Additive Schwarz Method for DG Discretization of Anisotropic Elliptic Problems

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1 Introduction

In the paper we consider a second order elliptic problem with discontinuous anisotropic coefficients defined on a polygonal region Ω. The problem is discretized by a Discontinuous Galerkin (DG) finite element method with triangular elements and piecewise linear functions. Our goal is to design and analyze an additive Schwarz method (ASM), see the book by Toselli and Widlund [4], for solving the resulting discrete problem with rate of convergence independent of the jumps of the coefficients. The method is two-level and without overlap of Ω_l, the substructures into which the original region Ω is partitioned. It is proved that the convergence of the method is independent of the jumps of the coefficients appearing on triangles inside of Ω_l, see [3]. It is the same for the jumps appearing on triangles which touch $\partial\Omega_l$ under additional assumptions on the coefficients, like monotonicity or quasi-monotonicity. The ASM discussed here is a generalization of method presented in [1]. Numerical experiments confirm the theoretical results.

The paper is organized as follows. In Section 2, differential and discrete DG problems are formulated. In Section 3, ASM for solving the discrete problem is designed and analyzed. Numerical experiments are presented in Section 4.

2 Differential and discrete DG problems

We consider the following elliptic problem: find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \qquad \forall v \in H_0^1(\Omega) \tag{1}$$

where

$$a(u,v) = \int_\Omega \rho(x) \nabla u \cdot \nabla v \, dx, \qquad f(v) = \int_\Omega f v \, dx,$$

$$\rho(x) = \begin{pmatrix} \rho_{11}(x) & \rho_{12}(x) \\ \rho_{21}(x) & \rho_{22}(x) \end{pmatrix}.$$

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We assume that Ω is a polygonal region, $f \in L^2(\Omega)$ and $\rho(x)$, the diffusivity tensor, is a symmetric matrix, uniformly positive definite with respect to x, and $\rho_{ij} \in L^\infty(\Omega), i.j = 1,2$. Under these assumptions problem (1) is well posed.

Let $\mathscr{T}^h(\Omega)$ be a triangulation of Ω with triangular elements K_i and the mesh parameter h. We assume that $\mathscr{T}^h(\Omega)$ is shape regular and quasiuniform. Let $X_i(K_i)$ denote a space of linear functions on K_i and

$$X_h(\Omega) = \Pi_{i=1}^N X_i(K_i), \quad \bar{\Omega} = \bigcup_{i=1}^N K_i$$

be the space in which problem (1) is approximated. Note that $X_h(\Omega) \not\subset H^1(\Omega)$ and its elements do not vanish on $\partial\Omega$, in general.

The discrete problem for (1) is of the form: find $u_h^* \in X_h(\Omega)$ such that

$$\hat{a}_h(u_h^*, v_h) = f(v_h), \qquad v_h \in X_h(\Omega), \tag{2}$$

where for $u, v \in X_h(\Omega), u = \{u_i\}_{i=1}^N, u_i \in X_i(K_i)$,

$$\hat{a}_h(u,v) = \sum_{i=1}^N \hat{a}_i(u,v), \qquad f(v) = \sum_{i=1}^N \int_{K_i} f v_i \, dx$$

and

$$\rho^{(i)} = \rho_{|K_i}, \qquad \rho^{(i)} = \{\rho_{kl}^{(i)}\}_{k,l=1}^2,$$

and $\rho_{kl}^{(i)}$ are constants on K_i which can always be assumed for linear elements. Here

$$\hat{a}_i(u,v) = a_i(u,v) + s_i(u,v) + p_i(u,v),$$

with symmetric forms

$$a_i(u,v) = \int_{K_i} \rho^{(i)} \nabla u_i \cdot \nabla v_i \, dx,$$

$$s_i(u,v) = \sum_{E_{ij} \subset \partial K_i} \int_{E_{ij}} \omega_i [n_i^T \rho^{(i)} \nabla u_i (v_j - v_i) + n_i^T \rho^{(i)} \nabla v_i (u_j - u_i)] \, ds,$$

$$p_i(u,v) = \sum_{E_{ij} \subset \partial K_i} \frac{\sigma}{h} \int_{E_{ij}} \gamma_{ij} (u_i - u_j)(v_i - v_j) \, ds$$

where $E_{ij} = E_{ji} = \partial K_i \cap \partial K_j, E_{ij} \subset \partial K_i$ and $E_{ji} \subset \partial K_j$; $n_i = n_{E_{ij}}$ is the unit normal vector to E_{ij} pointing from K_i to K_j;

$$\omega_i \equiv \omega_{E_{ij}} = \frac{\delta_{\rho^n}^{(j)}}{\delta_{\rho^n}^{(i)} + \delta_{\rho^n}^{(j)}}, \qquad \omega_j \equiv \omega_{E_{ji}} = \frac{\delta_{\rho^n}^{(i)}}{\delta_{\rho^n}^{(i)} + \rho_{\rho^n}^{(j)}}$$

and

$$\delta_{\rho^n}^{(i)} = n_i^T \rho^{(i)} n_i, \qquad \delta_{\rho^n}^{(j)} = n_j^T \rho^{(j)} n_j;$$

$\gamma_{ij} \equiv \gamma_{E_{ij}} = 2\delta_{\rho^n}^{(i)}\delta_{\rho^n}^{(j)}/(\delta_{\rho^n}^{(j)} + \delta_{\rho^n}^{(i)})$; σ is a positive (sufficiently large, cf. Lemma 1) penalty parameter, which ensures the ellipticity of $\hat{a}_i(\cdot,\cdot)$.

To analyze problem (2) we introduce some auxiliary bilinear forms and a broken norm. Let the elliptic symmetric form $d_h(\cdot,\cdot)$ be defined as

$$d_h(u,v) = \sum_{i=1}^{N} d_i(u,v), \quad d_i(u,v) = a_i(u,v) + p_i(u,v) \tag{3}$$

and let the weighted broken norm in $X_h(\Omega)$ be defined by

$$\| u \|_{1,h}^2 \equiv d_h(u,u) = \sum_{i=1}^{N} \{ \| (\rho^{(i)})^{1/2} \nabla u_i \|_{L^2(K_i)}^2 + \sum_{E_{ij} \subset \partial K_i} \frac{\sigma}{h} \gamma_{ij} \| u_i - u_j \|_{L^2(E_{ij})}^2 \}. \tag{4}$$

Lemma 1. *There exists $\sigma_0 > 0$ such that for $\sigma \geq \sigma_0$ there exist positive constants C_0 and C_1 independent of $\rho^{(i)}$ and h such tha*

$$C_0 d_i(u,u) \leq \hat{a}_i(u,u) \leq C_1 d_i(u,u)$$

and

$$C_0 d_h(u,u) \leq \hat{a}(u,u) \leq C_1 d_h(u,u)$$

for all $u \in X_h$.

For the proof we refer for example to [1] for isotropic cases and [2] for anisotropic cases.

Lemma 1 implies that the discrete problem (2) is well posed if the penalty parameter $\sigma \geq \sigma_0$. Below σ is fixed and assumed to satisfy the above condition.

The error bound is given by

Theorem 1. *Let u^* and u_h^* be the solutions of (1) and (2). For $u^*_{|K_i} \in H^2(K_i)$ holds*

$$\| u^* - u_h^* \|_{1,h}^2 \leq M h^2 \sum_{i=1}^{N} \lambda_{max}(\rho^{(i)}) |u^*|_{H^2(K_i)}^2$$

where M is independent of h, u^* and ρ_i; $\lambda_{max}(\rho^{(i)})$ is a maximum eigenvalue of $\rho^{(i)}$.

The proof follows from Lemma 1, for details see for example [2].

3 Additive Schwarz method

We design and analyze ASM for solving problem (2) following to the abstract theory of ASMs, see for example, [4].

3.1 Decomposition of $X_h(\Omega)$

Let

$$\bar{\Omega} = \bigcup_{l=1}^{L} \bar{\Omega}_l, \qquad \Omega_l \cap \Omega_m = \{\emptyset\}, \qquad l \neq m$$

where $\bar{\Omega}_l$ is a union of triangulation elements K_i and $H_l = diam(\Omega_l)$. The decomposition of $X_h(\Omega)$ is

$$X_h(\Omega) = X^{(0)}(\Omega) + X^{(1)}(\Omega) + \ldots + X^{(L)}(\Omega),$$

where for $l = 1, \ldots, L$

$$X^{(l)}(\Omega) = \{v = \{v_i\}_{i=1}^{N} \in X_h(\Omega) : v_i = 0 \text{ on } K_i \not\subset \Omega_l\}$$

and for $l = 0$

$$V^{(0)}(\Omega) = span\{\phi^{(l)}\}_{l=1}^{L}$$

with $\phi^{(l)} = 1$ on $\bar{\Omega}_l$ and $\phi^{(l)} = 0$ otherwise.

3.2 Inexact local solvers

For $u^{(l)} = \{u_i^{(l)}\}_{i=1}^{N} \in X^{(l)}(\Omega)$ and $v^{(l)} = \{v_i^{(l)}\}_{i=1}^{N} \in X^{(l)}(\Omega), l = 1, \ldots, L$, we define

$$b_l(u^{(l)}, v^{(l)}) = d_h(u^{(l)}, v^{(l)}).$$

The overlap between local subproblems is very small (only through the subdomain interface), reducing communication cost to a level similar to substructuring methods. Instead of solving exact subproblems with form $\hat{a}_h(\cdot,\cdot)$ on subdomains, we solve problems with simplified form $d_h(\cdot,\cdot)$. Note that on $X^{(l)}(\Omega) \times X^{(l)}(\Omega)$

$$d_h(u^{(l)}, v^{(l)}) = \sum_{K_i \subset \bar{\Omega}_l} \{(\rho^{(i)} \nabla u_i^{(l)}, \nabla v_i^{(l)})_{L^2(K_i)} + \sum_{E_{ij} \subset \partial K_i} \frac{\sigma}{h} \gamma_{ij} (u_i^{(l)} - u_j^{(l)}, v_i^{(l)} - v_j^{(l)})_{L^2(E_{ij})}\}.$$

For $l = 0$ and $u^{(0)} = \{u_i^{(0)}\}_{i=1}^{N} \in X^{(0)}(\Omega)$ and $v^{(0)} = \{v_i^{(0)}\}_{i=1}^{N} \in X^{(0)}(\Omega)$ we set

$$b_0(u^{(0)}, v^{(0)}) = d_h(u^{(0)}, v^{(0)}) \equiv \sum_{l=1}^{L} \frac{\sigma}{h} \sum_{E_{ij} \subset \partial \Omega_l} \gamma_{ij} (u_i^{(0)} - u_j^{(0)}, v_i^{(0)} - v_j^{(0)})_{L^2(E_{ij})}.$$

3.3 Operator equation

For $l = 0, \ldots, L$, let us define $T_l : X_h(\Omega) \to X^{(l)}(\Omega)$ by

$$b_l(T_l u, v) = \hat{a}_h(u,v), \quad v \in X^{(l)}(\Omega).$$

Then problem (2) is replaced by

$$Tu_h^* = g_h, \qquad g_h = \sum_{l=0}^{L} g_l, \qquad g_l = T_l u_h^*. \tag{5}$$

with $T = T_0 + T_1 + \ldots + T_L$. Note that in order to compute g_l we do not need to know u_h^*. From the theorem below it follows that problems (2) and (5) have the same unique solution.

3.4 Analysis

Let $\bar{\Omega}_l^h$ denote a layer around $\partial\Omega_l$. It is a union of $K_i \subset \bar{\Omega}_l$ which touch $\partial\Omega_l$ by edge or/and vertex.
Let

$$\bar{\alpha}_l := \max_{K_i \subset \bar{\Omega}_l^h} \lambda_{\max}(\rho^{(i)}), \qquad \underline{\alpha}_l := \min_{K_i \subset \bar{\Omega}_l^h} \lambda_{\min}(\rho^{(i)})$$

where $\lambda_{\max}(\rho^{(i)})$ and $\lambda_{\min}(\rho^{(i)})$ are maximum and minimum eigenvalues of $\rho^{(i)}$ on K_i.

Theorem 2 (main result). *For any $u \in X_h(\Omega)$ there holds*

$$C_2\beta^{-1}\hat{a}_h(u,u) \le \hat{a}_h(Tu,u) \le C_3\hat{a}_h(u,u) \tag{6}$$

where

$$\beta = \max_{1\le l\le L} \frac{\bar{\alpha}_l H_l^2}{\underline{\alpha}_l h^2}$$

and C_2 and C_3 are positive constants independent of $\rho^{(i)}, \bar{\alpha}_l$ and $\underline{\alpha}_l$ for $i = 1,\ldots,N$ and $l = 1,\ldots,L$.

To prove Theorem 2 we need to check three key assumptions of the abstract theory of ASMs, see Toselli and Widlund book [4]. The proof is omitted here due to the limit of pages and will be published elsewhere.

Remark 1. Note that the convergence of the method is independent of the jumps of $\rho^{(i)}$ on $\bar{\Omega}_l \backslash \Omega_l^h$ for all $l = 1,\ldots,L$, i.e. of the jumps of $\rho^{(i)}$ on K_i which do not touch $\partial\Omega_l$.

Remark 2. Let us mention several specific cases when the above estimate can be improved. When ρ is isotropic and subdomainwise constant, then we can prove that $\beta = \max_l (H_l/h)$ in (6). When $\bar{\alpha}_l$ and $\underline{\alpha}_l$ are the same order and $\underline{\alpha}_l \le \max_{K_i\subset\bar{\Omega}_l} \lambda_{\min}(\rho^{(i)})$, then $\beta = \max_l(H_l/h)$, i.e. the convergence is independent of the jumps of $\rho^{(i)}$. Estimate (6) can be also improved in the case when $\lambda_{\max}(\rho^{(i)})$ on K_i which touch $\partial\Omega_l$ by edges are monotonic or quasi-monotonic on $\partial\Omega_l$ for $l = 1,\ldots,L$.

4 Numerical experiments

Let us choose the unit square as the domain Ω and for some prescribed integer m divide it into $L = 2^m \times 2^m$ smaller squares Ω_l ($l = 1,\ldots,L$) of equal size. This decomposition of Ω is then further refined into a uniform triangulation $\mathscr{T}^h(\Omega)$ based on a square $2^M \times 2^M$ grid ($M \geq m$) with each square split into two triangles of identical shape. Hence, the fine mesh parameter $h = 2^{-M}$, while the coarse grid parameter is $H = 2^{-m}$. We discretize system (1) on the fine triangulation using method (2) with $\sigma = 7$.

In tables below we report the number of Preconditioned Conjugate Gradient iterations for operator T (defined in Section 3.3) which are required to reduce the initial Euclidean norm of the residual by a factor of 10^6 and (in parentheses) the condition number estimate for T. We consider two sets of test problems: with either anisotropic or discontinuous coefficients matrix ρ. We will always choose a random vector for the right hand side and a zero as the initial guess.

Discontinuous, elementwise constant isotropic coefficients. Let us consider diffusion coefficient of the form

$$\rho(x) = \rho_{11}(x) \cdot I \tag{7}$$

where ρ_{11} equals 1 on even numbered elements (of fine triangulation) and equals 10^{-2} on odd ones. Table 1 shows the dependence on the ratio between H and h in this case.

Fine (M) → ↓ Coarse (m)	2	3	4	5	6
2	33 (32)	82 (300)	133 (530)	164 (840)	237 (2000)
3		45 (41)	140 (370)	189 (700)	225 (1100)
4			48 (42)	155 (470)	186 (690)
5				41 (48)	155 (470)
6					49 (44)

Table 1 Dependence of the number of iterations and the condition number (in parentheses) on the ratio H/h, where $H = 2^{-m}$ and $h = 2^{-M}$. Isotropic, elementwise constant coefficient.

Next, let us fix the number of subdomains and the fine mesh size so that $M = 3$ and $m = 5$ and thus $H/h = 4$. Table 2 shows the dependence of the convergence rate and the condition number as we vary the value of ρ_{11} on odd-numbered triangles; on even triangles it remains equal to 1 as previously.

ρ_{11}	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
iter (cond)	61 (80)	72 (10^2)	167 ($6\cdot 10^2$)	335 ($5\cdot 10^3$)	485 ($5\cdot 10^4$)	613 ($5\cdot 10^5$)	743 ($5\cdot 10^6$)

Table 2 Dependence of the number of iterations and the condition number (in parentheses) on the discontinuity in the isotropic, elementwise constant coefficient. Fixed $H/h = 4$.

Indeed, the condition number estimates agree well with our theory regarding the dependence on the discontinuity of the coefficient. In our testcase the increase in the condition number is rather linear than quadratic in H/h, as reported in Table 1. This behaviour is in agreement with our Remark 2. Let us also explain that low iteration numbers in Table 2 are due to a very rapid residual in the residual during the initial phase of the iteration.

Discontinuous, domainwise constant isotropic coefficients. Here we consider ρ as in (7), with discontinuities aligned with an auxiliary partitioning of Ω into 4×4 squares. Precisely, we introduce a red–black checkerboard coloring of this partitioning and set $\rho = 1$ in red regions, and the value of ρ_{11} reported in Table 3 in black ones. In this way, our decomposition of the domain with $M = 5$ and $m = 3$ will always be aligned with the discontinuites and Table 3 shows the dependence on ρ_{11} in this case.

ρ_{11}	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
iter (cond)	61 (80)	60 (70)	58 (67)	58 (68)	62 (68)	64 (68)	67 (68)

Table 3 Dependence of the number of iterations and the condition number (in parentheses) on the discontinuity when the coefficient is isotropic and constant inside subdomains. Red–black 4×4 distribution of ρ, aligned with domain decomposition. Fixed $H/h = 4$.

As predicted in Remark 2, there is no dependence on the discontinuity in the coefficients in this case until the coefficient remains continuous (constant) inside subdomain. This behaviour is not observed when the red–black partitioning is not aligned with the subdomains Ω_l: corresponding numbers for a 3×3 partitioning are shown in Table 4.

ρ_{11}	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
iter (cond)	62 (80)	68 (130)	85 (710)	96 ($7\cdot 10^3$)	113 ($7\cdot 10^4$)	126 ($7\cdot 10^5$)	140 ($7\cdot 10^6$)

Table 4 Dependence of the number of iterations and the condition number (in parentheses) on the discontinuity when the coefficient is isotropic and discontinuous across subdomain boundaries. Red–black 3×3 distribution of ρ, not aligned with the domain decomposition. Fixed $H/h = 4$.

Anisotropic, discontinuous coefficients. Let us continue with the 4×4 red–black partitioning and let us set the coefficient matrix ρ equal to ρ^R in red regions and ρ^B in black ones, where

$$\rho^R(x) = \begin{pmatrix} 10+\rho_{22} & 0 \\ 0 & \rho_{22}, \end{pmatrix} \qquad \rho^B(x) = \begin{pmatrix} \rho_{22} & 0 \\ 0 & 10+\rho_{22} \end{pmatrix},$$

with constant ρ_{22} as specified in Table 5. In this way ρ is constant in both red and black regions, but it suffers from discontinuity across the partitioning borders; the jump is always equal to 10, while the anisotropy ratio is $1+10/\rho_{22}$. The condition numbers grow linearly with the growth of ρ_{22}, which agrees with Theorem 2.

ρ_{22}	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
iter (cond)	60 (82)	94 (210)	222 (10^3)	463 (10^4)	680 (10^5)	782 (10^6)	897 (10^7)

Table 5 Dependence on the anisotropy for discontinuous, piecewise constant coefficient. Fixed $H/h = 4$.

Anisotropic, constant coefficients. Finally, let us consider

$$\rho(x) = \begin{pmatrix} 1 & 0 \\ 0 & \rho_{22} \end{pmatrix}$$

with ρ_{22} constant throughout entire Ω, assuming values specified in Table 6.

ρ_{22}	10^0	10^1	10^2	10^3	10^4	10^5	10^6
iter (cond)	60 (82)	74 (10^2)	159 ($6 \cdot 10^2$)	159 ($6 \cdot 10^2$)	144 ($6 \cdot 10^2$)	143 ($6 \cdot 10^2$)	124 ($7 \cdot 10^2$)

Table 6 Dependence on the anisotropy. Fixed $H/h = 4$. Continuous, constant coefficient.

It turns out that after initial linear increase in the condition number for moderate ρ_{22}, the condition number is insensitive to further growth of the anisotropy ratio ρ_{22}. This observation can also be explained on the ground of our theory; the details will be provided elsewhere.

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