FETI solvers for non-standard finite element equations based on boundary integral operators

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1 Introduction

This paper is devoted to the construction and analysis of Finite Element Tearing and Interconnecting (FETI) methods for solving large-scale systems of linear algebraic equations arising from a new non-standard finite element discretization of the diffusion equation. This discretization technique uses PDE-harmonic trial functions in every element of a polyhedral mesh. The generation of the local stiffness matrices utilizes boundary element techniques. For these reasons, this non-standard finite element method can also be called a BEM-based FEM or Trefftz-FEM.

The FETI method was introduced by Farhat and Roux in [1] and has been generalized and analyzed by many people, see, e.g., [11] and [7] for the corresponding references. The Boundary Element Tearing and Interconnecting (BETI) method was later introduced by Langer and Steinbach [6] as the boundary element counterpart of the FETI method. The analysis of the convergence of the BETI method is heavily based on the spectral equivalences between FEM- and BEM-approximated Steklov-Poincaré operators. Similar techniques are used for the analysis of the BEM-based FETI methods considered in this paper. Due to space constraints, this analysis is however postponed to a forthcoming article. In the present work, we derive the solver, state the convergence results without proof, and present numerical results.

2 A skeletal variational formulation

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded Lipschitz domain, and let us consider the following diffusion problem in the standard weak form: find $u \in H^1(\Omega)$ such that u matches the given Dirichlet data g_D on Γ_D and satisfies the variational equation

$$\int_\Omega \alpha \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_{\Gamma_N} g_N v \, ds \; \forall v \in H^1_D(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\} \quad (1)$$

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where α is the uniformly positive and bounded diffusion coefficient, f is a given forcing term, $\Gamma_D \subseteq \partial\Omega$ is the Dirichlet boundary with positive surface measure, $\Gamma_N = \partial\Omega \setminus \overline{\Gamma_D}$ is the Neumann boundary with prescribed conormal derivative g_N.

Consider a decomposition $\mathscr{T}$ of the domain Ω into polytopal elements $T \in \mathscr{T}$. In contrast to a standard FEM method, we allow the mesh to consist of a mixture of rather general polygons (in 2d) or polyhedra (in 3d). We now require that the coefficient function α is piecewise constant with respect to $\mathscr{T}$, i.e., $\alpha|_T(x) \equiv \alpha_T \ \forall T \in \mathscr{T}$.

On every element T, we introduce the local harmonic extension operator $\mathscr{H}_T : H^{1/2}(\partial T) \to H^1(T)$ which maps any $g_T \in H^{1/2}(\partial T)$ to the unique weak solution $u_T \in H^1(T)$ of the local PDE $-\mathrm{div}(\alpha_T \nabla u_T) = 0$ with Dirichlet boundary condition $u_T|_{\partial T} = g_T$. Furthermore, we define the local *Steklov-Poincaré operator* $S_T : H^{1/2}(\partial T) \to H^{-1/2}(\partial T)$ by $S_T u_T = \gamma^1 \mathscr{H}_T u_T$, where γ^1 is the conormal derivative operator which takes the form $\gamma^1 = n \cdot \alpha \nabla$ for sufficiently regular arguments.

If we introduce the *skeleton* $\Gamma_S := \bigcup_{T\in\mathscr{T}} \partial T$ and denote by $H^{1/2}(\Gamma_S)$ the trace space of $H^1(\Omega)$-functions onto the skeleton, we can formulate the skeletal variational problem: find $u \in H^{1/2}(\Gamma_S)$ with $u|_{\Gamma_D} = g_D$ such that

$$a(u,v) = \langle F, v\rangle \quad \forall v \in \mathscr{W}_D = \{v \in \mathscr{W} = H^{1/2}(\Gamma_S) : v|_{\Gamma_D} = 0\}, \tag{2}$$

where the bilinear from $a(u,v)$ and the linear form $\langle F, v\rangle$ are defined by $a(u,v) = \sum_{T\in\mathscr{T}} \langle S_T u|_{\partial T}, v|_{\partial T}\rangle$ and $\langle F, v\rangle = \sum_{T\in\mathscr{T}} \left[\int_T f \mathscr{H}_T(v|_{\partial T})\,dx + \int_{\partial T\cap\Gamma_N} g_N v\,ds\right]$, respectively. It is easy to see that the skeletal variational formulation (2) is equivalent to the standard variational formulation (1) in the sense that the solution of the former is the skeletal trace of the solution of the latter [3].

3 Approximation of the Steklov-Poincaré operator

It is well-known [10] that the Steklov-Poincaré operator S_T can be expressed as

$$S_T = \alpha_T (V_T^{-1}(\tfrac{1}{2}I + K_T)) = \alpha_T (D_T + (\tfrac{1}{2}I + K_T')V_T^{-1}(\tfrac{1}{2}I + K_T))$$

in terms of the boundary integral operators defined on every element boundary ∂T,

$$V_T : H^{-1/2}(\partial T) \to H^{1/2}(\partial T), \quad K_T : H^{1/2}(\partial T) \to H^{1/2}(\partial T),$$
$$K_T' : H^{-1/2}(\partial T) \to H^{-1/2}(\partial T), \quad D_T : H^{1/2}(\partial T) \to H^{-1/2}(\partial T),$$

called, in turn, the *single layer potential*, *double layer potential*, *adjoint double layer potential*, and *hypersingular* operators. They are defined by means of the fundamental solution of the Laplace equation.

We construct a computable approximation as follows. We assume that each element boundary ∂T has a shape-regular mesh $\mathscr{F}_T$ which consists of line segments in $\mathbb{R}^2$ and of triangles in $\mathbb{R}^3$, and that these local meshes match across elements. On this mesh, we construct a space $\mathscr{Z}_T^h$ of piecewise constant functions and define, given $u \in H^{1/2}(\partial T)$, the discrete variable $w_T^h \in \mathscr{Z}_T^h$ by solving the discrete variational problem $\langle V_T w_T^h, z_T^h \rangle = \langle (\frac{1}{2}I + K_T)u, z_T^h \rangle$ for all $z_T^h \in \mathscr{Z}_T^h$. A computable approximation to S_T is then given by $\widetilde{S}_T u := \alpha_T (D_T u + (\frac{1}{2}I + K_T')w_T^h)$. The approximation $\widetilde{S}_T$ remains self-adjoint and its kernel is given by the constant functions, just as for S_T. Furthermore, it satisfies the spectral equivalence

$$\tilde{c}_T \langle S_T v, v \rangle \leq \langle \widetilde{S}_T v, v \rangle \leq \langle S_T v, v \rangle \qquad \forall v \in H^{1/2}(\partial T) \tag{3}$$

with $\tilde{c}_T \in (0, \frac{1}{4}]$. Replacing, in (2), S_T by its approximations $\widetilde{S}_T$, we obtain the inexact skeletal variational formulation: find $u \in H^{1/2}(\Gamma_S)$ with $u|_{\Gamma_D} = g_D$ such that

$$\widetilde{a}(u,v) := \sum_{T \in \mathscr{T}} \langle \widetilde{S}_T u|_{\partial T}, v|_{\partial T} \rangle = \langle F, v \rangle \qquad \forall v \in \mathscr{W}_D.$$

The positive constant $\tilde{c}_T$ in (3) depends on the geometry of the element T. For robust error estimates, it is necessary to bound $\tilde{c}_T$ from below uniformly for all elements. Recently, explicit bounds for these constants have been obtained, starting with a paper by Pechstein [8] which relied on the Jones parameter and a constant in an isoperimetric inequality. These results were employed in the rigorous *a priori* error analysis of the BEM-based FEM [3, 2] and have later been simplified in [4].

Theorem 1 ([4]). *Let $\Omega \subset \mathbb{R}^3$. Assume that there exists a shape-regular simplicial mesh $\Xi(\Omega')$ of an open, bounded superset $\Omega' \supset \overline{\Omega}$ of Ω such that each element $T \in \mathscr{T}$ is a union of simplices from $\Xi(\Omega')$, and the number of simplices per element T is uniformly bounded. Furthermore, assume that the boundary meshes $\mathscr{F}_T$, $T \in \mathscr{T}$, are shape-regular.*

Then, the contraction constants $\tilde{c}_T$, $T \in \mathscr{T}$, are uniformly bounded away from 0 *in terms of the mesh regularity parameters.*

4 Discretization

By assumption, $\mathscr{F} := \bigcup_{T \in \mathscr{T}} \mathscr{F}_T$ describes a shape-regular triangulation of the skeleton Γ_S. On this mesh, we construct the discrete trial space $\mathscr{W}^h \subset H^{1/2}(\Gamma_S)$ of piecewise linear, continuous functions on the skeleton and set $\mathscr{W}_D^h := \mathscr{W}^h \cap \mathscr{W}_D$. After this discretization, we aim to find $u^h \in \mathscr{W}^h$ with $u^h|_{\Gamma_D} = g_D$ such that

$$\widetilde{a}(u^h, v^h) = \langle F, v^h \rangle \qquad \forall v^h \in \mathscr{W}_D^h. \tag{4}$$

Rigorous error estimates of optimal order for this discretized variational problem can be found in [3, 2]. Equivalently, (4) can be written as an operator equation

$$Au^h = F \tag{5}$$

with $A : \mathscr{W}^h \to (\mathscr{W}_D^h)^*$. The associated stiffness matrix in the canonical nodal basis shares many properties with the stiffness matrix obtained from a standard finite element method like sparsity, symmetry and positive definiteness.

5 A FETI solver

In the following, we derive a solution method for (5) based on the ideas of the FETI substructuring approach, originally proposed by Farhat and Roux [1]. Our derivation closely follows that of the classical FETI method. Thus, we refer to the monographs [11] and [7] and the references therein for further details and proofs.

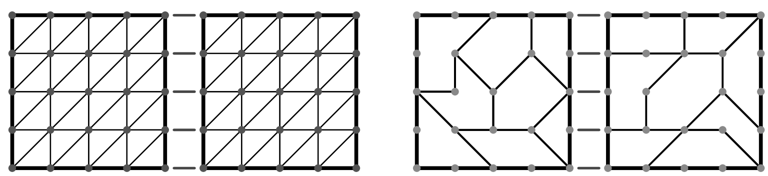

Fig. 1 Sketch of domain decomposition approach in 2D for a rectangular domain with $N = 2$ subdomains. *Left:* FETI substructuring. *Right:* FETI-like substructuring for the BEM-based FEM.

We decompose Ω into non-overlapping subdomains $(\Omega_i)_{i=1}^N$ in agreement with the polyhedral mesh $\mathscr{T}$, that is, $\overline{\Omega}_i = \bigcup_{T \in \mathscr{T}_i} \overline{T}$ with an associate decomposition $(\mathscr{T}_i)_{i=1}^N$. We set $H_i := \operatorname{diam} \Omega_i$ and $H := \max_{i=1}^N H_i$. Every subdomain Ω_i has an associated skeleton $\bigcup_{T \in \mathscr{T}_i} \partial T$ and discrete skeletal trial spaces $\mathscr{W}^h(\Omega_i)$ and $\mathscr{W}_D^h(\Omega_i)$, constructed as in Section 4. In the following, we assume that the problem has been homogenized with respect to the given Dirichlet data g_D, such that $u^h \in \mathscr{W}_D^h$.

Both the operator A and the functional F in (5) can be written as a sum of local contributions $A_i : \mathscr{W}^h(\Omega_i) \to \mathscr{W}^h(\Omega_i)^*$ and $f_i \in \mathscr{W}^h(\Omega_i)^*$ such that $\sum_{i=1}^N A_i(u|_{\Omega_i}) = \sum_{i=1}^N f_i$, where here and in the sequel we drop the superscript h since all functions are discrete from now on. Indeed, all relevant functions live in spaces of piecewise linear functions which have natural nodal bases. Therefore, we will not distinguish in the following between functions and the coefficient vectors representing them with respect to the nodal basis, nor between operators and their matrix representations.

We introduce the Schur complement $\widetilde{S}_i = A_{i,\Gamma\Gamma} - A_{i,\Gamma I} A_{i,II}^{-1} A_{i,I\Gamma}$ of the subdomain stiffness matrix A_i. The blocks $A_{i,\Gamma\Gamma}, A_{i,\Gamma I}, A_{i,I\Gamma}, A_{i,II}$ are chosen such that the subscripts Γ and I correspond to the boundary and inner degrees of freedom, i.e.,

$$A_i w = \begin{bmatrix} A_{i,\Gamma\Gamma} & A_{i,\Gamma I} \\ A_{i,I\Gamma} & A_{i,II} \end{bmatrix} \begin{bmatrix} w_\Gamma \\ w_I \end{bmatrix}.$$

Eliminating the interior unknowns in (5) yields the equivalent minimization problem

$$u = \arg\min_{v\in\mathscr{W}_D^h(\Gamma_S^H)} \frac{1}{2}\sum_{i=1}^{N}\langle \widetilde{S}_i v|_{\partial\Omega_i}, v|_{\partial\Omega_i}\rangle - \sum_{i=1}^{N}\langle g_i, v|_{\partial\Omega_i}\rangle, \tag{6}$$

where $\Gamma_S^H = \bigcup_{i=1}^N \partial\Omega_i$ is the coarse skeleton, $\mathscr{W}_D^h(\Gamma_S^H)$ is the trace space of discrete functions $\mathscr{W}_D^h(\Omega)$ onto Γ_S^H, and g_i is a suitably adjusted forcing term.

Let $\mathscr{W}^h(\partial\Omega_i) := \{v|_{\partial\Omega_i} : v\in\mathscr{W}^h(\Omega_i)\}$ denote a space of discrete boundary functions. We then introduce the broken space $Y := \prod_{i=1}^N Y_i$ with $Y_i := \{v\in\mathscr{W}^h(\partial\Omega_i) : v|_{\Gamma_D} = 0\}$. In order to enforce continuity of the functions in Y, we introduce the jump operator $B : Y \to \mathbb{R}^{N_\Lambda}$, where $N_\Lambda \in \mathbb{N}$ is the total number of constraints. Here we assume fully redundant constraints, i.e., for every node on a subdomain interface, constraints corresponding to all neighboring subdomains are introduced. This choice implies that B is not surjective, and we define the space of Lagrange multipliers as the range $\Lambda := \mathrm{Range}\,B \subseteq \mathbb{R}^{N_\Lambda}$ and consider B as a mapping $Y \to \Lambda$.

Using the jump operator, we rewrite (6) as $u = \arg\min_{y\in\ker B} \frac{1}{2}\sum_{i=1}^N\langle \widetilde{S}_i y_i, y_i\rangle - \sum_{i=1}^N\langle g_i, y_i\rangle$. Introducing Lagrange multipliers to enforce the constraint $By = 0$, we obtain the saddle point formulation

$$\begin{bmatrix} \widetilde{S} & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}, \tag{7}$$

for $u\in Y$ and $\lambda\in\Lambda$, with the block matrices and vectors $\widetilde{S} = \mathrm{diag}(\widetilde{S}_1,\dots,\widetilde{S}_N)$, $B = (B_1,\dots,B_N)$, $u = (u_1,\dots,u_N)^\top$, $g = (g_1,\dots,g_N)^\top$. From (7), we see that the local skeletal functions u_i satisfy the relationship

$$\widetilde{S}_i u_i = g_i - B_i^\top\lambda. \tag{8}$$

For a *non-floating* domain Ω_i, that is, one that shares a part of the Dirichlet boundary such that $\partial\Omega_i\cap\Gamma_D \neq \emptyset$, $\widetilde{S}_i$ is positive definite and thus invertible. For a *floating* domain Ω_i, the kernel of $\widetilde{S}_i$ consists only of the constant functions, and we parameterize it by the operator $R_i : \mathbb{R} \to \ker\widetilde{S}_i \subset Y_i$ which maps a scalar to the corresponding constant function. Under the condition that the right-hand side is orthogonal to the kernel, i.e.,

$$\langle g_i - B_i^\top\lambda, R_i\zeta\rangle = 0 \qquad \forall\zeta\in\mathbb{R}, \tag{9}$$

the local problem (8) is solvable and we have $u_i = \widetilde{S}_i^\dagger(g_i - B_i^\top\lambda) + R_i\xi_i$ with some $\xi_i\in\mathbb{R}$. Here, $\widetilde{S}_i^\dagger$ denotes a pseudo-inverse of $\widetilde{S}_i$. For non-floating domains Ω_i, we set $\widetilde{S}_i^\dagger = \widetilde{S}_i^{-1}$.

We set $Z := \prod_{i=1}^N \mathbb{R}^{\dim(\ker\widetilde{S}_i)}$ and introduce the operator $R : Z \to Y$ by $(R\xi)|_{\Omega_i} := R_i\xi_i$ for floating Ω_i and $(R\xi)|_{\Omega_i} := 0$ for non-floating Ω_i. The local solutions u can then be expressed by

$$u = \widetilde{S}^\dagger(g - B^\top\lambda) + R\xi \tag{10}$$

under the compatibility condition $R^\top B^\top\lambda = R^\top g$ derived from (9). Inserting (10) into the second line of (7) yields $B\widetilde{S}^\dagger g - B\widetilde{S}^\dagger B^\top\lambda + BR\xi = 0$, and together with the compatibility condition and using the notations $F = B\widetilde{S}^\dagger B^\top$ and $G = BR$, we obtain the dual saddle point problem

$$\begin{bmatrix} F & -G \\ G^\top & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \xi \end{bmatrix} = \begin{bmatrix} B\widetilde{S}^\dagger g \\ R^\top g \end{bmatrix}. \tag{11}$$

With a self-adjoint operator $Q : \Lambda \to \Lambda$ which is assumed to be positive definite on the range of G and which will be specified later, we define the projector $P = I - QG(G^\top QG)^{-1}G^\top$ from Λ onto the subspace $\Lambda_0 := \ker G^\top \subset \Lambda$ of admissible increments. The choice $\lambda_g := QG(G^\top QG)^{-1}R^\top g \in \Lambda$ ensures that $G^\top \lambda_g = R^\top g$, and thus, with $\lambda = \lambda_0 + \lambda_g$, we can homogenize (11) such that we only search for a $\lambda_0 \in \Lambda_0$ with

$$F\lambda_0 - G\xi = B\widetilde{S}^\dagger g - F\lambda_g. \tag{12}$$

Applying the projector $P^\top$ to this equation and noting that $P^\top G = 0$, we obtain the following formulation of the dual problem: find $\lambda_0 \in \Lambda_0$ such that

$$P^\top F\lambda_0 = P^\top(B\widetilde{S}^\dagger g - F\lambda_g) = P^\top B\widetilde{S}^\dagger(g - B^\top \lambda_g). \tag{13}$$

It can be shown that $P^\top F$ is self-adjoint and positive definite on Λ_0. Thus, the problem (13) has a unique solution which may be computed by CG iteration in the subspace Λ_0. Once $\lambda = \lambda_0 + \lambda_g$ has been computed, we see that applying $(G^\top QG)^{-1}G^\top Q$ to (12) yields $\xi = (G^\top QG)^{-1}G^\top QB\widetilde{S}^\dagger(B^\top \lambda - g)$. The unknowns u_i may then be obtained by solving the local problems (10), and the unknowns in the interior of each Ω_i may be recovered by solving local Dirichlet problems.

Preconditioners for FETI are typically constructed in the form PM^{-1} with a suitable operator $M^{-1} : \Lambda \to \Lambda$. The FETI Dirichlet preconditioner adapted to our setting, is given by the choice $M^{-1} = B\widetilde{S}B^\top$ and works well for constant or mildly varying coefficient α. In this case, the choice $Q = I$ works satisfactorily.

To deal with coefficient jumps, we need to employ a *scaled* or *weighted jump operator* as introduced in [9] and analyzed in [5]. We restrict ourselves to the case of subdomain-wise constant coefficient α, i.e., $\alpha(x) = \alpha_i$ for $x \in \Omega_i$.

Let $x^h \in \partial\Omega_i$ refer to a boundary node. We introduce weighted counting functions δ_j via piecewise linear interpolation on the facets of the coarse skeleton Γ_S^H of the nodal values defined by $\delta_j(x^h) = \alpha_j / (\sum_{k \in \{1,\ldots,N\}: x^h \in \partial\Omega_k} \alpha_k)$ for $x^h \in \partial\Omega_j$ and 0 otherwise, $j = 1,\ldots,N$. We introduce diagonal scaling matrices $D_i : \Lambda \to \Lambda$, $i = 1,\ldots,N$, operating on the space of Lagrange multipliers. Consider two neighboring domains Ω_i and Ω_j sharing a node $x^h \in \partial\Omega_i \cap \partial\Omega_j$. Let $k \in \{1,\ldots,N_\Lambda\}$ denote the index of the Lagrange multiplier associated with this node and pair of subdomains. Then, the k-th diagonal entry of D_i is set to $\delta_j(x^h)$, and the k-th diagonal entry of D_j to $\delta_i(x^h)$. Diagonal entries of D_i not associated with a node on $\partial\Omega_i$ are set to 0.

The *weighted jump operator* $B_D : Y \to \Lambda$ is now given by $B_D = [D_1B_1,\ldots,D_NB_N]$, and the weighted Dirichlet preconditioner by $M_D^{-1} = B_D\widetilde{S}B_D^\top$. In this case, a possible choice for Q is simply $Q = M_D^{-1}$. Alternatively, Q can be replaced by a suitable diagonal matrix as described in [5].

6 Convergence Analysis

The convergence analysis proceeds by the idea of spectral equivalences between the BEM-based FEM Schur complements $\widetilde{S}_i$ and the Schur complements which occur in a standard one-level FETI method, allowing us to transfer the known condition estimates from the FETI literature to our case. This is similar to the approach used in the analysis of the BETI method [6]. For space reason, we cannot give this analysis here, and it must be postponed to a forthcoming paper. Here we only state the main results. Under standard assumptions, we can prove the condition number estimate

$$\kappa(P^\top F|_{\Lambda_0}) \leq C(\overline{\alpha}/\underline{\alpha}) \max_{i=1,\ldots,N}(H_i/h_i)$$

for the non-preconditioned case, where $\overline{\alpha} = \max_{x\in\Omega} \alpha(x)$, $\underline{\alpha} = \min_{x\in\Omega} \alpha(x)$, and the constant C depends only on mesh regularity parameters. For the preconditioned case, with the choice $Q = M_D^{-1}$, we have the condition number estimate

$$\kappa(PM_D^{-1}P^\top F|_{\Lambda_0}) \leq C\,(1+\log(\max_{i=1,\ldots,N}(H_i/h_i)))^2.$$

7 Numerical experiments

We solve the pure Dirichlet boundary value problem $-\Delta u = 0$ in Ω and $u(x) = -(2\pi)^{-1}\log|x-x^\star|$ on $\partial\Omega$. The 2d domain Ω (Figure 2, left) is discretized by an irregular polygonal mesh. The source point $x^\star = (-1,1)$ lies outside of Ω.

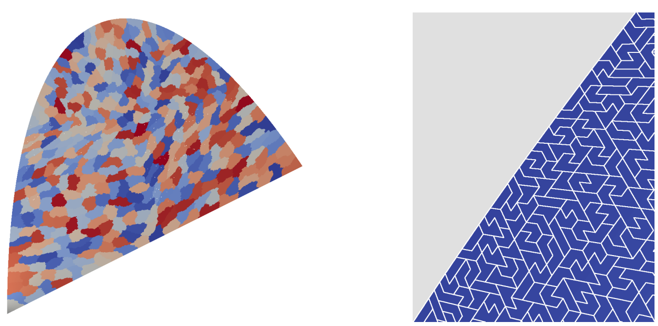

Fig. 2 *Left:* Ω partioned into $N = 400$ subdomains. *Right:* Zoom into the polygonal mesh.

The polygonal mesh $\mathscr{T}$ is constructed by applying the graph partitioner METIS to a standard triangular mesh consisting of 524,288 triangles, resulting in a polygonal mesh with 99,970 elements, most of which are unions of 5 or 6 triangles, cf. Figure 2, right. The domain decomposition $\{\Omega_i\}$ is obtained by applying METIS a second time on top of the mesh $\mathscr{T}$, see Figure 2, left.

We use the Dirichlet preconditioner with multiplicity scaling and a suitable diagonal matrix for Q as described in [5], and solve the dual system by the corresponding PCG iteration. In Table 1, we give the number of CG iterations required to reduce the initial residual by a factor of 10^{-8} without and with Dirichlet preconditioner, and provide some CPU times for varying number N of subdomains.

N	total time	avg. loc. time	#iter	# Lagrange
25	32.23 / **20.49**	0.0776 / **0.0759**	133 / **29**	5875
50	30.19 / **19.10**	0.0317 / **0.0310**	135 / **30**	8962
100	26.64 / **17.70**	0.0135 / **0.0131**	131 / **31**	13012
200	23.69 / **17.41**	0.0059 / **0.0057**	134 / **36**	19056
400	21.06 / **16.13**	0.0027 / **0.0026**	123 / **34**	27324
800	20.23 / **17.68**	0.0013 / **0.0013**	109 / **36**	39304
1600	22.19 / **20.96**	0.0006 / **0.0006**	095 / **35**	56632

Table 1 Results of the non-preconditioned (left) / **preconditioned (right)** CG solver. Columns: number of subdomains, total CPU time for the solution in seconds, averaged time for solving the local problems in seconds, number of iterations, number of Lagrange multipliers.

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