Preconditioning for Mixed Finite Element Formulations of Elliptic Problems

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Summary. In this paper, we discuss a preconditioning technique for mixed finite element discretizations of elliptic equations. The technique is based on a block-diagonal approximation of the mass matrix which maintains the sparsity and positive definiteness of the corresponding Schur complement. This preconditioner arises from the multipoint flux mixed finite element method and is robust with respect to mesh size and is better conditioned for full permeability tensors than a preconditioner based on a diagonal approximation of the mass matrix.

1 Introduction

Consider the mixed formulation of a second order linear elliptic equation. Introducing a flux variable, we solve for a scalar potential p and a vector function $\mathbf{u}$ that satisfy

$$\mathbf{u} = -\mathbb{K}\nabla p \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega, \tag{2}$$

$$p = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where Ω is a polygonal domain with Lipschitz continuous boundary and $\mathbb{K}$ is a symmetric and uniformly positive definite tensor with $L^\infty(\Omega)$ components. Homogeneous Dirichlet boundary conditions are considered for the simplicity of the presentation.

Mixed finite element methods lead to the non-singular indefinite system:

$$\mathbb{M}\begin{pmatrix} U \\ P \end{pmatrix} := \begin{pmatrix} \mathbb{A} & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix}\begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}, \tag{4}$$

where the matrix $\mathbb{A}$ is a symmetric and positive definite.

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In this paper, we consider preconditioners of the form:

$$\widetilde{\mathbb{M}} := \begin{pmatrix} \widetilde{\mathbb{A}} & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix}. \quad (5)$$

The applicability of this type preconditioner is due to the fact that

- $\widetilde{\mathbb{A}}$ is easily invertible.
- The Schur complement of the preconditioner $\widetilde{M}$ is sparse and positive definite, and can be solved easily.

One way is choosing $\widetilde{\mathbb{A}}$ as a diagonal matrix. In [1], $\widetilde{\mathbb{A}}$ is given as $\omega\mathbb{I}$. The global parameter ω is chosen to minimize the spectral radius of $\mathbb{I} - \widetilde{\mathbb{M}}^{-1}\mathbb{M}$. In [5], the diagonal matrix is optimally scaled at element level and a precise upper bound of the spectral radius has been shown: $\rho(\mathbb{I} - \widetilde{\mathbb{M}}^{-1}\mathbb{M}) \leq 1/2$. In other words, the preconditioner is independent of both the mesh size and the tensor $\mathbb{K}$. This uniformity is derived when the problem has a diagonal $\mathbb{K}$ and is discretized by the lowest order Raviart-Thomas [8] mixed finite element on rectangular grids. For other mixed finite element spaces or full tensor $\mathbb{K}$, the uniformity result is not clearly understood. Alternatively, a simple parameter-free choice for $\widetilde{\mathbb{A}}$, $\widetilde{\mathbb{A}} = \mathrm{Diag}(\mathbb{A})$, can be used.

Another approach is to take $\widetilde{\mathbb{A}}$ as a block-diagonal matrix which guarantees that the corresponding Schur complement matrix is sparse and positive definite. Multipoint flux mixed finite element (MFMFE) methods [6, 9–12] give matrices of the form (5), where the flux variable can be locally eliminated due to the block-diagonal structure of $\widetilde{\mathbb{A}}$. The corresponding Schur complement gives a cell-centered stencil for the scalar variable. In this paper, we study the performance of this MFMFE operator as a preconditioner. The Schur complement of MFMFE has a 9-point stencil on logically rectangular grids and with full tensor $\mathbb{K}$ in contrast to 5-point stencil which arises if $\widetilde{\mathbb{A}}$ is a diagonal matrix. Our numerical result indicates that the MFMFE method gives a better preconditioner than the diagonal preconditioner ($\widetilde{\mathbb{A}} = \mathrm{Diag}(\mathbb{A})$). A natural extension of this work is the use of approximate preconditioners based on algebraic multigrid for MFMFE as described in [2, 7] and will be the subject of future work.

The rest of the paper is organized as follows. Mixed finite element formulation is described in Sect. 2. A block type preconditioner is discussed in Sect. 3. Finally in Sect. 4, numerical experiments are given.

2 Mixed Finite Element Formulation

Define $H(\mathrm{div};\Omega) := \left\{\mathbf{v} \in (L^2(\Omega))^d : \nabla\cdot\mathbf{v} \in L^2(\Omega)\right\}$ and let $(\cdot,\cdot)$ denote the inner product in $L^2(\Omega)$. Let $X \lesssim (\gtrsim)\ Y$ denote that there exists a constant C, independent of the mesh size h, such that $X \leq (\geq)\ CY$. The notation $X \eqsim Y$ means that both $X \lesssim Y$ and $X \gtrsim Y$ hold.

Let $\mathscr{T}_h$ be a finite element partition of the domain Ω consisting of either triangles or quadrilaterals. We assume that $\mathscr{T}_h$ is shape-regular in the sense of Ciarlet [4].

The finite element spaces on any physical element $E \in \mathscr{T}_h$ are defined via the Piola transformation

$$\mathbf{v} \leftrightarrow \hat{\mathbf{v}} : \hat{\mathbf{v}} = \frac{1}{J_E}\mathbb{DF}_E \hat{\mathbf{v}} \circ F_E^{-1},$$

and the scalar transformation

$$w \leftrightarrow \hat{w} : w = \hat{w} \circ F_E^{-1},$$

where F_E denotes a mapping from the reference element $\hat{E}$ to the physical element E, $\mathbb{DF}_E$ is the Jacobian of F_E, and J_E is its determinant. The finite element spaces V_h and W_h on $\mathscr{T}_h$ are given by

$$V_h = \left\{\mathbf{v} \in H(\mathrm{div};\Omega) : \quad \mathbf{v}|_E \leftrightarrow \hat{\mathbf{v}},\ \hat{\mathbf{v}} \in \hat{V}(\hat{E}), \quad \forall E \in \mathscr{T}_h\right\},$$
$$W_h = \left\{w \in L^2(\Omega) : \quad w|_E \leftrightarrow \hat{w},\ \hat{w} \in \hat{W}(\hat{E}), \quad \forall E \in \mathscr{T}_h\right\},$$

where $V(\hat{E})$ and $\hat{W}(\hat{E})$ are the lowest order Brezzi-Douglas-Marini (BDM_1) spaces on the reference element $\hat{E}$. Definitions of Piola transformation and BDM_1 spaces yield $V_h \subset H(\mathrm{div};\Omega)$ and $W_h \subset L^2(\Omega)$.

The finite element method reads: find $\mathbf{u}_h \in V_h$ and $p_h \in W_h$, such that

$$(\mathbb{K}^{-1}\mathbf{u}_h, \mathbf{v}) - (p_h, \nabla \cdot \mathbf{v}) = 0, \qquad \forall \mathbf{v} \in V_h, \tag{6}$$
$$-(\nabla \cdot \mathbf{u}_h, w) = -(f, w) \quad \forall w \in W_h. \tag{7}$$

The method (6) and (7) can have a second order convergence for the flux and first order convergence for the scalar potential [3] if $\mathbf{u}$ and p are sufficiently regular.

3 Preconditioning the Mixed Finite Element System

3.1 Multipoint Flux Mixed Finite Element

A family of multipoint flux mixed finite element (MFMFE) methods on various grids has been developed and analyzed [6, 9–12]. The method is defined as: find $\mathbf{u}_h \in V_h$ and $p_h \in W_h$, such that

$$(\mathbb{K}^{-1}\mathbf{u}_h, \mathbf{v})_Q - (p_h, \nabla \cdot \mathbf{v}) = 0, \qquad \forall \mathbf{v} \in V_h, \tag{8}$$
$$-(\nabla \cdot \mathbf{u}_h, w) = -(f, w) \quad \forall w \in W_h, \tag{9}$$

where the finite element spaces are BDM_1 on triangular and rectangular meshes. Compared to the BDM_1 finite element method, a specific numerical quadrature rule is employed. It is defined as:

$$(\mathbb{K}^{-1}\mathbf{q}, \mathbf{v})_Q = \sum_{E \in \mathscr{T}_h} (\mathbb{K}^{-1}\mathbf{q}, \mathbf{v})_{Q,E} \equiv \sum_{E \in \mathscr{T}_h} \mathrm{Trap}\left(\mathscr{K}\hat{\mathbf{q}}, \hat{\mathbf{v}}\right)_{\hat{E}}, \tag{10}$$

where $\mathscr{K}$ on each $\hat{E}$ is defined as

$$\mathscr{K} = \frac{1}{J_E} \mathbb{DF}_E^T \mathbb{K}^{-1}(F_E(\hat{x}))\mathbb{DF}_E, \tag{11}$$

and the trapezoidal rule on $\hat{E}$ is denoted as

$$\text{Trap}(\hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{E}} \equiv \frac{|\hat{E}|}{m} \sum_{i=1}^{m} \hat{\mathbf{q}}(\hat{\mathbf{r}}_i) \cdot \hat{\mathbf{v}}(\hat{\mathbf{r}}_i), \tag{12}$$

with $\{\hat{\mathbf{r}}_i\}_{i=1}^m$ being vertices of $\hat{E}$ and m being the number of vertices of $\hat{E}$.

The degrees of freedom for the flux variable are chosen as the normal components at two vertices on each edge. More specifically, denote the basis functions associated with $\hat{\mathbf{r}}_i$ by $\hat{\mathbf{v}}_{ij}, j = 1, 2$: $(\hat{\mathbf{v}}_{ij} \cdot \hat{\mathbf{n}}_{ij})(\hat{\mathbf{r}}_i) = 1$, $(\hat{\mathbf{v}}_{ij} \cdot \hat{\mathbf{n}}_{ik})(\hat{\mathbf{r}}_i) = 0, k \neq j$, and $(\hat{\mathbf{v}}_{ij} \cdot \hat{\mathbf{n}}_{lk})(\hat{\mathbf{r}}_l) = 0, l \neq i$, k = 1, 2. As a consequence, the quadrature rule (10) couples only the two basis functions associated with a vertex. For example, on the unit square

$$\begin{aligned} &(\mathscr{K}\hat{\mathbf{v}}_{11}, \hat{\mathbf{v}}_{11})_{\hat{Q},\hat{E}} = \frac{\mathscr{K}_{11}(\hat{\mathbf{r}}_1)}{4}, \quad (\mathscr{K}\hat{\mathbf{v}}_{11}, \hat{\mathbf{v}}_{12})_{\hat{Q},\hat{E}} = \frac{\mathscr{K}_{21}(\hat{\mathbf{r}}_1)}{4}, \\ &(\mathscr{K}\hat{\mathbf{v}}_{11}, \hat{\mathbf{v}}_{ij})_{\hat{Q},\hat{E}} = 0, \ \ i \neq 1, j = 1, 2. \end{aligned} \tag{13}$$

where $\mathscr{K}_{ij}$ denotes i-th row and j-th column of the matrix function $\mathscr{K}$. This localization property on interactions between the flux basis functions gives the assembled mass matrix in (8) has a block diagonal structure with one block per grid vertex.

We denote the algebraic system arising from (8) and (9) as

$$\begin{pmatrix} \mathbb{A}_Q & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}, \tag{14}$$

where $\mathbb{A}_Q$ is block diagonal. The approximate flux, U, can be easily eliminated via

$$U = -\mathbb{A}_Q^{-1}\mathbb{B}^T P. \tag{15}$$

The resulting Schur complement system

$$\mathbb{B}\mathbb{A}_Q^{-1}\mathbb{B}^T P = -F, \tag{16}$$

is symmetric positive definite and sparse. On rectangular grids, Eq. (16) has a 5-point stencil for a diagonal tensor $\mathbb{K}$ and 9-point stencil for the full tensor. The Schur complement system can be solved using classical algebraic multigrid methods. The flux variable is then obtained easily by (15) due to the block diagonal structure of $\mathbb{A}_Q$.

The following result concerns the convergence of the MFMFE methods. Let $W^{k,\infty}_{\mathscr{T}_h}$ consist of functions ϕ such that $\phi|_E \in W^{k,\infty}(E)$ for all $E \in \mathscr{T}_h$.

Theorem 1 ([6, 10–12]). *Let $\mathscr{T}_h$ consist of simplices, h^2-parallelograms, h^2-parallelepipeds or triangular prisms. If $\mathbb{K}^{-1} \in W^{1,\infty}_{\mathscr{T}_h}$, then, the flux $\mathbf{u}_h$ and scalar p_h of the MFMFE method* (8)–(9) *satisfies*

$$\|\mathbf{u} - \mathbf{u}_h\| \lesssim h\|\mathbf{u}\|_1, \ \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\| \lesssim h\|\nabla \cdot \mathbf{u}\|_1, \ \|p - p_h\| \lesssim h(\|\mathbf{u}\|_1 + \|p\|_1).$$

Compared to the second order L^2 convergence of the flux variable in the BDM_1 mixed method, the MFMFE has a first order convergence for the flux variable due to the numerical quadrature. However the MFMFE method is a solver friendly scheme since the MFMFE method can be reduced to a cell-centered stencil in terms of the scalar variable without solving a saddle-point problem.

3.2 Multipoint Flux Mixed Finite Element as a Preconditioner

The MFMFE method may be used as a preconditioner to the BDM_1 mixed finite element method by choosing $\widetilde{\mathbb{A}} = \mathbb{A}_Q$.

Lemma 1. *The condition number of* $\widetilde{\mathbb{A}}^{-1}\mathbb{A}$ *is independent of the mesh size.*

Proof. It has been shown [6, 11, 12] that the bilinear form $(\mathbb{K}^{-1}\cdot,\cdot)_Q$ is an inner product in $\mathbf{V}_h$ and $(\mathbb{K}^{-1}\mathbf{q},\mathbf{q})_Q^{1/2}$ is a norm equivalent to the L^2 norm. Thus

$$(\mathbb{K}^{-1}\mathbf{q},\mathbf{q})_Q \approx \|\mathbf{q}\|^2 \approx (\mathbb{K}^{-1}\mathbf{q},\mathbf{q}), \quad \forall \mathbf{q} \in \mathbf{V}_h. \qquad \square \tag{17}$$

The preconditioner of the form (5) has been analyzed by Ewing, Lazarov, Lu and Vassilevski.

Theorem 2 ([5]). *The eigenvalues of* $\widetilde{\mathbb{M}}^{-1}\mathbb{M}$ *are real and positive and lie in the interval* $[\lambda_{min},\lambda_{max}]$*, where* λ_{min} *and* λ_{max} *are the extreme eigenvalues of* $\widetilde{\mathbb{A}}^{-1}\mathbb{A}$*.*

By Lemma 1 and Theorem 2, we have the following corollary.

Corollary 1. *The preconditioned system of* BDM_1 *mixed finite element method with MFMFE as a preconditioner is positive definite. The condition number is independent of the mesh size.*

4 Numerical Results

4.1 Example 1

In this example, we consider (1)–(3) on the computational domain shown in Fig. 1 (left) with $p = 0$ on $\partial\Omega$ and $f = 1$.

First, we use the MFMFE method as a preconditioner for the BDM_1 mixed finite element method with $\mathbb{K} = \mathbb{I}$. The result is presented in Table 1 where we can clearly see that the preconditioner is robust with respect to the mesh size h. Next, we consider the heterogeneous permeability field shown in Fig. 1 (right) which is generated using geostatistical techniques (kriging) with a longer correlation length in the horizontal direction. In Table 2 we see that the preconditioner is not only robust with respect to mesh size, but also with respect to the heterogeneities in the permeability.

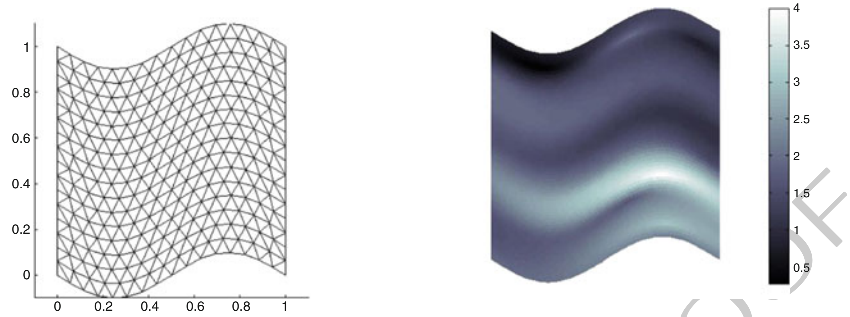

Fig. 1. The triangular mesh used in Example 1 with $h \approx 1/16$ (*left*) and the log of the heterogeneous permeability field (*right*)

h	Degrees of Freedom	$\text{cond}(\widetilde{\mathbb{M}}^{-1}\mathbb{M})$
$1/8$	512	13.43
$1/16$	2048	15.84
$1/32$	8192	15.61
$1/64$	32768	15.63

Table 1. Performance of the MFMFE preconditioner with a homogeneous permeability field.

h	Degrees of Freedom	$\text{cond}(\widetilde{\mathbb{M}}^{-1}\mathbb{M})$
$1/8$	512	20.07
$1/16$	2048	21.61
$1/32$	8192	16.61
$1/64$	32768	14.27

Table 2. Performance of the MFMFE preconditioner with a heterogeneous permeability field.

4.2 Example 2

In this example, we consider (1)–(3) with $\Omega = [0,1] \times [0,1]$ and

$$\mathbb{K} = \begin{pmatrix} 1+\alpha & 1-\alpha \\ 1-\alpha & 1+\alpha \end{pmatrix},$$

with $0 < \alpha \leq 1$. We use uniform rectangular meshes and our objective is to demonstrate that the MFMFE preconditioner is more robust as $\alpha \to 0$. In Tables 3 and 4 we present the results using the diagonal preconditioner ($\widetilde{\mathbb{A}} = \text{Diag}(\mathbb{A})$) and the MFMFE preconditioner respectively. We see that both preconditioners are robust with respect to h, but degrade as $\alpha \to 0$, but the MFMFE preconditioner degrades at a much slower rate.

α	$h=1/4$	$h=1/8$	$h=1/16$	$h=1/32$
1	22.43	22.32	22.32	22.32
1E-1	1.06E2	9.95E2	1.06E2	1.06E2
1E-2	7.00E2	6.97E2	6.97E2	6.97E2
1E-3	9.51E3	9.41E3	9.75E3	8.42E3

Table 3. Performance of a diagonal preconditioner with respect to h and α.

α	$h=1/4$	$h=1/8$	$h=1/16$	$h=1/32$
1	22.42	22.32	22.32	22.32
1E-1	32.07	32.09	32.26	32.09
1E-2	51.01	50.06	50.39	50.39
1E-3	5.20E2	6.96E2	8.10E2	8.21E2

Table 4. Performance of the MFMFE preconditioner with respect to h and α.

5 Conclusions

The purpose of this paper is to investigate the performance of the multipoint flux mixed finite element as a preconditioner for the saddle-point system for the full BDM_1 mixed finite element approximation. Numerical results indicate that the MFMFE preconditioner is robust with respect to the mesh size and performs better than the preconditioner based on the diagonal mass matrix.

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