Robust isogeometric Schwarz preconditioners for composite elastic materials

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1 Introduction

In this paper, we study Overlapping Schwarz preconditioners for the system of linear elasticity for composite materials discretized with Isogeometric Analysis (IGA). This is an innovative numerical methodology, introduced by Hughes et al. [10, 6], where the geometry description of the PDE domain is adopted from a Computer Aided Design (CAD) parametrization usually based on Non-Uniform Rational B-Splines (NURBS). In IGA, these NURBS basis functions representing the CAD geometry are also used as the PDEs discrete basis, following an isoparametric paradigm. Since its introduction, IGA techniques have been studied and applied in diverse fields, see e.g. [6].

In our previous Domain Decomposition (DD) works for IGA scalar elliptic problems, we studied Overlapping Additive Schwarz (OAS) methods [2] and Balancing Domain Decomposition by Constraints (BDDC) methods [3], providing optimal and quasi-optimal convergence rate bounds for isogeometric DD methods, together with the required theoretical foundation, technical tools and numerical validation. Other DD IGA works have explored numerically dual primal Finite Element Tearing and Interconnecting (FETI-DP) methods for 2D elliptic problems [11] and have studied multigrid methods for the 2D and 3D Laplacian [9] and Schwarz methods in the case of two subdomains with non-matching grids [5].

Here we study Isogeometric OAS preconditioners for the system of linear elasticity for compressible composite materials. An extension to mixed methods for almost incompressible elastic materials can be found in [4].

We consider the linear elastic deformation of a body Ω in $\mathbb{R}^d$, $d = 2,3$, with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$. The body is clamped on Γ_D and it is subjected to a given traction $g : \Gamma_N \to \mathbb{R}^d$ on Γ_N, as well as to a body force density $f : \Omega \to \mathbb{R}^d$. The displacement field $u : \Omega \to \mathbb{R}^d$ satisfies the system

$$\begin{cases} \operatorname{div}\mathbb{C}\varepsilon(u) + f = 0 & \text{in } \Omega \\ u = 0 \text{ on } \Gamma_D \quad \text{and} \quad \mathbb{C}\varepsilon(u)\cdot n = g \text{ on } \Gamma_N \end{cases} \tag{1}$$

Here, ε is the symmetric gradient operator, n is the unit outward normal at each point of the boundary, $\mathbb{C}\tau = 2\mu\tau + \lambda \operatorname{tr}(\tau)I$ for all second order tensors τ, where $\operatorname{tr}(\tau)$ is

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the trace of τ, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \mu = \frac{E}{2(1+\nu)}$ are the Lamé constants, E is the Young modulus and ν the Poisson's ratio. Given loadings $f \in [L^2(\Omega)]^d$ and $g \in [L^2(\Gamma_N)]^d$, define

$$< \psi, v >= (f, v)_\Omega + (g, v)_{\Gamma_N} \qquad \forall v \in [H^1(\Omega)]^d, \tag{2}$$

where $(\cdot,\cdot)_\Omega$, $(\cdot,\cdot)_{\Gamma_N}$ indicate as usual the L^2 scalar product respectively on Ω and Γ_N. The variational formulation of problem (1) then reads:

$$\begin{cases} \text{Find } u \in [H^1_{\Gamma_D}(\Omega)]^d \text{ such that:} \\ a(u,v) =< \psi, v > \qquad \forall v \in [H^1_{\Gamma_D}(\Omega)]^d, \end{cases} \tag{3}$$

where $[H^1_{\Gamma_D}(\Omega)]^d = \{v \in [H^1(\Omega)]^d \mid v|_{\Gamma_D} = 0\}$ and

$$a(w,v) = \int_\Omega 2\mu\, \varepsilon(w) : \varepsilon(v)\, dx + (\lambda\, \mathrm{div} w, \mathrm{div} v)_\Omega \qquad \forall w, v \in [H^1_{\Gamma_D}(\Omega)]^d. \tag{4}$$

2 Isogeometric discretization of linear elasticity

We discretize the elasticity system (3) with IGA based on B-splines and NURBS basis functions, see e.g. [6]. Considering for simplicity the two-dimensional case, the bivariate B-spline discrete space is defined as

$$\widehat{\mathscr{S}_h} = \mathrm{span}\{B^{p,q}_{i,j}(\xi,\eta),\ i = 1,\ldots,n,\ j = 1,\ldots,m\}, \tag{5}$$

where the bivariate B-spline basis functions $B^{p,q}_{i,j}(\xi,\eta) = N^p_i(\xi)\, M^q_j(\eta)$ are defined by tensor product of one-dimensional B-splines functions $N^p_i(\xi)$ and $M^q_j(\eta)$ of degree p and q, respectively. Analogously, the NURBS space is the span of NURBS basis functions defined in 1D as

$$R^p_i(\xi) = \frac{N^p_i(\xi)\omega_i}{\sum_{\hat{i}=1}^n N^p_{\hat{i}}(\xi)\omega_{\hat{i}}} = \frac{N^p_i(\xi)\omega_i}{w(\xi)}, \tag{6}$$

(with weight function $w(\xi) = \sum_{\hat{i}=1}^n N^p_{\hat{i}}(\xi)\omega_{\hat{i}} \in \widehat{\mathscr{S}_h}$), and in 2D by tensor product

$$R^{p,q}_{i,j}(\xi,\eta) = \frac{B^{p,q}_{i,j}(\xi,\eta)\omega_{i,j}}{\sum_{\hat{i}=1}^n \sum_{\hat{j}=1}^m B^{p,q}_{\hat{i},\hat{j}}(\xi,\eta)\omega_{\hat{i},\hat{j}}} = \frac{B^{p,q}_{i,j}(\xi,\eta)\omega_{i,j}}{w(\xi,\eta)}, \tag{7}$$

where $w(\xi,\eta)$ is the weight function and $\omega_{i,j} = (C^\omega_{i,j})_3$ the weights associated with a $n \times m$ net of control points $C_{i,j}$. The discrete space of NURBS scalar fields on the domain Ω is defined, component by component as the span of the *push-forward* of the NURBS basis functions (7)

$$\mathscr{N}_h := \mathrm{span}\{R^{p,q}_{i,j} \circ F^{-1},\ \text{with } i = 1,\ldots,n; j = 1,\ldots,m\}, \tag{8}$$

with $F : \widehat{\Omega} \rightarrow \Omega$ the geometrical map between parameter and physical spaces

$$F(\xi,\eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}^{p,q}(\xi,\eta) C_{i,j}. \tag{9}$$

Taking into account the boundary conditions, if for simplicity we consider the case $\Gamma_D = \partial\Omega$, we define the spline space in parameter space as

$$\widehat{V}_h = [\widehat{\mathscr{S}_h} \cap H_0^1(\widehat{\Omega})]^d = [\mathrm{span}\{B_{i,j}^{p,q}(\xi,\eta),\ i = 2,\ldots,n-1,\ j = 2,\ldots,m-1\}]^d.$$

and the NURBS space in physical space as

$$V_h = [\mathscr{N}_h \cap H_0^1(\Omega)]^d = [\mathrm{span}\{R_{i,j}^{p,q} \circ F^{-1}, \text{ with } i = 2,\ldots,n-1;\ j = 2,\ldots,m-1\}]^d. \tag{10}$$

The IGA formulation of problem (3) then reads:

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that:} \\ a(u_h, v_h) = < \psi, v_h > \qquad \forall v_h \in V_h. \end{cases} \tag{11}$$

3 Isogeometric Overlapping Schwarz preconditioners

We refer to the monographs [12, 13] for a general introduction to Overlapping Schwarz methods. We describe first the subdomain and subspace decompositions in 1D and then extend them by tensor products to 2D and 3D. The decomposition is first built for the underlying space of spline functions in parameter space, and then easily extended to the NURBS space in the physical domain.

1D B-spline decomposition. From the full set of knots $\{\xi_1 = 0,...,\xi_{n+p+1} = 1\}$, we select a subset $\{\xi_{i_k}, k = 1,\ldots,N+1\}$ of (non repeated) interface knots with $\xi_{i_1} = 0, \xi_{i_{N+1}} = 1$. This subset of interface knots defines a decomposition of the closure of the reference interval

$$\overline{(\widehat{I})} = [0,1] = \overline{\Big(\bigcup_{k=1,..,N} \widehat{I}_k\Big)}, \text{ with } \widehat{I}_k = (\xi_{i_k}, \xi_{i_{k+1}}),$$

that we assume to have a similar characteristic diameter $H \approx H_k = \mathrm{diam}(\widehat{I}_k)$. The interface knots are thus given by ξ_{i_k} for $k = 2,..,N$. For each of the interface knots ξ_{i_k} we choose an index $2 \leq s_k \leq n-1$ (strictly increasing in k) that satisfies $s_k < i_k < s_k + p + 1$, so that the support of the basis function $N_{s_k}^p$ intersects both $\widehat{I}_{k-1}$ and $\widehat{I}_k$. Note that at least one such s_k exists; if it is not unique, any choice can be made.

We then define an overlapping decomposition of $\widehat{I}$ in the following way. Let $r \in \mathbb{N}$ be an integer (called the overlap index) counting the basis functions shared by adjacent subdomains, defined as

$$\widehat{V}_k = [\mathrm{span}\{N_j^p(\xi),\ s_k - r \le j \le s_{k+1} + r\}]^d \qquad k = 1,2,..,N, \tag{12}$$

with the exception that $2 \le j \le s_2 + r$ for the space $\widehat{V}_1$ and $s_N - r \le j \le n-1$ for the space $\widehat{V}_N$. These subspaces form an overlapping decomposition of the spline space $\widehat{V}_h$. For $r = 0$ we have the minimal overlap consisting of just one common basis function between subspaces, while more generally $2r+1$ represents the number of basis functions in common (in the univariate case) among "adjacent" local subspaces. We now define the extended subdomains $\widehat{I}'_k$ by

$$\widehat{I}'_k = \bigcup_{N_j^p \in \widehat{V}_k} \mathrm{supp}(N_j^p) = (\xi_{s_k - r}, \xi_{s_{k+1}+r+p+1}), \tag{13}$$

with the analogous exception for $\widehat{I}'_1$, $\widehat{I}'_N$,

We consider two choices for the *coarse space* $\widehat{V}_0$.

a) A nested coarse space defined by introducing a (open) coarse knot vector $\xi_0 = \{\xi_1^0 = 0, ..., \xi_{N_c+p+1}^0 = 1\}$ corresponding to a coarse mesh determined by the subdomains $\widehat{I}_k$, i.e.

$$\xi_0 = \{\xi_1, \xi_2, \ldots, \xi_p, \xi_{i_1}, \xi_{i_2}, \xi_{i_3}, \ldots, \xi_{i_N}, \xi_{i_{N+1}}, \xi_{i_{N+1}+1}, \xi_{i_{N+1}+2} \ldots, \xi_{i_{N+1}+p}\},$$

such that the distance between adjacent distinct knots is of order H, $\xi_1 = \cdots = \xi_p = \xi_{i_1} = 0$ and $\xi_{i_{N+1}} = \xi_{i_{N+1}+1} = \cdots = \xi_{i_{N+1}+p} = 1$. A coarse spline space is then defined as

$$\widehat{V}_0 := [\widehat{\mathscr{S}_H}]^d = [\mathrm{span}\{N_i^{0,p}(\xi),\ i = 2, ..., N_c - 1\}]^d,$$

with the same degree p of $\widehat{\mathscr{S}_h}$ and is thus a subspace of $[\widehat{\mathscr{S}_h}]^d$.

b) A non-nested coarse space, of smaller dimension than in case a), is defined as

$$\widehat{V}_0 := [\widehat{\mathscr{S}_H}]^d = [\mathrm{span}\{N_i^{0,1}(\xi),\ i = 2, ..., N_c - 1\}]^d,$$

where now note that $p = 1$ and the coarse knot vector (and N_c) is changed accordingly

$$\xi_0 = \{\xi_1, \xi_{i_1}, \xi_{i_2}, \xi_{i_3}, \ldots, \xi_{i_N}, \xi_{i_{N+1}}, \xi_{i_{N+1}+1}\},$$

with $\xi_1 = \xi_{i_1} = 0$ and $\xi_{i_{N+1}} = \xi_{i_{N+1}+1} = 1$. The construction above gives the standard piecewise linear space on the coarse subdivision.

2D, 3D B-spline decomposition. By tensor product (here in 2D for simplicity), we define subdomains, overlapping subdomains and extended supports by

$$\widehat{\Omega}_{kl} = \widehat{I}_k \times \widehat{I}_l, \qquad \widehat{\Omega}'_{kl} = \widehat{I}'_k \times \widehat{I}'_l, \qquad 1 \le k \le N,\ 1 \le l \le M,$$

(where $\widehat{I}_k = (\xi_{i_k}, \xi_{i_{k+1}}), \widehat{I}_l = (\eta_{j_l}, \eta_{j_{l+1}})$). Moreover, we take the indices $\{s_k\}_{k=2}^N$ associated to $\{\xi_{i_k}\}_{k=2}^N$ and the analogous indices $\{\bar{s}_l\}_{l=2}^M$ associated to $\{\eta_{j_l}\}_{l=2}^M$. The local and coarse subspaces are then defined by

$$\widehat{V}_{kl} = [\text{span}\{B^{p,q}_{i,j}(\xi,\eta),\ s_k - r \le i \le s_{k+1}+r,\ \overline{s}_l - r \le j \le \overline{s}_{l+1}+r\ \}]^d,$$
$$\widehat{V}_0 = [\text{span}\{\overset{\circ}{B}{}^{p,q}_{i,j}:\ \overset{\circ}{B}{}^{p,q}_{i,j}(\xi,\eta) := N^{0,p}_i(\xi)M^{0,q}_j(\eta),\ i=1,...,N_c, j=1,...,M_c\}]^d,$$

with the usual modification for boundary subdomains and where $\overset{\circ}{B}{}^{p,q}_{i,j}$ are the coarse basis functions.

2D, 3D NURBS decomposition. The subdomains in physical space are defined as the image of the subdomains in parameter space with respect to the mapping F

$$\Omega_{kl} = F(\widehat{\Omega}_{kl}), \quad \Omega'_{kl} = F(\widehat{\Omega}'_{kl}).$$

The local subspaces and the coarse space are, up to the usual modification for the boundary subdomains,

$$V_{kl} = [\text{span}\{R^{p,q}_{i,j}\circ F^{-1},\ s_k - r \le i \le s_{k+1}+r,\ \overline{s}_l - r \le j \le \overline{s}_{l+1}+r\ \}]^d,$$
$$V_0 = [\text{span}\{\overset{\circ}{R}{}^{p,q}_{i,j}\circ F^{-1} := (\overset{\circ}{B}{}^{p,q}_{i,j}/w)\circ F^{-1},\ i=1,...,N_c, j=1,...,M_c\}]^d,$$

where we recall that w is the weight function, see (7).

Overlapping Schwarz preconditioners. Given the local and coarse embedding operators $I_{kl}: V_{kl} \to V_h$, $k=1,..,N$, $l=1,..,M$ and $I_0: V_0 \to V_h$, the discrete space V_h can be decomposed into coarse and local space as

$$V_h = I_0 V_0 + \sum_{k,l} I_{kl} V_{kl}.$$

Define the local projections $\widetilde{T}_{kl}: V_h \to V_{kl}$ by

$$a(\widetilde{T}_{kl}u, v) = a(u, I_{kl}v) \quad \forall v \in V_{kl},$$

and the coarse projection $\widetilde{T}_0: V_h \to V_0$ by

$$a(\widetilde{T}_0 u, v) = a(u, I_0 v) \quad \forall v \in V_0.$$

Defining $T_{kl} = I_{kl}\widetilde{T}_{kl}$ and $T_0 = I_0\widetilde{T}_0$, our two-level Overlapping Additive Schwarz (OAS) operator is then

$$T_{OAS} := T_0 + \sum_{k=1}^{N}\sum_{l=1}^{M} T_{kl}. \tag{14}$$

The matrix form of this operator is $T_{OAS} = B_{OAS}\mathscr{A}$, where $\mathscr{A}$ is the stiffness matrix and B_{OAS} is the Additive Schwarz preconditioner

$$B_{OAS} = R_0^T A_0^{-1} R_0 + \sum_{k=1}^{N}\sum_{l=1}^{M} R_{kl}^T A_{kl}^{-1} R_{kl}. \tag{15}$$

Here, R_{kl} are restriction matrices with $0,1$ entries returning the coefficients of the basis functions belonging to the local spaces V_{kl} and A_{kl} are the local stiffness matrices restricted to the subspace V_{kl}. If the coarse space is nested into the fine space, R_0^T is the coarse-to-fine interpolation matrix and A_0 is the coarse stiffness matrix associated with the coarse space V_0. If the coarse space is non-nested, R_0^T is the coarse-to-fine L^2-projection matrix and the coarse space stiffness matrix is given by $A_0 = R_0 \mathscr{A} R_0^T$.

A convergence rate bound. Given the overlap index r defined before (12), we define the overlap parameter

$$\gamma = h(2r+2), \tag{16}$$

that is related to the width δ of the overlapping region by the bounds $\gamma = h(2r+2) \leq \delta \leq h(2r+p+1) \leq \frac{p+1}{2}\gamma$. Assuming that a) the parametric mesh is quasi-uniform, and b) the overlap index r is bounded from above by a fixed constant, we have the following result (see [4]).

Theorem 1. *The condition number of the 2-level additive Schwarz preconditioned operator T_{OAS} defined in (14), with either nested or non-nested coarse space, is bounded by*

$$\kappa_2(T_{OAS}) \leq C\left(1+\frac{H}{\gamma}\right),$$

where $\gamma = h(2r+2)$ is the overlap parameter defined in (16) and C is a constant independent of h, H, N, γ (but not of p, k).

4 Numerical results

In this section, we test the convergence properties of the isogeometric OAS preconditioner defined in (15) for linear elasticity problems on 3D domains. The IGA discretization with mesh size h, polynomial degree p, regularity k, is carried out by using the Matlab isogeometric library GeoPDEs [7]. The domain is decomposed into N overlapping subdomains of characteristic size H and overlap index r. The resulting linear system is solved by PCG with the isogeometric OAS preconditioner, with zero initial guess and a stopping criterion of 10^{-6} reduction of the relative residual.

Table 1 shows the scalability of the proposed isogeometric OAS preconditioner for a reference cubic domain decomposed into an increasing number of subdomainscho N of fixed subdomain size $H/h = 4$ (scaled speedup test), $p = 3,\ k = 2$, overlap $r = 0$ and $r = 1$, and both nested (left) and non-nested (right) coarse spaces. In addition to scalability, the results show that the two coarse spaces have similar performances and both improve when increasing the overlap size.

Table 2 illustrates the robustness of the OAS preconditioner for composite materials where the Young modulus E presents discontinuities across subdomain boundaries. The deformed 3D domain is a twisted bar shown in Fig. 1 (right), discretized by $16 \times 16 \times 8$ fine elements, $N = 4 \times 4 \times 2$ subdomains, and NURBS with $p = 3$ and $k = 2$ (except at the subdomain interfaces where $k = 0$). In the central jump test, the jump region consists of the $2 \times 2 \times 2$ central subdomains. Outside the jump

	nested coarse space				non-nested coarse space			
	$r=0$		$r=1$		$r=0$		$r=1$	
N	$\kappa_2=\lambda_{\max}/\lambda_{\min}$	n_{it}	$\kappa_2=\lambda_{\max}/\lambda_{\min}$	n_{it}	$\kappa_2=\lambda_{\max}/\lambda_{\min}$	n_{it}	cho $\kappa_2=\lambda_{\max}/\lambda_{\min}$	n_{it}
$2\times2\times2$	$16.3=8.03/0.49$	22	$9.1=8.25/0.91$	19	$17.2=8.03/0.47$	23	$9.3=8.25/0.89$	21
$3\times3\times3$	$18.5=8.04/0.43$	25	$11.2=9.31/0.83$	22	$22.8=8.04/0.35$	28	$12.8=9.68/0.76$	25
$4\times4\times4$	$19.8=8.04/0.41$	26	$11.9=9.47/0.80$	23	$20.1=8.04/0.40$	27	$12.0=9.47/0.79$	24
$5\times5\times5$	$20.2=8.04/0.40$	26	$12.1=9.52/0.79$	23	$20.5=8.04/0.39$	27	$12.4=9.53/0.77$	25
$6\times6\times6$	$20.4=8.05/0.40$	26	$12.3=9.56/0.78$	23	$20.6=8.05/0.39$	27	$12.5=9.56/0.76$	25

Table 1 Scalability of OAS preconditioner with nested (left) and non-nested (right) coarse space: condition number $\kappa_2(T_{OAS})$, extremal eigenvalues $\lambda_{\max}$, $\lambda_{\min}$ and PCG iteration counts n_{it} as a function of the number of subdomains N. Cubic domain, fixed $H/h=4$, $p=3$, $k=2$, $E=6e+6$, $\nu=0.3$.

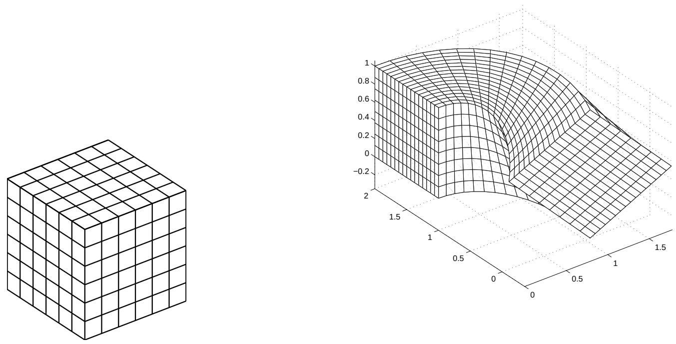

Fig. 1 3D domains used in the numerical tests.

region, $E=6e+3$ and $\nu=0.3$, while inside such region E has the value indicated in Table 2. In the checkerboard test, E alternates between the values $E=6e+3$ and $E=6e+7$, while $\nu=0.3$ everywhere. The results show that the unpreconditioned PCG deteriorate when E jumps towards $6e+7$, while the 2-level OAS preconditioner is very robust for jumps in E.

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Jumping coefficient E, twisted quarter-ring domain							
		unpreconditioned		1-level OAS		2-level OAS	
	E	$\kappa_2 = \frac{\lambda_{max}}{\lambda_{min}}$	n_{it}	$\kappa_2 = \frac{\lambda_{max}}{\lambda_{min}}$	n_{it}	$\kappa_2 = \frac{\lambda_{max}}{\lambda_{min}}$	n_{it}
	$6e+1$	$1.01e+6 = \frac{8.48e+3}{8.40e-3}$	6029	$263.96 = \frac{8.00}{3.03e-2}$	57	$22.40 = \frac{8.47}{0.38}$	28
central	$6e+3$	$1.24e+4 = \frac{8.74e+3}{0.70}$	691	$261.04 = \frac{8.00}{3.06e-2}$	66	$25.79 = \frac{8.34}{0.32}$	30
jump	$6e+5$	$9.92e+5 = \frac{7.08e+5}{0.71}$	5793	$215.73 = \frac{8.00}{3.71e-2}$	55	$26.35 = \frac{8.58}{0.32}$	29
	$6e+7$	$7.85e+7 = \frac{7.08e+7}{0.90}$	20625	$191.83 = \frac{8.00}{4.17e-2}$	54	$30.93 = \frac{8.62}{0.28}$	30
checkerboard E		$8.10e+7 = \frac{3.29e+7}{0.41}$	20625	$70.32 = \frac{8.00}{0.11}$	32	$19.21 = \frac{8.50}{0.44}$	24

Table 2 OAS robustness with respect to jump discontinuities in E. Outside the central jump region of $2 \times 2 \times 2$ subdomains $E = 6e+3$ and $\nu = 0.3$. In the checkerboard test for E, $E = 6e+3$ or $E = 6e+7$. Condition number κ_2, extremal eigenvalues $\lambda_{max}, \lambda_{min}$ and iteration counts n_{it}. Fixed fine mesh $16 \times 16 \times 8$, $N = 4 \times 4 \times 2$, $H/h = 4$, $p = 3$, $k = 2$.

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