers needed for reducing the energy norm error by a factor of 10^{-6} for five random choices of v_h and then average them. The numbers are collected in Tables 1 and 2. Also, to illustrate the practical performance of our preconditioner, such iteration numbers needed for reducing the energy norm error by a factor of 10^{-2} with 16 subdomains are reported in Table 3. They show that the bound for the condition number of BA_h is independent of h.

We also compute, in the case of 4 and 16 subdomains, the maximum eigenvalue, the minimum eigenvalue, and the condition number of the preconditioned system for the fine mesh $h = 2^{-6}$ and various overlaps among subdomains by using Lanczos methods. The results are tabulated in Tables 4 and 5. They show that the maximum eigenvalue is bounded and the minimum eigenvalue increases as the overlap among subdomains decreases.

Table 1. Average number of iterations for reducing the energy norm error by a factor of 10^{-6} with $H = 1/2$ and $J = 4$

	$h = 2^{-2}$	$h = 2^{-3}$	$h = 2^{-4}$	$h = 2^{-5}$	$h = 2^{-6}$
$\delta = 2^{-2}$	17	17	17	15	15
$\delta = 2^{-3}$	-	20	20	19	17
$\delta = 2^{-4}$	-	-	26	25	24
$\delta = 2^{-5}$	-	-	-	47	45
$\delta = 2^{-6}$	-	-	-	-	93

Table 2. Average number of iterations for reducing the energy norm error by a factor of 10^{-6} with $H = 1/4$ and $J = 16$

	$h = 2^{-3}$	$h = 2^{-4}$	$h = 2^{-5}$	$h = 2^{-6}$
$\delta = 2^{-3}$	27	29	27	24
$\delta = 2^{-4}$	-	28	26	24
$\delta = 2^{-5}$	-	-	42	39
$\delta = 2^{-6}$	-	-	-	83

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Table 3. Average number of iterations for reducing the energy norm error by a factor of 10^{-2} with $H = 1/4$ and $J = 16$

	$h = 2^{-3}$	$h = 2^{-4}$	$h = 2^{-5}$	$h = 2^{-6}$
$\delta = 2^{-3}$	6	6	5	5
$\delta = 2^{-4}$	-	5	5	4
$\delta = 2^{-5}$	-	-	5	4
$\delta = 2^{-6}$	-	-	-	5

Table 4. $\lambda_{\max}(BA_h), \lambda_{\min}(BA_h)$ and $\kappa(BA_h)$ with $H = 1/2, h = 2^{-6}$ and $J = 4$

H/δ	$\lambda_{\max}(BA_h)$	$\lambda_{\min}(BA_h)$	$\kappa(BA_h)$
2	4.8394	0.4259	1.1363×10^1
4	4.8029	0.3045	1.5775×10^1
8	4.7526	0.1279	3.7149×10^1
16	4.6600	0.0247	1.8850×10^2
32	4.5849	0.0036	1.2895×10^3

Table 5. $\lambda_{\max}(BA_h), \lambda_{\min}(BA_h)$ and $\kappa(BA_h)$ with $H = 1/4, h = 2^{-6}$ and $J = 16$

H/δ	$\lambda_{\max}(BA_h)$	$\lambda_{\min}(BA_h)$	$\kappa(BA_h)$
2	6.5195	0.1811	3.5992×10^1
4	4.8740	0.1633	2.9852×10^1
8	4.6968	0.0631	7.4402×10^1
16	4.5865	0.0103	4.4698×10^2

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