ASM-BDDC Preconditioners with variable polynomial degree for CG- and DG-SEM

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1 Introduction

Discontinuous Galerkin (DG) methods for partial differential equations are well suited to treat nonconforming meshes and inhomogeneous polynomial orders required by hp-adaptivity. Their elementwise formulation permit us to consider complex meshes and the relaxation of the continuity constraints allows the polynomial order to be refined locally. However, DG discretizations lead to large and ill-conditioned algebraic systems. In this paper, we study a quasi-optimal preconditioner for the spectral element version of Discontinuous Galerkin methods. In particular, we focus on the interior penalty formulation of such DG schemes. For a review of the different classes of DG methods, the reader is referred to [2].

Recent endeavors in the domain decomposition community have lead to the development of additive [7] and multiplicative [1] Schwarz preconditioners for DG. Among additive Schwarz solvers, nonoverlapping methods such as BDDC (Balancing Domain Decomposition by Constraints) or FETI-DP (Dual-Primal Finite Element Tearing and Interconnecting) for DG have been designed [6] considering only variations on the subdomain size H or the element size h in a finite element context. Based on the pioneer work by [5, 9] and later [10], the BDDC algorithm was recently generalized to CG-SEM (continuous Galerkin spectral elements) in [12, 8]. Following the work in [11], and more recently [3], we make use of the ASM (*Auxiliary Space Method*) to derive a preconditioner for DG-SEM. The paper is organized as follows.

First, we generalize the BDDC preconditioners for CG-SEM studied in [12] to inhomogeneous polynomial distributions, where polynomial degrees is allowed to vary in different elements but we enforce the polynomial degree of the basis functions to match at the interface between elements.

Second, the ASM is presented and applied to derive a solver for DG-SEM based on the previous continuous solver. Once the Schur complement for the continuous problem is solved, the global continuous solution is readily obtained using exact local solvers. The discontinuous solution is then obtained solving the ASM problem. The resulting preconditioner is proved to have the same performances of the BDDC preconditioners for CG-SEM if the polynomial jumps are smooth enough.

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In the last section, we present numerical simulations showing the robustness of the extended BDDC preconditioner with respect to polynomial jumps. The ASM-BDDC is finally tested by varying the number of spectral elements per subdomain H/h, the polynomial degree p and the viscosity coefficients.

The present work is an extension of [4].

2 Balancing Domain Decomposition by Constraints with inhomogeneous polynomial degrees

We consider the second-order elliptic problem with homogeneous Dirichlet boundary conditions

$$-\nabla\cdot(\mu\nabla u) = f \quad \text{in } \Omega\,, \qquad u = 0 \quad \text{on } \partial\Omega\,, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ ($d = 2,3$) is a bounded domain with Lipschitz boundary. Problem (1) admits a unique weak solution in $H_0^1(\Omega)$ if we assume that $f \in L^2(\Omega)$ and $\mu \in L^\infty(\Omega)$, with $\mu \geq \mu_0$ a.e. in Ω for a suitable constant $\mu_0 > 0$.

2.1 CG-SEM discretization for elliptic problems

Given a partition of $\overline{\Omega} = \bigcup_{k=1}^N \Omega_k$ into spectral elements, we define the continuous Galerkin space $\mathscr{V}_\delta^C = \{v : \Omega \to \mathbb{R} \mid \forall k,\ v_{|\Omega_k} \in \mathbb{P}_{p_k}(\Omega_k),\ v \in C^0(\Omega)\}$, that is the space of continuous elementwise polynomial functions. Problem (1) in its weak form is then:
Find $u \in H_0^1(\Omega)$ such that

$$a_c(u,v) = L(v) \quad \forall v \in \mathscr{V}_\delta^C, \tag{2}$$

where

$$a_c(u,v) = \sum_k \int_{\Omega_k} \mu(x)\nabla u\cdot\nabla v dx, \quad L(v) = \sum_k \int_{\Omega_k} f v dx.$$

Considering elliptic coefficients μ that are constant on each spectral element *i.e.* $\mu_{|\Omega_k} = \mu_k$, the bilinear form of problem (2) can be written

$$a_c(u,v) = \sum_k \mu_k a_c^k(u,v). \tag{3}$$

2.2 CG-SEM with locally varying polynomial degrees

The definition of $\mathscr{V}_\delta^C$ allows the polynomial degree to vary inside an element. However, the continuity constraint forces the polynomial degrees to match at the interface between two spectral elements, in the direction parallel to the interface. Therefore, the polynomial degree at the interface is enforced by the spectral element carrying the lowest polynomial degree. For a given polynomial order $\mathbf{p}$ on Ω_k, we introduce the nodal basis functions $\{\psi_{i_n}\}_{i_n=0\cdots p_n}$ formed by the (p_n+1) Lagrange interpolants at the Gauss-Legendre-Lobatto (GLL) nodes $\{x_{i_n}\}_{i_n=0\cdots p_n}$ in the n-th dimension. Considering a node $\mathbf{x}\in\Omega_k$, the following two configurations can occur:

- $\mathbf{x}\in\Omega_k/\partial\Omega_k$. In this case the basis function $\phi_{\mathbf{j}}$ relative to $\mathbf{x}$ is obtained by tensorial product of one-dimensional basis functions and $\phi_{\mathbf{j}}(\mathbf{x}) = \Pi_{n=1}^d\ \psi_{j_n}(x_n)$.
- $\mathbf{x}\in\partial\Omega_k$. In this case, $\mathbf{x}$ lies on a face $F=\Omega_k\cap\Omega_{k'}$ normal to, lets say, the q-th dimension. The basis function $\phi_{\mathbf{j}}$ relative to $\mathbf{x}$ is built as $\phi_{\mathbf{j}}(\mathbf{x}) = \psi_{j_q}(x_q)\Pi_{n\neq q}\psi_{j_n}^{\perp}(x_n^{\perp})$. The functions $\left\{\psi_{j_n}^{\perp}\right\}$ — defined as the Lagrange interpolants at the GLL nods $\left\{x_n^{\perp}\right\}$ — are obtained by linear combinations of the $\left\{\psi_{j_n}\right\}$
$$\psi_{j_n}^{\perp}(x) = \sum_{i_m}\psi_{j_n}^{\perp}(x_m)\psi_{i_m}(x) = \sum_{i_m}\mathscr{C}_{nm}^k\psi_{i_m}(x).$$
The nodes $\left\{x_n^{\perp}\right\}$ are given by the lowest GLL quadrature on the face F: $p_F=\min(p_k,p_{k'})$.

Problem (2) is now brought into the algebraic form

$$A\mathbf{u} = f, \tag{4}$$

where $A=\sum_{n=1}^N \mathscr{P}_n^t A^n\mathscr{P}_n$ and $\{A^n\}$ are the matrices representing the bilinear forms $a_c^n(.,.)$ of problem (3). The $\{\mathscr{P}_n\}$ are defined in terms of the coefficient $\{\mathscr{C}_{ij}^n\}$

$$\mathscr{P}_n = \begin{bmatrix} I & 0 \\ 0 & \mathscr{C}^n \end{bmatrix},$$

provided that internal unknowns are all ordered before those of the interface. In the next section, we present the continuous solver relative to this algebraic system.

2.3 BDDC as a preconditioner for the Schur complement

In this section, we assume that the domain Ω is decomposed into nonoverlapping subdomains $\Omega=\bigcup_k\Omega^{(k)}$. Each subdomain $\Omega^{(k)}$ has diameter H_k and is composed of several spectral elements $\Omega^{(k)}=\bigcup_{m=1}^{N_k}\Omega^m$ having diameter h_k — we assume without loss of generality that the partition is spatially uniform inside a subdomain — so that H_k/h_k quantifies the number of spectral elements along a subdomain edge.

By partitioning the local degrees of freedom into interior (I) and interface (Γ) sets, and by further partitioning the latter into dual (Δ) and primal (Π) degrees of freedom, then the matrix $A^{(n)}$ relative to the restriction of $a_c(.,.)$ to the n-th subdomain $\Omega^{(n)}$ can be written as

$$A^{(n)} = \begin{bmatrix} A_{II}^{(n)} & A_{\Gamma I}^{(n)^T} \\ A_{\Gamma I}^{(n)} & A_{\Gamma\Gamma}^{(n)} \end{bmatrix} = \begin{bmatrix} A_{II}^{(n)} & A_{\Delta I}^{(n)^T} & A_{\Pi I}^{(n)^T} \\ A_{\Delta I}^{(n)} & A_{\Delta\Delta}^{(n)} & A_{\Pi\Delta}^{(n)^T} \\ A_{\Pi I}^{(n)} & A_{\Pi\Delta}^{(n)} & A_{\Pi\Pi}^{(n)} \end{bmatrix}. \tag{5}$$

The choice of primal and dual variables is discussed in [12]. In two dimensions, the primal variables reduce to the vertices of the subdomains while the dual ones correspond to the unknowns lying on an interface between two subdomains. Using the scaled restriction matrices defined in [12] and keeping the same notations, the BDDC preconditioner for the Schur complement of system (4) can be written as

$$M^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}, \tag{6}$$

where

$$\widetilde{S}_\Gamma^{-1} = R_{\Gamma\Delta}^T \left(\sum_{n=1}^N \begin{bmatrix} 0 & R_\Delta^{(n)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(n)} & A_{\Delta I}^{(n)^T} \\ A_{\Delta I}^{(n)} & A_{\Delta\Delta}^{(n)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ R_\Delta^{(n)} \end{bmatrix} \right) R_{\Gamma\Delta} + \Phi S_{\Pi\Pi}^{-1} \Phi^T, \tag{7}$$

with the coarse matrix

$$S_{\Pi\Pi} = \sum_{n=1}^N R_\Pi^{(n)^T} \left(A_{\Pi\Pi}^{(n)} - \begin{bmatrix} A_{\Pi I}^{(n)} & A_{\Pi\Delta}^{(n)} \end{bmatrix} \begin{bmatrix} A_{II}^{(n)} & A_{\Delta I}^{(n)^T} \\ A_{\Delta I}^{(n)} & A_{\Delta\Delta}^{(n)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(n)^T} \\ A_{\Pi\Delta}^{(n)^T} \end{bmatrix} \right) R_\Pi^{(n)}$$

and a matrix Φ mapping interface variables to primal degrees of freedom, given by

$$\Phi = R_{\Gamma\Pi}^T - R_{\Gamma\Delta}^T \sum_{n=1}^N \begin{bmatrix} 0 & R_\Delta^{(n)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(n)} & A_{\Delta I}^{(n)^T} \\ A_{\Delta I}^{(n)} & A_{\Delta\Delta}^{(n)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(n)^T} \\ A_{\Pi\Delta}^{(n)^T} \end{bmatrix} R_\Pi^{(n)}.$$

Equation (7) means that we solve on each subdomain a problem with Neumann data for the dual variables and a coarse problem with matrix $S_{\Pi\Pi}$ for the primal variables.

Theorem 1. *The condition number κ_2 of the BDDC and FETI-DP preconditioned systems in 2D, using at least one primal vertex for each subdomain edge $F_\Omega \subseteq \Gamma$, satisfies the following bound:*

$$\kappa_2(M^{-1}\widehat{S}) \leq C \left(1 + \log \left(H \max_{F_K \subseteq \Gamma} \frac{p_{F_K}^2}{h_{F_K}} \right) \right)^2, \tag{8}$$

where p_{F_K} *is the polynomial degree over an element edge* F_K *(we recall that if* $F_K = \partial K \bigcap \partial K'$*, then* $p_{F_K} = \min(p_K, p_{K'})$ *and the constant* $C > 0$ *is independent of* p_{F_K}, h_{F_K}, H *and the values of the coefficient* μ *of the elliptic operator.*

This result (see [4] for a proof) states in particular that the preconditioned problem is scalable in the number of subdomains and robust with respect to jumps in the elliptic coefficients. Once we have a preconditioner for the Schur complement of the CG-SEM problem, we are able to build a global preconditioner for DG *via* the Auxiliary Space Method. This is the object of the next section.

3 Preconditioning DG with ASM-BDDC

3.1 DG-SEM discretization for elliptic problems

We recall that the weak form of problem (1) obtained choosing as Galerkin space $\mathscr{V}_\delta = \{v : \Omega \to \mathbb{R} \mid \forall k,\ v_{|\Omega_k} \in \mathbb{P}_{p_k}(\Omega_k),\ v \in L^2(\Omega)\}$, that is the space of discontinuous elementwise polynomial functions is given by:
Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = L(v) \quad \forall v \in \mathscr{V}_\delta, \tag{9}$$

where the bilinear form defined on $\mathscr{V}_\delta \times \mathscr{V}_\delta$ is

$$\begin{aligned} a_\delta(u,v) = & \sum_{K \in \mathscr{K}} \int_K \mu \nabla u \cdot \nabla v - \sum_{F \in \mathscr{F}} \mu_F \int_F \{\{\nabla u\}\}_F [\![v]\!]_F + \{\{\nabla v\}\}_F [\![u]\!]_F \\ & + \sum_{F \in \mathscr{F}} \eta_F \mu_F \int_F [\![u]\!]_F [\![v]\!]_F \ , \end{aligned}$$

as well as the linear form $F(v) = \int_\Omega f v$ defined on $\mathscr{V}_\delta$. The jump $[\![.]\!]_F$ and average $\{\{.\}\}_F$ operators are the standard ones defined e.g. in [2] and the coefficients η_F and μ_F are defined as in [6]. Choosing an appropriate basis of $\mathscr{V}_\delta$, problem (9) is brought into its algebraic form and we are ready to apply the ASM preconditioning technique.

3.2 The auxiliary space method (ASM)

The Auxiliary Space Method (ASM) [11] gives a general framework for designing preconditioners of nonconforming discretizations, provided preconditioners for some related conforming discretizations are available. Hereafter, we recall the ASM formulation tailored to the current situation of interest, referring e.g. to [3] for the most general setting. We assume there exists a symmetric bilinear form $b_\delta(u,v)$ on $V_\delta \times V_\delta$ and a linear operator $Q_\delta^c : V_\delta \to V_\delta^c$ such that

$$a_\delta(v,v) \lesssim b_\delta(v,v) \quad \forall v \in V_\delta \tag{10}$$

and

$$b_\delta(v - Q^c_\delta v, v - Q^c_\delta v) \lesssim a_\delta(v,v) \qquad \forall v \in V_\delta\ . \tag{11}$$

Here and in the sequel, the symbol $\lesssim$ means $\leq c$ for a constant c bounded independently of δ in the admissible range of variability of δ. This implies the following algebraic results. Let $\mathbb{A}$ and B denote the matrices associated with the forms a_δ and b_δ once a basis in V_δ has been chosen; similarly, let A denote the matrix associated with the form $a = a_\delta$ restricted to V^c_δ, once a basis in V^c_δ has been chosen. Let Z be the matrix representing the inclusion $V^c_\delta \subset V_\delta$ in the chosen bases. In addition, assume that P_B^{-1} is a symmetric preconditioner for B and P_A^{-1} is a symmetric preconditioner for A, such that the following eigenvalue bounds hold:

$$\lambda_{\max}(P_B^{-1}B),\ \lambda_{\max}(P_A^{-1}A)\ \leq\ \Lambda_{\max}, \qquad \lambda_{\min}(P_B^{-1}B),\ \lambda_{\min}(P_A^{-1}A)\ \geq\ \Lambda_{\min}\ .$$

Then,

$$P_{\mathbb{A}}^{-1} := P_B^{-1} + Z P_A^{-1} Z^T \tag{12}$$

is a symmetric preconditioner for $\mathbb{A}$, such that

$$\kappa_2(P_{\mathbb{A}}^{-1}\mathbb{A}) \leq \frac{\Lambda_{\max}}{\Lambda_{\min}}\ . \tag{13}$$

Now, we choose for P_A^{-1} the global BDDC-based preconditioner defined according to [13]

$$P_A^{-1} = \begin{pmatrix} I & -A_{II}^{-1}A_{I\Gamma} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{II}^{-1} & 0 \\ 0 & M^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -A_{\Gamma I}A_{II}^{-1} & I \end{pmatrix}, \tag{14}$$

where M^{-1} is the BDDC preconditioner of equation (6). The subscript Γ means that we consider the unknowns lying on the Schur skeleton while the subscript I is linked to internal unknowns (inside a subdomain). In the last section, we present some numerical results showing the robustness of preconditioners (6) and (12).

4 Numerical results and conclusion

We present two test cases that illustrate the robustness and quasi-optimality of both preconditioners P_A^{-1} and $P_{\mathbb{A}}^{-1}$. First, the number of spectral elements is fixed and we consider both jumping elliptic coefficients and polynomial degrees, see Figure 1. The results are presented in Table 1, where it is shown that the condition number $\kappa_2(P_A^{-1}A)$ is quite insensitive to moderate jumps in the polynomial degree such as $p \to p+2 \to p+4$. Discontinuities in the elliptic coefficients are managed quite well by the ASM-BDDC preconditioner for minor variations in the polynomial degree. We also study the sensitivity of κ_2 to simultaneous variations in h and p. In particular, setting $H = 1$ (that is the continuous solver is exact), Table 2 shows that the condition number of the ASM-BDDC remains $O(1)$ in agreement with bound (8) in Theorem 1. Lastly, a case with rectangular spectral elements is investigated, see Figure 2. We consider a diadic evolution of spectral elements width h as 2^{-i} for $i = 1, \cdots, 5$ with a uniform polynomial degree. The results are presented in Figure 2 for both $k_2(P_{\mathbb{A}}^{-1}\mathbb{A})$ and $\kappa_2(P_A^{-1}A)$.

As a conclusion, this paper presents a new way of preconditioning DG-SEM systems based on an available preconditioner for CG-SEM. The ASM applied to such a global BDDC-based preconditioner provides a solver for DG that is still $0(H\log(\max\frac{p_K^2}{h_K}))$ but it also introduces a dependence on the maximal polynomial jump and elliptic coefficients. However, we show numerically that for moderate polynomial jumps, the preconditioner is scalable and quasi-optimal.

(a) (b)

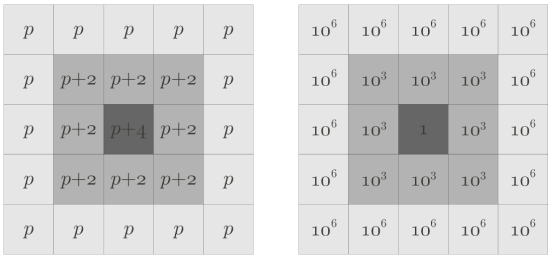

Fig. 1 Test case with 5×5 subdomains. Polynomial degrees and elliptic coefficients obey a pyramidal distribution, where p increases toward the center of the domain (a), the elliptic coefficient μ decreases (b).

Table 1 Condition numbers for increasing polynomial degree p with nonuniform (uniform in brackets) polynomial distribution and jumping elliptic coefficients given in Fig. 1, with 5×5 subdomains and $H/h = 1$.

Degree p	BDDC $\kappa_2(P_A^{-1}A)$	ASM-BDDC $\kappa_2(P_{\mathbb{A}}^{-1}\mathbb{A})$
2	2.34 (1.47)	5.95 (5.09)
4	3.37 (2.64)	6.31 (5.71)
6	4.20 (3.56)	6.54 (6.20)
8	4.89 (4.33)	6.70 (6.50)
10	5.49 (4.99)	6.83 (6.70)
12	6.02 (5.56)	6.94 (6.82)

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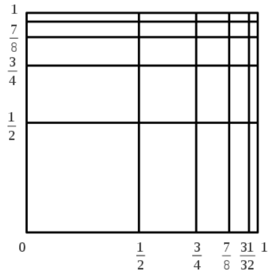
Table 2 Condition number of the preconditioned DG matrix for increasing polynomial degree p with uniform polynomial distribution and increasing h, so that the ratio p^2/h is maintained approximatively constant. Uniform elliptic coefficients $\mu_K = 1$. Results for one subdomain $H = 1$.

Degree p	# elements	$\frac{p^2}{h}$	ASM $\kappa_2(P_{\mathbb{A}}^{-1}\mathbb{A})$
2	25^2	100	5.10
3	10^2	90	5.44
4	6^2	96	5.82
5	4^2	100	6.07
6	3^2	108	6.25



p	$\kappa_2(P_A^{-1}A)$	$\kappa_2(P_{\mathbb{A}}^{-1}\mathbb{A})$
2	4.12	14.13
4	4.09	13.46
6	4.75	12.89
8	5.43	12.50
10	6.07	12.24
12	6.65	12.08

Fig. 2 Test case with uniform polynomial degree and diadic mesh in h. The ratio H/h is kept equal to 1, meaning one element per subdomain. Condition number of the preconditioned DG matrix for this configuration. Uniform elliptic coefficients $\mu_K = 1$. Results for one element per subdomain $H/h = 1$.

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