Conservative inexact solvers for porous media flow

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Abstract The governing equations for flow and transport in porous media are derived assuming conservation of mass. To ensure stability of the simulations significant attention is given to ensure that the discrete system retains the conservation property. Due to discretization errors and parameter uncertainty it is natural to consider an inexact solution strategy for the resulting system of equations. However most linear solvers are not designed by the same principles as the underlying discretization and will thus not produce inexact solutions that preserve the conservation property. In this work we illustrate how inexact yet conservative linear solvers can be realized for porous media applications. The linear solver is formulated as a multi-level control volume methods and produces a conservative flux field for all approximated solutions.

Key words: inexact solvers, control volume methods, porous media flow

1 Introduction

Simulation models of flow and transport in geological porous media are characterized by a high degree of uncertainty due to both discretization errors and incomplete measurements of physical parameters. In the context of linear solvers this seemingly mandates the use of inexact strategies, where a solution is sought with an accuracy similar to that of the overall computational model. Since the solution of linear systems often consumes a substantial part of the total simulation time, inexact solvers can yield considerable computational savings. However the derivation of the continuous model is based on conservation of mass, and this property must be preserved in the discrete system for the results to be physically meaningful. The discretization schemes commonly applied are conservative by construction, but unless the linear solver is designed specifically to produce solutions that, even if inexact, conserve mass the inexact solution may not yield a stable overall simulation strategy. For this reason linear systems are commonly solved to an accuracy that is much higher than mandated by known discretization errors and parameter uncertainties.

The key to producing physically meaningful inexact solutions is to design the linear solver by the same principles as the discretization scheme. Herein we will explore these ideas in the context of two-phase flow in a horizontal porous media. The phases denoted water (w) and oil (o) are immiscible and incompressible with a

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velocity given by Darcy's law

$$\mathbf{u}_\alpha = -\lambda_\alpha \mathbf{K}\nabla p, \qquad \alpha = w, o. \tag{1}$$

Here the phase mobilities λ_w and λ_o, represent fluid viscosity and rock-fluid interaction. Furthermore $\mathbf{K}$ is the permeability and p is the fluid pressure. Of particular importance to this paper are the properties of the permeability, which commonly possesses sharp contrasts of several orders of magnitude and spatial correlation structures on a continuum of length scales. Conservation of mass for each phase is expressed as

$$\phi \frac{\partial S_\alpha}{\partial t} + \nabla \cdot \mathbf{u}_\alpha = q_\alpha, \qquad \alpha = w, o, \tag{2}$$

where ϕ represents porosity, S_α is the volume fraction of phase α and q_α is the source term. The saturations are assumed to fill the pore volume, that is $S_w + S_o = 1$. Thus when (2) for the two phases are added to get an equation for conservation of total mass, the saturations are eliminated. This gives a linear elliptic equation for the pressure, which can be written

$$\nabla \cdot \mathbf{u}_T = -\nabla \cdot (\lambda_T \mathbf{K}\nabla p) = q_T. \tag{3}$$

Here $\mathbf{u}_T = \mathbf{u_w} + \mathbf{u_o}$ is the total velocity, $\lambda_T = \lambda_w + \lambda_o$ is the total mobility and $q_T = q_w + q_o$ is the total source term.

2 Discretization

In the rest of the paper we describe the construction of an inexact linear solver for (3) which preserves the conservation property of $\mathbf{u}_T$. The solver is formulated in terms of a novel multi-level control volume method which is briefly described next. More details can be found in [6].

2.1 A hierarchy of control volume discretizations

In applications conservation of mass is considered an essential property that should be preserved during discretization. To that end a cell centered control volume method is applied for the spatial discretization. A discrete Darcy's law is constructed as in [1]

$$\mathbf{u}_{h,\alpha} = -\lambda_\alpha^U T_h p_h, \tag{4}$$

where $\mathbf{u}_{h,\alpha}$ is the discrete phase velocities for phase α, T_h is a matrix of transmissibilities and p_h is a cell centered approximation of the pressure. The mobilities, λ_α^U, are discretized by phase-wise upstream weighting. A discrete equation for the pressure is found by

$$D_h((\lambda_w^U + \lambda_o^U)T_h p_h) = A_h p_h = q_h, \tag{5}$$

where D_h is the discrete divergence, A_h is the system matrix and q_h represents discrete sources. We note that (5) can be considered a Petrov-Galerkin discretization of (3), with piece-wise constants on the cells as test functions and shape functions defined by the specific control volume method. When (5) has been solved for p_h, (2) for the water phase is discretized by an explicit method with upstream weighting of the mobilities.

The sharp contrasts and long correlation structures of the permeability is reflected in the discretization matrix A_h, thus solving (5) is time consuming. Discretization errors and uncertainties in the permeability make the linear system a prime candidate for an inexact linear solver. However, (5) was derived by requiring conservation of mass, and unless this is reflected in the inexact solution, conservation errors will in worst case grow exponentially in the time propagation of (2). The linear solver should therefore be constructed to produce a discrete flux field that, even if inexact, satisfies (5). Furthermore an efficient solution strategy for (5) should invoke coarse solvers to account for the global dependencies of the equation.

An inexact two-level method which retains the conservation property can be realized within the framework of the multiscale finite volume (MSFV) method [3], see also [7]. The domain is partitioned into a coarse grid and a coarse shape function ψ_H is constructed for each coarse cell to account for fine-scale variabilities in the permeability. Coarse test functions ϕ_H are defined as piece-wise constants on the coarse cells. A coarse linear system is then defined as

$$(\Phi_H^T A_h \Psi_H) p_H = A_H p_H = \Phi_H^T q_h. \tag{6}$$

Here Φ_H and Ψ_H are column matrices of test and shape functions, respectively, and A_H is the coarse discretization. It is important to note the similarity between (5) and (6), in that both are obtained by applying Petrov-Galerkin techniques. In this way the coarse linear system retains the conservation property of the fine-scale discretization. Specifically it will produce conservative coarse fluxes in the sense that the fluxes into a coarse cell match the sources within the cell. When projected to the fine scale the inexact fluxes will not be conservative. This is remedied by a post-processing step where local fine-scale problems are solved within each coarse cell [3]. The boundary conditions are the projection of the conservative coarse fluxes to the fine scale.

2.2 Multi-level flux post-processing

The two-level method outlined above amounts to an inexact linear solver that can also be applied as a preconditioner within an iterative solver. However it is natural to seek multi-level methods to realize efficient residual smoothing strategies. Also when multiple grid levels are available, adaptive upscaling can be applied during the simulation. Finally, the MSFV method is known to be unstable in cases where the coarse grid does not follow anisotropy patterns in the permeability [5]. This can be remedied by an unstructured coarsening strategy that is currently under development but for this approach to be robust multiple coarsening steps with mild upscaling ratios should be applied.

Since (6) has the same properties as (5) in terms of sparsity pattern and conservation property, a further coarsening of the system can easily be constructed by recursion. However, for the multi-level method to be applicable as a conservative inexact linear solver, multi-level post-processing is needed, and specifically local Neumann problems must be solved. For the coarser levels the discretization of Neumann boundary conditions is not available, and this has in practice limited control volume linear solvers to two grid levels. In the following we will outline how the multi-level post-processing can be realized, a thorough explanation is given in [6].

As for the two-level method, the post-processing is performed by solving local problems that are confined to single cells on the coarser level. When conservative fluxes on coarse faces are known these can be mapped to any finer level via the shape functions, specifically they can be mapped one level down to form boundary conditions for the local problems. In this way the flux discretization on coarse boundaries is replaced by known fluxes. However there will be faces interior to the coarse cell with exterior cells in their flux discretization, in conflict with the goal of a local post-processing. The exterior cells are eliminated by considering groups of cells that are centered around vertexes on the boundary of the coarse cell and have common support for their basis functions, as illustrated in Fig. 1. The exterior cells can be replaced by the known fluxes over the boundary by formulating and solving a local linear system. When the number of exterior cells and the number of known fluxes are equal, the elimination is straightforward. If there are more exterior cells

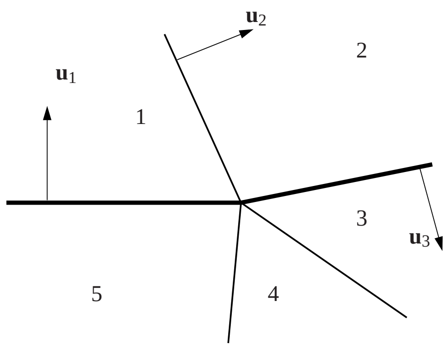


Fig. 1 Parts of cells with common support for their basis functions centered around a vertex at the boundary of a coarse cell. Fluxes (arrows) and cells close to the boundary of a coarse cell (bold). Cells 3-5 are outside the coarse cell and must be eliminated from the flux expression for $\mathbf{u}_2$ using $\mathbf{u}_1$ and $\mathbf{u}_3$ (which are known) and their sub-fluxes.

than there are boundary conditions (respectively 3 and 2 in Fig. 1), additional equations can be obtained by splitting the boundary fluxes into sub-fluxes on a finer grid level and computing higher order moments of the fluxes based on these. Note that on the finest level the elimination is straightforward since a boundary discretization is available there; thus the splitting into sub-fluxes is available when needed. A linear system is then solved around all vertexes on the boundary, and the results are used to formulate a local system within the coarse cell that is solved to get conservative fluxes.

This methodology provides conservative fluxes for all faces on all grid levels even if the accompanying pressure is inexact. We make two comments on the approach: firstly the only pair of pressure and fluxes which satisfies both the discrete flux law (4) and the conservation equation (5) is the exact solution. The post-processed fluxes possess the conservation property, but they cannot be computed from the inexact pressures via (4). The post-processed fluxes can be thought of as being exact for a modified permeability field, in accordance with an uncertainty in this parameter. Secondly the post-processing is not applicable unless the inexact solution preserves the conservation property of the continuous problem. This not only requires the construction of coarse problems as described above, but also a careful treatment of the right hand side of the linear system. To be specific, the right hand side should be coarsened according to the Schur complement formulation of the multi-level method [8]. The multi-level method with this special coarsening can be applied as a correction to the residual of any inexact solution. The corrected solution will in general still be inexact, but it will possess the structure necessary to apply the post-processing.

2.3 Error control

With the post-processing outlined above, we can obtain solutions that are inexact but still honor the conservation property. There are two natural criteria for controlling the linear solver. The simplest option is to terminate the iterations when a desired reduction of the relative residual is achieved and then apply post-processing to obtain a mass conserving flux field. However, even though the post-processing produces a velocity field without conservation errors a reduction of the relative residual gives little control of the accuracy of the fluxes. A more nuanced notion of error can be derived from [4], where we find the expression

$$\|\mathbf{K}^{-1/2}(\mathbf{u}-\mathbf{u}_h^*)\| \leq \inf_{s\in H^1} \|\mathbf{K}^{-1/2}(\mathbf{u}_h^* - \mathbf{K}\nabla s)\| + \sup_{\beta\in H^1, \|\beta=1\|} (\nabla\cdot(\mathbf{u}-\mathbf{u}_h^*),\beta), \quad (7)$$

where $\mathbf{u}$ is the true flux and $\mathbf{u}_h^*$ is the post-processed flux field. The last term evaluates to zero since the post-processed and exact fluxes have the same divergence. The triangular inequality applied on the first term gives

$$\|\mathbf{K}^{-1/2}(\mathbf{u}-\mathbf{u}_h^*)\| \leq \|\mathbf{K}^{-1/2}(\mathbf{u}_h^* - \mathbf{K}\nabla p_h^*)\| + \inf_{s\in H^1} \|\mathbf{K}^{1/2}\nabla(p_h^* - s)\|, \qquad (8)$$

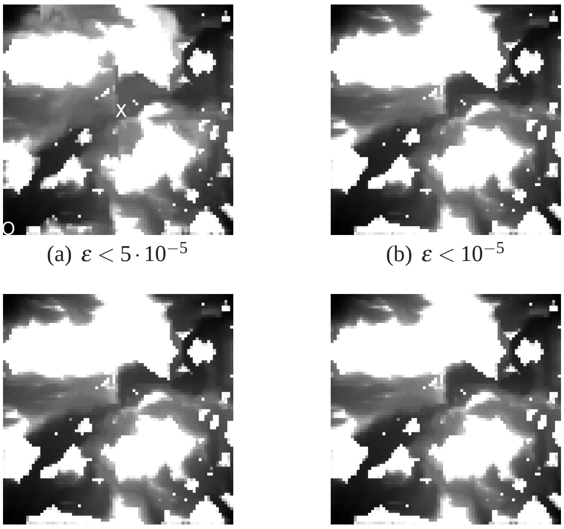
with p_h^* representing the inexact pressure. The first term on the right hand side of (8) is immediately computable, and can be interpreted as the error stemming from the linear solver. We denote this term e_{LS}. The second term is identified as the discretization error, denoted e_d. To give reasonable estimates for the gradient of p_h^* in heterogeneous media, we compute this from face pressures that are reconstructed from the fine-scale discretization. The estimate (8) can employed to control the linear solution process by terminating the iterations when the error from the linear solver is smaller than the discretization error, at which point it can be argued that it makes little sense to improve the inexact solution.

3 Numerical results

In this section we illustrate the utility of the conservative framework by coupling an inexact multi-level linear solver for the pressure equation to a non-linear transport problem. The computational grid is Cartesian, with 3^4 cells in each direction. The permeability is taken from the bottom layer of the 10th SPE comparative solution project (SPE10) [2], which is characterized by long and highly permeable channels and sharp contrasts of 6 orders of magnitude. The medium is initially filled with oil. Water is injected in the lower left corner of the grid, and a production well is placed in the middle of the domain.

The phase velocities in (4) are discretized on the fine-scale grid by a two-point flux approximation. Periodic boundary conditions are assigned for simplicity. Three levels of coarsening are applied, each with a ratio of 3 in each direction, and a direct solver is invoked on the coarsest grid. Thus the coarse operator constitutes a four-level multi-grid method. Updates of the saturation feed back to the pressure equation via the mobilities, which are set to $\lambda_w = S_w^3$ and $\lambda_o = 10S_o^2$, and thus the velocity field must be updated regularly. The pressure time step is fixed at a tenth of the total simulation time, while the time step for the saturation equation is decided by the CFL criterion.

To solve the pressure equation, GMRES iteration preconditioned by the multi-level method is applied. Four criteria for terminating the iterative solver are considered: Two consider the reduction of the relative residual, ε, and terminate the iterations when $\varepsilon < 5\cdot 10^{-5}$ and $\varepsilon < 10^{-5}$, respectively. The third criterion requires that $e_{LS} < e_d$, which in this case corresponds to a value of ε of $10^{-6} - 10^{-8}$. All these estimates apply post-processing to ensure the approximated flux field is conservative. Finally, we consider a solver with the same preconditioner, but where post-processing is not applied after the iterations. In this case the fluxes must be brought sufficiently close to being conservative by iterating on the solution. Note that this is the strategy applied by a traditional linear solver. For the present setup, a value of $\varepsilon < 10^{-10}$ is needed to avoid severe stability issues due to conservation errors.



(a) $\varepsilon < 5 \cdot 10^{-5}$ (b) $\varepsilon < 10^{-5}$

(c) $e_{LS} < e_d$ (d) $\varepsilon < 10^{-10}$, no p.p.

Fig. 2 Saturation profiles obtained with different stopping criteria for the linear solvers. Water (light) is injected into a domain initially filled with oil (dark). Injection (O) and production (X) wells are marked in (a). Periodic boundary conditions are applied.

Table 1 Total number of GMRES iterations needed to achieve desired tolerance level.

$\varepsilon < 5 \cdot 10^{-5}$	$\varepsilon < 10^{-5}$	$e_{LS} < e_d$	$\varepsilon < 10^{-10}$, no p.p.
190	200	212	293

Snapshots of the saturation distributions with the respective control parameters are shown in Fig. 2. All simulations predict the same large-scale pattern, and it is only the loosest tolerance for the pressure solver that yields notable differences in the saturation profile. The computational gains from applying post processing can be seen from the number of iterations shown in Tab. 1. We observe that there is considerable room for computational savings without sacrificing significant accuracy of the transport solution. We reiterate that this is due to the post-processing, which facilitates inexact yet conservative flux fields. Some caution is needed when deciding the stopping criterion for the linear solver as the accuracy necessary to get reasonable transport solution is highly dependent on the simulation setup. Note that if the post-processing is not applied the accuracy to produce a flux field that makes the transport solver behaves stable increases significantly. The tolerance necessary will be different for other simulations, and in practice the only options to obtain stable simulations are to iterate until the exact solution is found, or to apply an inexact solver and somehow tackle conservation errors in the transport solver. We also remark that the performance of all preconditioners suffers from the Cartesian coarse grids that leads to strong heterogeneities within the coarse cells. This will be amended by an unstructured coarsening procedure currently under development.

4 Concluding remarks

In this paper we have considered the application of an inexact linear solver for porous media flow with the special property that it provide a set of fluxes that exactly satisfy a conservation law, even if the associated pressure that drives the flux was approximated. The solver was formulated as a multi-level control volume discretization, and we considered the coupling of the solver with a non-linear transport problem. Since the approximated flux field possessed the conservation property, considerable computational savings were possible without sacrificing stability or significant accuracy in the transport simulation.

For simulation of realistic applications there will always be a trade-off between accuracy and computational effort, and this balance is particularly well articulated when control parameters for linear solvers are decided. We have shown in this paper that the linear solver should not be considered a stand-alone part of the overall simulation tool. Instead it should be in accordance with the same principles as guided the choice of the disrcetization scheme. The resulting solver will provide solutions that even if approximated are physically meaningful, enhancing the robustness of the simulator.

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