3-D FETI-DP preconditioners for composite finite element-discontinuous Galerkin methods

Maksymilian Dryja[1] and Marcus Sarkis[2]

Key words: FETI-DP, Discontinuous Galerkin, Nonmatching Grids, Discontinuous Coefficients, Domain Decomposition, Schwarz Methods

1 Introduction

In this paper a Nitsche-type discretization based on discontinuous Galerkin (DG) method for an elliptic three-dimensional problem with discontinuous coefficients is considered. The problem is posed on a polyhedral region Ω which is a union of N disjoint polyhedral subdomains Ω_i of diameter $O(H_i)$ and we assume that this partition is geometrically conforming. Inside each subdomain, a conforming finite element space on a quasiuniform triangulation with mesh size $O(h_i)$ is introduced. Large discontinuities on the coefficients and nonmatching meshes are allowed to occur only across $\partial\Omega_i$. In order to deal with the nonconformity of the FE spaces across subdomain interfaces, a discrete problem is formulated using the symmetric interior penalty DG method only on the subdomain interfaces. For solving the resulting discrete system, FETI-DP type of methods are designed and fully analyzed. This paper extends the 2-D results in [2] to 3-D problems.

2 Differential and discrete problems

Consider the following problem: *Find* $u_{ex}^* \in H_0^1(\Omega)$ *such that*

$$a(u_{ex}^*, v) = f(v) \quad \text{for all} \quad v \in H_0^1(\Omega), \tag{1}$$

where

$$a(u,v) := \sum_{i=1}^{N} \int_{\Omega_i} \rho_i(x) \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) := \int_{\Omega} f v \, dx.$$

To simplify the presentation, we assume that $\rho_i(x)$ is equal to positive constant ρ_i.

[1]Department of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland, e-mail: dryja@mimuw.edu.pl ·[2] Mathematical Sciences Department, Worcester Polytechnic Institute, 100 Institute Road, Worcester, MA 01609, USA, and and Instituto de Matemática Pura e Aplicada - IMPA, Estrada Dona Castorina 110, CEP 22460-320, Rio de Janeiro, Brazil, e-mail: msarkis@wpi.edu

We now consider the discrete problem associated to (1). Let $X_i(\Omega_i)$ be the regular finite element (FE) space of piecewise linear and continuous functions in Ω_i and define

$$X(\Omega) = \prod_{i=1}^{N} X_i(\Omega_i) \equiv X_1(\Omega_1) \times X_2(\Omega_2) \times \cdots \times X_N(\Omega_N).$$

We note that we do not assume that the functions in $X_i(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$.

Let us denote $\bar{F}_{ij} := \partial\Omega_i \cap \partial\Omega_j$ as a face of $\partial\Omega_i$ and $\bar{F}_{ji} := \partial\Omega_j \cap \partial\Omega_i$ as a face of $\partial\Omega_j$. In spite of the common face F_{ij} and F_{ji} being geometrically the same, they will be treated separately since we consider different triangulations on $\bar{F}_{ij} \subset \partial\Omega_i$ with a mesh parameter h_i and on $\bar{F}_{ji} \subset \partial\Omega_j$ with a mesh parameter h_j. We denote the interior h_i-nodes of F_{ij} and the h_j-nodes of F_{ji} by F_{ijh} and F_{jih}, respectively.

Let us denote by $\mathscr{F}_i^0$ the set of indices j of Ω_j which has a common face F_{ji} with Ω_i. To take into account also of these faces of Ω_i which belong to $\partial\Omega$, we introduce a set of indices $\mathscr{F}_i^\partial$ to refer theses faces. The set of indices of all faces of Ω_i is denoted by $\mathscr{F}_i := \mathscr{F}_i^0 \cup \mathscr{F}_i^\partial$. A discrete problem is obtained by a composite FE/DG method, see [1], is of the form: *Find* $u^* = \{u_i^*\}_{i=1}^N \in X(\Omega)$ *where* $u_i \in X_i(\Omega_i)$*, such that*

$$a_h(u^*, v) = f(v) \quad \text{for all} \quad v = \{v_i\}_{i=1}^N \in X(\Omega), \tag{2}$$

where

$$a_h(u,v) := \sum_{i=1}^{N} a_i'(u,v), \quad f(v) := \sum_{i=1}^{N} \int_{\Omega_i} f v_i \, dx,$$

$$a_i'(u,v) := \{a_i(u,v) + p_i(u,v)\} + s_i(u,v) \equiv \{d_i(u,v)\} + s_i(u,v), \tag{3}$$

where

$$a_i(u,v) := \int_{\Omega_i} \rho_i \nabla u_i \cdot \nabla v_i \, dx,$$

$$p_i(u,v) := \sum_{j \in \mathscr{F}_i} \int_{F_{ij}} \frac{\delta}{l_{ij}} \frac{\rho_i}{h_{ij}} (u_j - u_i)(v_j - v_i) \, ds,$$

and

$$s_i(u,v) := \sum_{j \in \mathscr{F}_i} \int_{F_{ij}} \frac{1}{l_{ij}} \left(\rho_i \frac{\partial u_i}{\partial n}(v_j - v_i) + \rho_i \frac{\partial v_i}{\partial n}(u_j - u_i) \right) ds.$$

Here, when $j \in \mathscr{F}_i^0$, we set $l_{ij} = 2$ and let $h_{ij} := 2h_i h_j/(h_i + h_j)$, i.e., the harmonic average of h_i and h_j. When $j \in \mathscr{F}_i^\partial$, we denote the boundary faces $F_{ij} \subset \partial\Omega_i$ by $F_{i\partial}$ and set $l_{i\partial} = 1$ and $h_{i\partial} = h_i$, and on the artificial face $F_{ji} \equiv F_{\partial i}$, we set $u_\partial = 0$ and $v_\partial = 0$. The partial derivative $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega_i$ and δ is the sufficiently large penalty parameter. For details on accuracy and well-posedness, see [2, 1] and references there in. In particular, we show that exists positive constants γ_0 and γ_1, which do not depend on the ρ_i, h_i and H_i, such that

$$\gamma_0 a_h(u,u) \le \sum_{i=1}^{N} d_i(u,u) \le \gamma_1 a_h(u,u) \quad \text{for all} \quad u \in X(\Omega).$$

3 Schur complement systems and discrete harmonic extensions

This section is similar to Section 3 in [2] with a few natural changes when passing from the 2-D to the 3-D case, and we refer to that for more details.

- Define the sets Ω_i', Γ_i, Γ_i', I_i, Γ, Γ', I and Ω' by

$$\Omega_i' = \overline{\Omega}_i \bigcup \{\cup_{j\in\mathscr{F}_i^0}\bar{F}_{ji}\}, \quad \Gamma_i = \overline{\partial\Omega_i\setminus\partial\Omega}, \quad \Gamma_i' = \Gamma_i \bigcup \{\cup_{j\in\mathscr{F}_i^0}\bar{F}_{ji}\},$$

$$\Gamma = \bigcup_{i=1}^N \Gamma_i, \quad \Gamma' = \prod_{i=1}^N \Gamma_i', \quad I_i = \Omega_i'\setminus\Gamma_i', \quad I = \prod_{i=1}^N I_i \quad \text{and} \quad \Omega' = \prod_{i=1}^N \Omega_i'.$$

- Define the space $W_i(\Omega_i')$ by

$$W_i(\Omega_i') = X_i(\Omega_i) \times \prod_{j\in\mathscr{F}_i^0} X_i(\bar{F}_{ji}), \quad \text{where} \quad X_i(\bar{F}_{ji}) = X_j(\Omega_j)_{|\bar{F}_{ji}}.$$

  A function $u_i \in W_i(\Omega_i')$ will be represented as

$$u_i = \{(u_i)_i, \{(u_i)_j\}_{j\in\mathscr{F}_i^0}\},$$

  where $(u_i)_i := u_{i|\overline{\Omega}_i}$ (u_i restricted to $\overline{\Omega}_i$) and $(u_i)_j := u_{i|\bar{F}_{ji}}$ (u_i restricted to $\bar{F}_{ji}$).
- For the definition of the discrete harmonic extension operators $\mathscr{H}_i'$ and $\mathscr{H}_i$ (elimination of I_i variables) with respect to the bilinear forms a_i' and a_i, see [2].
- The matrices A_i' and S_i' are defined by

$$a_i'(u_i, v_i) = \langle A_i' u_i, v_i\rangle \quad u_i, v_i \in W_i(\Omega_i'), \quad a_i'(u_i, v_i) = \langle S_i' u_i, v_i\rangle \quad u_i, v_i \in W_i(\Gamma_i').$$

- $W_i(\Gamma_i') \subset W_i(\Omega_i')$ denotes the $\mathscr{H}_i'$-discrete harmonic functions.
- Define $W(\Omega') = \prod_{i=1}^N W_i(\Omega_i')$ and $W(\Gamma') = \prod_{i=1}^N W_i(\Gamma_i')$.
- Let the subspace $\hat{W}(\Omega') \subset W(\Omega')$ consist of functions $u = \{u_i\}_{i=1}^N \in W(\Omega')$ which are continuous on Γ, that is, for all $1 \le i \le N$ satisfy

$$(u_i)_i(x) = (u_j)_i(x) \quad \text{for all} \ x \in \bar{F}_{ij} \ \text{for all} \ j \in \mathscr{F}_i^0$$

  and

$$(u_i)_j(x) = (u_j)_j(x) \quad \text{for all} \ x \in \bar{F}_{ji} \ \text{for all} \ j \in \mathscr{F}_i^0.$$

  We note that $\hat{W}(\Omega')$ can be identified to $X(\Omega)$.
- $\hat{W}(\Gamma')$ denotes the subspace of $\hat{W}(\Omega')$ of $\mathscr{H}_i'$-discrete harmonic functions.
- The rest of Section 3 in [2] remains the same for 3-D problems. In particular, by eliminating the interior variables I from the system (2), we obtain

$$\hat{S}u_\Gamma^* = g_\Gamma. \tag{4}$$

  We note that $\hat{S}$ can be assembled from S_i', i.e., $\hat{S} = \sum_{i=1}^N R_{\Gamma_i'}^T S_i' R_{\Gamma_i'}$, where $R_{\Gamma_i'}$ is the restriction operator from Γ to Γ_i'.

4 FETI-DP with corners, average edges and faces constraints

We now design a FETI-DP method for solving (4). We follow to the abstract approach described in pages 160-167 in [3].

Let us define the set of indices $\mathscr{E}_i^0$ of pairs (j,k) of Ω_j and Ω_k, $j \neq k$, for which $\bar{E}_{ijk} := \partial F_{ij} \cap \partial F_{ik}$, for $j,k \in \mathscr{F}_i^0$, is an edge of $\partial \Omega_i$. In spite of the common edges E_{ijk}, E_{jik}, and E_{kij} being geometrically the same, we treat them separately since we consider different triangulations on $E_{ijk} \subset \partial \Omega_i$ with a mesh parameter h_i, $E_{jik} \subset \partial \Omega_j$ with a mesh parameter h_j and $E_{kij} \subset \partial \Omega_k$ with a mesh parameter h_k. We denote the interior edge nodes of these triangulations by E_{ijkh}, E_{jikh} and E_{kijh}, respectively.

Let us introduce the nodal points associated to the corner unknowns by

$$\mathscr{V}_i := \{\cup_{(j,k)\in \mathscr{E}_i^0} \partial E_{ijk}\} \ \text{ and } \ \mathscr{V}_i' := \{\mathscr{V}_i \bigcup \{\cup_{(j,k)\in \mathscr{E}_i^0} \partial E_{jik} \cup \partial E_{kij}\}\}.$$

We say that $u = \{u_i\}_{i=1}^N \in W(\Omega')$ is continuous at the corners $\mathscr{V}_i$ if

$$(u_i)_i(x) = (u_j)_i(x) = (u_k)_i(x) \quad \text{at all } \ x \in \mathscr{V}_i.$$

Definition 1. (Subspaces $\tilde{W}(\Omega')$ and $\tilde{W}(\Gamma')$). The $\tilde{W}(\Omega')$ consists of functions $u = \{u_i\}_{i=1}^N \in W(\Omega')$ for which, for all $1 \leq i \leq N$, the following conditions are satisfied:

- At all corners $\mathscr{V}_i$, u is continuous.
- On all edges E_{ijk} for $(j,k) \in \mathscr{E}_i^0$

$$(\bar{u}_i)_{i,E_{ijk}} = (\bar{u}_j)_{i,E_{ijk}} = (\bar{u}_k)_{i,E_{ijk}}.$$

- On all faces F_{ij} for $j \in \mathscr{F}_i^0$

$$(\bar{u}_i)_{i,F_{ij}} = (\bar{u}_j)_{i,F_{ij}},$$

where

$$(\bar{u}_i)_{i,E_{ijk}} = \frac{1}{|E_{ijk}|}\int_{E_{ijk}} (u_i)_i ds, \quad (\bar{u}_j)_{i,F_{ij}} = \frac{1}{|F_{ij}|}\int_{F_{ij}} (u_j)_i ds.$$

The $\tilde{W}(\Gamma')$ denotes the subspace of $\tilde{W}(\Omega')$ of functions which are discrete harmonic in the sense of $\mathscr{H}_i'$. It is easy to see that $\hat{W}(\Gamma') \subset \tilde{W}(\Gamma') \subset W(\Gamma')$.

Let $\tilde{A}$ be the stiffness matrix which is obtained by assembling the matrices A_i' for $1 \leq i \leq N$, from $W(\Omega')$ to $\tilde{W}(\Omega')$. We represent $u \in \tilde{W}(\Omega')$ as $u = (u_I, u_\Pi, u_\triangle)$ where the subscript I refers to the interior degrees of freedom at the nodal points on I, the Π refers to the degrees of freedom at the corners $\{\mathscr{V}_i\}_{i=1}^N$ and edges and faces averages, and the $\triangle$ refers to the remaining degrees of freedom, i.e., the nodal values on $\{\Gamma_i' \backslash \mathscr{V}_i'\}_{i=1}^N$ with edges and faces average equal to zero. For details on $\tilde{A}$, see (4.5) in [2], and its Schur complement $\tilde{S}$ (after eliminating the I and Π degrees of freedom from $\tilde{A}$), see (4.6) in [2].

A vector $u \in \tilde{W}(\Gamma')$ can uniquely be represented by $u = (u_\Pi, u_\triangle)$, therefore, we can represent

$$\tilde{W}(\Gamma') = \hat{W}_{\Pi}(\Gamma') \times W_{\triangle}(\Gamma'),$$

where $\hat{W}_{\Pi}(\Gamma')$ refers to the Π-degrees of freedom of $\tilde{W}(\Gamma')$ while $W_{\triangle}(\Gamma')$ to the $\triangle$-degrees of freedom of $\tilde{W}(\Gamma')$. The vector space $W_{\triangle}(\Gamma')$ can be decomposed as

$$W_{\triangle}(\Gamma') = \prod_{i=1}^{N} W_{i,\triangle}(\Gamma_i'),$$

where the local space $W_{i,\triangle}(\Gamma_i')$ refers to the degrees of freedom associated to the nodes of $\Gamma_i' \backslash \mathscr{V}_i'$ for $1 \leq i \leq N$ with zero averages on F_{ij} and F_{ji}, for $i \in \mathscr{F}_i^0$, and on E_{ijk}, E_{jik} and E_{kij}, for $(j,k) \in \mathscr{E}_i^0$.

The jump operator $B_{\triangle} : W_{\triangle}(\Gamma') \to U_r$

$$B_{\triangle} = (B_{\triangle}^{(1)}, B_{\triangle}^{(2)}, \cdots, B_{\triangle}^{(N)})$$

is defined as follows. Each $B_{\triangle}^{(i)}$ maps $W_{\triangle}(\Gamma')$ to $U_{i,r}$ (jumps on edges and faces), where $v_i = B^{(i)} u_{\triangle}$ is defined by:

- For each face F_{ij} for $j \in \mathscr{F}_i^0$, let

$$v_i(x) = (u_{i,\triangle})_i(x) - (u_{j,\triangle})_i(x) \quad \text{for all } \ x \in F_{ijh}.$$

- For each edge E_{ijk} for $(j,k) \in \mathscr{E}_i^0$, let $v_i = \{v_{i,1}, v_{i,2}\}$, where

$$v_{i,1}(x) = (u_{i,\triangle})_i(x) - (u_{j,\triangle})_i(x) \quad \text{for all } \ x \in E_{ijkh},$$

$$v_{i,2}(x) = (u_{i,\triangle})_i(x) - (u_{k,\triangle})_i(x) \quad \text{for all } \ x \in E_{ijkh}.$$

Let $U_r = (U_{1,r}, \cdots, U_{N,r})$ where $U_{i,r}$ is the range of $B_{\triangle}^{(i)}$, and note that the $U_{i,r}$ also has zero average on edges and faces. The space U_r will also be denoted as the space of Lagrange multipliers. We note that by setting $B_{\triangle}^{(i)} u_{\triangle} = 0$, we have one constraint for each node on F_{ijh} and two constraints for each node on E_{ijkh}. The saddle point problem is defined as in [2], except that here we replace $\hat{W}_{\triangle}$ by U_r, and the problem (4) is reduced to: *Find* $u_{\triangle}^* \in W_{\triangle}(\Gamma')$ *and* $\lambda^* \in U_r$ *such that*

$$\begin{cases} \tilde{S} u_{\triangle}^* & + B_{\triangle}^T \lambda^* = \tilde{g}_{\triangle} \\ B_{\triangle} u_{\triangle}^* & = \ 0. \end{cases}$$

Hence, it reduces to

$$F \lambda^* = g, \tag{5}$$

where

$$F := B_{\triangle} \tilde{S}^{-1} B_{\triangle}^T, \qquad g := B_{\triangle} \tilde{S}^{-1} \tilde{g}_{\triangle}.$$

4.1 Dirichlet Preconditioner

We now define the FETI-DP preconditioner for F, see (5). Let $S'_{i,\triangle}$ be the Schur complement of S'_i restricted to $W_{i,\triangle}(\Gamma'_i) \subset W_i(\Gamma'_i)$, and define $S'_\triangle = \mathrm{diag}\{S'_{i,\triangle}\}_{i=1}^N$.

Let us introduce diagonal scaling matrices $D_i : U_{i,r} \to U_{i,r}$, for $1 \le i \le N$ as follows. For $\beta \in [1/2, \infty)$, define the diagonal entry of D_i by:

- For each face F_{ij} for $j \in \mathscr{F}_i^0$, let

$$D_i(x) = \rho_j^\beta(\rho_i^\beta + \rho_j^\beta)^{-1} =: \gamma_{ji} \quad \text{for all } x \in F_{ijh}.$$

- For each edge E_{ijk} for $(j,k) \in \mathscr{E}_i^0$, let $D_i = \{D_{i,1}, D_{i,2}\}$, where

$$D_{i,1}(x) = \rho_j^\beta(\rho_i^\beta + \rho_j^\beta + \rho_k^\beta)^{-1} =: \gamma_{jik} \quad \text{for all } x \in E_{ijkh},$$

$$D_{i,2}(x) = \rho_k^\beta(\rho_i^\beta + \rho_j^\beta + \rho_k^\beta)^{-1} =: \gamma_{kij} \quad \text{for all } x \in E_{ijkh}.$$

We now introduce $B_{D,\triangle} : U_r \to U_r$ by $B_{D,\triangle} = (D_1 B_\triangle^{(1)}, \cdots, D_N B_\triangle^{(N)})$ and the operator $P_\triangle : W_\triangle(\Gamma') \to W_\triangle(\Gamma')$ by $P_\triangle := B_{D,\triangle}^T B_\triangle$. We can check that for $u_\triangle = \{u_{i,\triangle}\}_{i=1}^N \in W_\triangle(\Gamma')$, that $v_\triangle := P_\triangle u_\triangle$ satisfies:

$$(v_{i,\triangle})_i = \gamma_{ji}[(u_{i,\triangle})_i - (u_{j,\triangle})_i] \quad \text{on} \quad F_{ijh}, \tag{6}$$

$$(v_{j,\triangle})_i = \gamma_{ij}[(u_{j,\triangle})_i - (u_{i,\triangle})_i] \quad \text{on} \quad F_{ijh}, \tag{7}$$

$$(v_{i,\triangle})_i = \gamma_{jik}[(u_{i,\triangle})_i - (u_{j,\triangle})_i] + \gamma_{kij}[(u_{i,\triangle})_i - (u_{k,\triangle})_i] \quad \text{on} \quad E_{ijkh}, \tag{8}$$

$$(v_{j,\triangle})_i = \gamma_{ijk}[(u_{j,\triangle})_i - (u_{i,\triangle})_i] + \gamma_{kij}[(u_{j,\triangle})_i - (u_{k,\triangle})_i] \quad \text{on} \quad E_{ijkh}, \tag{9}$$

$$(v_{k,\triangle})_i = \gamma_{ijk}[(u_{k,\triangle})_i - (u_{i,\triangle})_i] + \gamma_{jik}[(u_{k,\triangle})_i - (u_{j,\triangle})_i] \quad \text{on} \quad E_{ijkh}. \tag{10}$$

We note from [(6) - (7)] that on F_{ijh} it holds

$$[(v_{i,\triangle})_i - (v_{j,\triangle})_i] = [(u_{i,\triangle})_i - (u_{j,\triangle})_i], \tag{11}$$

and from [(8) - (9)] + [(8) - (10)] that on E_{ijkh} it holds

$$[(v_{i,\triangle})_i - (v_{j,\triangle})_i] + [(v_{i,\triangle})_i - (v_{k,\triangle})_i] = [(u_{i,\triangle})_i - (u_{j,\triangle})_i] + [(u_{i,\triangle})_i - (u_{k,\triangle})_i],$$

and it follows that $B_\triangle P_\triangle = B_\triangle$ and $P_\triangle^2 = P_\triangle$.

In the FETI-DP method, the preconditioner M^{-1} is defined as follows:

$$M^{-1} = B_D S'_\triangle B_D^T = \sum_{i=1}^N D_i B_\triangle^{(i)} S'_{i,\triangle} (B_\triangle^{(i)})^T D_i.$$

The main result of this paper is the following:

Theorem 1. *For any $\lambda \in U_r$, it holds that*

$$\langle M\lambda, \lambda \rangle \le \langle F\lambda, \lambda \rangle \le C(1+\log \frac{H}{h})^2 \langle M\lambda, \lambda \rangle,$$

where C is a positive constant independent of h_i, H_i, λ and the jumps of ρ_i. Here and below, $\log(\frac{H}{h}) := \max_{i=1}^N \log(\frac{H_i}{h_i})$.

Proof. Using the same algebraic arguments as in [2], it reduces to Lemma 1. The proof of Lemma 1 for the 3-D case is new and given with details below.

Lemma 1. *For any $u_\triangle \in W_\triangle(\Gamma')$, it holds that*

$$\|P_\triangle u_\triangle\|^2_{S'_\triangle} \le C(1+\log \frac{H}{h})^2 \|u_\triangle\|^2_{\tilde{S}}, \tag{12}$$

where C is a positive constant independent of h_i, H_i, $u_\triangle$ and the jumps of ρ_i.

Proof. Given $u_\triangle \in W_\triangle(\Gamma')$, let $u = (u_\Pi, u_\triangle) \in \tilde{W}(\Gamma')$ be the solution of

$$\langle \tilde{S} u_\triangle, u_\triangle \rangle = \min \langle S'w, w\rangle =: \langle S'u, u\rangle, \tag{13}$$

where the minimum is taken over $w = (w_\Pi, w_\triangle) \in \tilde{W}(\Gamma')$ such that $w_\Pi \in \hat{W}_\Pi(\Gamma')$ and $w_\triangle = u_\triangle$. Hence, we can replace $\|u_\triangle\|_{\tilde{S}}$ in (12) by $\|u\|_{S'}$.

Let us represent the u defined above as $\{u_i\}_{i=1}^N \in W(\Gamma')$ where $u_i \in W_i(\Gamma_i')$. Let $v \in \tilde{W}(\Gamma')$ be equal to $P_\triangle u_\triangle$ at the $\triangle$-nodes and equal to zero at the Π-nodes, i.e., $v = 0$ on $\mathscr{V}_i'$ for $1 \le i \le N$ and zero average on faces and edges. Let us represent v as $\{v_i\}_{i=1}^N \in W(\Gamma')$, where $v_i \in W_i(\Gamma_i')$. We have

$$\|P_\triangle u_\triangle\|^2_{S'_\triangle} = \|v\|^2_{S'} = \sum_{i=1}^N \|v_i\|^2_{S'_i}$$

in view of the definition of $S'_{i,\triangle}$ and $S_\triangle$, see (4.18), (3.5) and (4.6) in [2]. Hence, to prove the lemma it remains to show that

$$\sum_{i=1}^N \|v_i\|^2_{S'_i} \le C(1+\log \frac{H}{h})^2 \|u\|^2_{S'}$$

since by (13) we obtain (12). By Corollary 3.2 in [2] we need to show

$$\sum_{i=1}^N \tilde{d}_i(v_i, v_i) \le C(1+\log \frac{H}{h})^2 \sum_{i=1}^N \tilde{d}_i(u_i, u_i),$$

where, see (2.9) in [2], $\tilde{d}_i(v_i, v_i) = d_i(\mathscr{H}_i v_i, \mathscr{H}_i v_i)$ and

$$\tilde{d}_i(v_i, v_i) = \rho_i \parallel \nabla (\mathscr{H}_i v_i)_i \parallel^2_{L^2(\Omega_i)} + \sum_{j \in \mathscr{F}_i} \frac{\rho_i \delta}{l_{ij} h_{ij}} \parallel (v_i)_i - (v_i)_j \parallel^2_{L^2(F_{ij})}. \tag{14}$$

Here, $(v_i)_i = (\mathscr{H}_i v_i)_i$ and $(u_i)_i = (\mathscr{H}_i u_i)_i$ inside of the subdomains Ω_i.

To estimate the terms of the right-hand side (RHS) of (14) we represent $(v_i)_i$ as

$$(v_i)_i = \sum_{F_{ij} \subset (\partial\Omega_i \setminus \partial\Omega)} \theta_{F_{ij}}(v_i)_i + \sum_{E_{ijk} \subset \partial\Omega_i} \theta_{E_{ijk}}(v_i)_i \tag{15}$$

and $\mathscr{H}_i$ is discrete harmonic on Ω_i. Here, $\theta_{F_{ij}}(v_i)_i := I^{h_i}(\vartheta_{F_{ij}}(v_i)_i)$ and $\theta_{E_{ijk}}(v_i)_i := I^{h_i}(\vartheta_{E_{ijk}}(v_i)_i)$, where $\vartheta_{F_{ij}}$ and $\vartheta_{E_{ijk}}$ are the standard face and edge cutoff functions and I^{h_i} the finite element interpolator. We note that we do not have any vertex terms in (15) since $(v_i)_i = 0$ on $\mathscr{V}_i$. From now on, we denote $\nabla(\mathscr{H}_\ell w_\ell)_\ell$ by $\nabla(w_\ell)_\ell$ for $\ell = i, j, k$ and $w = v, u$. Hence, using (15), we have

$$\|\nabla(v_i)_i\|^2_{L^2(\Omega_i)} \le C\{\sum_{j \in \mathscr{F}_i^0} \|\theta_{F_{ij}}(v_i)_i\|^2_{H^{1/2}_{00}(F_{ij})} + \sum_{(j,k) \in \mathscr{E}_i^0} \|\theta_{E_{ijk}}(v_i)_i\|^2_{L^2(E_{ijk})}\} \tag{16}$$

by well-known estimates, see [3]. Note that (16) is also valid for substructures Ω_i which intersect $\partial\Omega$ by using the same arguments as for the 2-D case; see [2]. Using (6), $(\bar{u}_i)_{i,F_{ij}} = (\bar{u}_j)_{i,F_{ij}}$ and Lemma 4.26 in [3], we obtain

$$\begin{aligned} &\rho_i \parallel \theta_{F_{ij}}(v_i)_i \parallel^2_{H^{1/2}_{00}(F_{ij})} = \rho_i \gamma_{ji}^2 \parallel \theta_{F_{ij}}[(u_i)_i - (u_j)_i] \parallel^2_{H^{1/2}_{00}(F_{ij})} \\ &\le C\rho_i \gamma_{ji}^2 (1 + \log \frac{H_i}{h_i})^2 |(u_i)_i - (u_j)_i|^2_{H^{1/2}(F_{ij})}. \end{aligned} \tag{17}$$

Let $Q_{i,F_{ij}}$ be the L^2-projection onto $X_i(F_{ij})$, the restriction of $X_i(\Omega_i)$ on $\bar{F}_{ij}$. Using the triangle and inverse inequalities, and the $H^{1/2}$- and L^2-stability of the $Q_{i,F_{ij}}$ projection, we have

$$\begin{aligned} &|(u_i)_i - (u_j)_i|^2_{H^{1/2}(F_{ij})} \\ &\le C\{|Q_{i,F_{ij}}[(u_i)_i - (u_j)_j]|^2_{H^{1/2}(F_{ij})} + |Q_{i,F_{ij}}[(u_j)_j - (u_j)_i]|^2_{H^{1/2}(F_{ij})} \\ &\le C\,\{|(u_i)_i|^2_{H^1(\Omega_i)} + |(u_j)_j|^2_{H^1(\Omega_j)} + \frac{1}{h_i}\|(u_j)_j - (u_j)_i\|^2_{L^2(F_{ij})}\}. \end{aligned} \tag{18}$$

Substituting (18) into (17) and using $\rho_i \gamma_{ji}^2 \le \min\{\rho_i, \rho_j\}$ if $\beta \in [1/2, \infty)$, we obtain

$$\begin{aligned} &\rho_i \|\theta_{F_{ij}}(v_i)_i\|^2_{H^{1/2}_{00}(F_{ij})} \le \\ &\le C(1 + \log \frac{H_i}{h_i})^2 \quad \{\rho_i |(u_i)_i|^2_{H^1(\Omega_i)} + \rho_j |(u_j)_j|^2_{H^1(\Omega_j)} + \frac{\rho_j}{h_i}\|(u_j)_j - (u_j)_i\|^2_{L^2(F_{ij})}\} \\ &\le C(1 + \log \frac{H_i}{h_i})^2 \{\tilde{d}_i(u_i, u_i) + \tilde{d}_j(u_j, u_j)\}. \end{aligned} \tag{19}$$

We now estimate the second term of (16). Using (8), we have

$$\rho_i \|\theta_{E_{ijk}}(v_i)_i\|^2_{L^2(E_{ijk})} \le 2\rho_i\{\gamma_{jik}^2 \|(u_i)_i - (u_j)_i\|^2_{L^2(E_{ijk})} + \gamma_{kij}^2 \|(u_i)_i - (u_k)_i\|^2_{L^2(E_{ijk})}\}.$$

Using $(\bar{u}_i)_{i,E_{ijk}} = (\bar{u}_j)_{i,E_{ijk}}$ and Lemma 4.17 in [3], and the same arguments given in (18), and $\rho_i \gamma_{jik}^2 \le \min\{\rho_i, \rho_j\}$ for $\beta \in [1/2, \infty)$, we obtain

$$\rho_i \gamma_{jik}^2 \|(u_i)_i - (u_j)_i\|_{L^2(E_{ijk})}^2 \le C(1+\log\frac{H_i}{h_i})\rho_i \gamma_{jik}^2 |(u_i)_i - (u_j)_i|_{H^{1/2}(F_{ij})}^2 \quad (20)$$

$$\le C(1+\log\frac{H_i}{h_i})\{\rho_i |(u_i)_i|_{H^1(\Omega_i)}^2 + \rho_j |(u_j)_j|_{H^1(\Omega_i)}^2 + \frac{\rho_j}{h_i}\|(u_j)_j - (u_j)_i\|_{L^2(F_{ij})}^2\}$$

$$\le C(1+\log\frac{H_i}{h_i})\{\tilde{d}_i(u_i,u_i) + \tilde{d}_j(u_j,u_j)\}$$

and similarly

$$\rho_i \gamma_{kij}^2 \|(u_i)_i - (u_k)_i\|_{L^2(E_{ijk})}^2 \le C(1+\log\frac{H_i}{h_i})\{\tilde{d}_i(u_i,u_i) + \tilde{d}_k(u_k,u_k)\}. \quad (21)$$

Hence, by adding (20) and (21), we obtain

$$\rho_i \|\theta_{E_{ijk}}(v_i)_i\|_{L^2(E_{ijk})}^2 \le C(1+\log\frac{H_i}{h_i})\{\tilde{d}_i(u_i,u_i) + \tilde{d}_j(u_j,u_j) + \tilde{d}_k(u_k,u_k)\}. \quad (22)$$

Substituting (19) and (22) into (16), we get

$$\rho_i \|\nabla(v_i)_i\|_{L^2(\Omega_i)}^2 \le C(1+\log\frac{H_i}{h_i})^2\{\tilde{d}_i(u_i,u_i) + \tilde{d}_j(u_j,u_j) + \tilde{d}_k(u_k,u_k)\}. \quad (23)$$

We now estimate the second term of the RHS of (14). Note that $(v_i)_i$ and $(v_i)_j$ are defined on different meshes. In addition, the nodal values of $(v_i)_i(x)$, are defined by different formulas if a node x belongs to F_{ijh} or to $E_{ijkh} \subset \partial F_{ij}$, see (6) and (8). The same holds for $(v_i)_j(x)$. These issues must be taken into account when estimating the second terms of the RHS of (14). We have

$$\begin{aligned}\|(v_i)_i - (v_i)_j\|_{L^2(F_{ij})}^2 &\le 2\{\|(v_i)_i - Q_{i,F_{ij}}(v_i)_j\|_{L^2(F_{ij})}^2 + \|(v_i)_j - Q_{i,F_{ij}}(v_i)_j\|_{L^2(F_{ij})}^2\} \\ &\equiv 2\{I + II\}. \end{aligned} \quad (24)$$

Using (15) and that $w_i = \theta_{F_{ij}} w_i + \theta_{\partial F_{ij}} w_i$ for $w_i \in X_i(\Omega_i)_{|\bar{F}_{ij}}$, we have

$$\begin{aligned} I &\le C\{\|\theta_{F_{ij}}[(v_i)_i - Q_{i,F_{ij}}(v_i)_j]\|_{L^2(F_{ij})}^2 + \|\theta_{\partial F_{ij}}[(v_i)_i - Q_{i,F_{ij}}(v_i)_j]\|_{L^2(F_{ij})}^2\} \\ &\equiv C\{I_{F_{ij}} + I_{\partial F_{ij}}\}. \end{aligned} \quad (25)$$

To estimate $I_{F_{ij}}$, we first represent $(v_i)_j = \theta_{F_{ji}}(v_i)_j + \theta_{\partial F_{ji}}(v_i)_j$ to obtain

$$\begin{aligned} I_{F_{ij}} &\le 2\{\|\theta_{F_{ij}}\{(v_i)_i - Q_{i,F_{ij}}\theta_{F_{ji}}(v_i)_j\}\|_{L^2(F_{ij})}^2 + \|\theta_{F_{ij}} Q_{i,F_{ij}}\theta_{\partial F_{ji}}(v_i)_j)\|_{L^2(F_{ij})}^2\} \\ &\equiv 2\{I_{Fij}^{(1)} + I_{Fij}^{(2)}\}. \end{aligned} \quad (26)$$

Using (6) and (7), we have

$$I^{(1)}_{Fij} \leq C\gamma_{ji}^2 \|\theta_{F_{ij}}\{[(u_i)_i - (u_j)_i] - Q_{i,F_{ij}}\theta_{F_{ji}}[(u_i)_j - (u_j)_j]\}\|^2_{L^2(F_{ij})}$$

and by adding and subtracting $\theta_{F_{ij}} Q_{i,F_{ij}} \theta_{\partial F_{ji}}[(u_i)_j - (u_j)_j]$, we obtain

$$\begin{aligned}
I^{(1)}_{Fij} &\leq C\gamma_{ji}^2\{\|\theta_{F_{ij}}\{[(u_i)_i - (u_j)_i] - Q_{i,F_{ij}}[(u_i)_j - (u_j)_j])\}\|^2_{L^2(F_{ij})} + \\
&+ \|\theta_{F_{ij}} Q_{i,F_{ij}} \theta_{\partial F_{ji}}[(u_i)_j - (u_j)_j])\}\|^2_{L^2(F_{ij})}\} \\
&\leq C\gamma_{ji}^2\{\|(u_i)_i - Q_{i,F_{ij}}(u_i)_j\|^2_{L^2(F_{ij})} + \|(u_j)_i - Q_{i,F_{ij}}(u_j)_j\|^2_{L^2(F_{ij})} + \\
&+ \sum_{E_{jik} \subset \partial F_{ji}} h_j \|(u_i)_j - (u_j)_j\|^2_{L^2(E_{jik})}\} \leq C\gamma_{ji}^2\{\|(u_i)_i - (u_i)_j\|^2_{L^2(F_{ij})} + \\
&+ \|(u_j)_i - (u_j)_j\|^2_{L^2(F_{ij})} + h_j(1 + \log\frac{H_j}{h_j})|(u_i)_j - (u_j)_j|^2_{H^{1/2}(F_{ji})}\} \\
&\leq C\gamma_{ji}^2\{\|(u_i)_i - (u_i)_j\|^2_{L^2(F_{ij})} + \|(u_j)_i - (u_j)_j\|^2_{L^2(F_{ij})} + \\
&+ (1 + \log\frac{H_j}{h_j})(h_j|(u_i)_i|^2_{H^1(\Omega_i)} + h_j|(u_j)_j|^2_{H^1(\Omega_j)} + \|(u_i)_i - (u_i)_j\|^2_{L^2(F_{ij})}\},
\end{aligned} \tag{27}$$

where we have used the L^2-stability of $Q_{i,F_{ij}}$ and $\theta_{F_{ji}}$, the constraint $(\bar{u}_i)_{j,E_{jik}} = (\bar{u}_j)_{j,E_{jik}}$ and Lemma 4.17 in [3]. For the last inequality of (27), we have used a similar argument as in (18).

To estimate $I^{(2)}_{F_{ij}}$, first note that

$$I^{(2)}_{F_{ij}} \leq Ch_j\|(v_i)_j\|^2_{L^2(\partial F_{ji})} \leq Ch_j \sum_{E_{jik} \subset \partial F_{ji}} \|(v_i)_j\|^2_{L^2(E_{jik})} \tag{28}$$

and using the definition of $(v_j)_i$, see (9), we have

$$\|(v_i)_j\|^2_{L^2(E_{jik})} \leq 2\{\gamma_{jik}^2\|(u_i)_j - (u_j)_j\|^2_{L^2(E_{jik})} + \gamma_{kij}^2\|(u_i)_j - (u_k)_j\|^2_{L^2(E_{jik})}\}. \tag{29}$$

The first term of the RHS of (29) is estimated as in (20) while the second term as

$$\begin{aligned}
&h_j\|(u_i)_j - (u_k)_j\|^2_{L^2(E_{jik})} \leq 2h_j\{\|(u_i)_j - (u_j)_j\|^2_{L^2(E_{jik})} + \|(u_j)_j - (u_k)_j\|^2_{L^2(E_{jik})}\} \\
&\leq C(1 + \log\frac{H_j}{h_j})\{h_j|(u_i)_i|^2_{H^1(\Omega_i)} + h_j|(u_j)_j|^2_{H^1(\Omega_j)} + \|(u_i)_j - (u_i)_i\|^2_{L^2(F_{ji})} \\
&\qquad + h_j|(u_k)_k|^2_{H^1(\Omega_j)} + \|(u_k)_j - (u_k)_k\|^2_{L^2(F_{jk})}\}.
\end{aligned} \tag{30}$$

Substituting (29) and (30) into (28) and adding with (27), see (26), we obtain

$$\frac{\rho_i \delta}{l_{ij} h_{ij}} I_{F_{ij}} \leq C(1 + \log\frac{H}{h})\{\frac{h_j}{h_{ij}}\tilde{d}_i(u_i, u_i) + \frac{h_j}{h_{ij}}\tilde{d}_j(u_j, u_j) + \sum_{E_{ijk} \subset \partial F_{ij}} \frac{h_j}{h_{ij}}\tilde{d}_k(u_k, u_k)\}.$$

We now estimate $I_{\partial F_{ij}}$, see (25). Note that $(\bar{v}_i)_{j,F_{ji}} = 0$ implies a zero average of $Q_{i,F_{ij}}(v_i)_j$ on F_{ij}. We also have $(\bar{v}_j)_{i,F_{ij}} = 0$. Using previous arguments, we obtain

$$
\begin{aligned}
I_{\partial F_{ij}} &\le Ch_i \|(v_i)_i - Q_{i,F_{ij}}(v_i)_j\|^2_{L^2(\partial F_{ij})} \\
&\le Ch_i\{\|(v_i)_i\|^2_{L^2(\partial F_{ij})} + \|Q_{i,F_{ij}}\theta_{F_{ji}}(v_i)_j\|^2_{L^2(\partial F_{ij})} + \|Q_{i,F_{ij}}\theta_{\partial F_{ji}}(v_i)_j\|^2_{L^2(\partial F_{ij})}\} \\
&\le C\sum_{E_{ijk}\subset\partial F_{ij}}\{h_i\|(v_i)_i\|^2_{L^2(E_{ijk})} + h_i\|Q_{i,F_{ij}}\theta_{F_{ji}}(v_i)_j\|^2_{L^2(E_{ijk})} + h_j\|(v_i)_j\|^2_{L^2(E_{jik})}\} \\
&\equiv C\sum_{E_{ijk}\subset\partial F_{ij}}\{I^{(1)}_{E_{ijk}} + I^{(2)}_{E_{ijk}} + I^{(3)}_{E_{ijk}}\}. \qquad (31)
\end{aligned}
$$

It is not hard to see, using the same argument as priviously, that

$$
\begin{aligned}
I^{(1)}_{E_{ijk}} &= h_i\|\gamma_{jik}[(u_i)_i - (u_j)_i] + \gamma_{kij}[(u_i)_i - (u_k)_i]\|^2_{L^2(E_{ijk})} \qquad (32)\\
&\le C(1+\log\frac{H_i}{h_i})\{\gamma^2_{jik}(h_i|(u_i)_i|^2_{H^1(\Omega_i)} + h_i|(u_j)_j|^2_{H^1(\Omega_j)} + \|(u_j)_j - (u_j)_i\|^2_{L^2(F_{ij})}) \\
&+ \gamma^2_{kij}(h_i|(u_i)_i|^2_{H^1(\Omega_i)} + h_i|(u_k)_k|^2_{H^1(\Omega_k)} + \|(u_k)_k - (u_k)_i\|^2_{L^2(F_{ik})})\},
\end{aligned}
$$

$$
\begin{aligned}
I^{(2)}_{E_{ijk}} &\le Ch_i\gamma^2_{ji}\|Q_{i,F_{ij}}\theta_{F_{ji}}[(u_j)_j - (u_i)_j]\|^2_{L^2(E_{ijk})} \qquad (33)\\
&\le Ch_i\gamma^2_{ji}\{\|Q_{i,F_{ij}}[(u_j)_j - (u_i)_j]\|^2_{L^2(E_{ijk})} + \|Q_{i,F_{ij}}\theta_{\partial_{F_{ji}}}[(u_j)_j - (u_i)_j]\|^2_{L^2(E_{ijk})}\} \\
&\le C\gamma^2_{ji}\{h_i(1+\log\frac{H_i}{h_i})|(u_j)_j - (u_i)_j\|^2_{H^{1/2}(F_{ji})} + h_j\|(u_j)_j - (u_i)_j\|^2_{L^2(E_{ijk})}\} \\
&\le C\gamma^2_{ji}(h_i+h_j)(1+\log\frac{H}{h})|(u_j)_j - (u_i)_j|^2_{H^{1/2}(F_{ji})} \\
&\le C\gamma^2_{ji}(h_i+h_j)(1+\log\frac{H}{h})\{|(u_i)_i|^2_{H^1(\Omega_i)} + |(u_j)_j|^2_{H^1(\Omega_j)} + \frac{1}{h_i}\|(u_i)_i - (u_i)_j\|^2_{L^2(F_{ij})}\},
\end{aligned}
$$

$$
\begin{aligned}
I^{(3)}_{E_{ijk}} &\le Ch_j\|(\gamma_{jik}[(u_i)_j - (u_j)_j] + \gamma_{kij}[(u_i)_j - (u_k)_j])\|^2_{L^2(E_{jik})} \le C(1+\log\frac{H_j}{h_j}) * \\
&\{(\gamma^2_{jik} + \gamma^2_{jik})(h_j|(u_i)_i|^2_{H^1(\Omega_i)} + h_j|(u_j)_j|^2_{H^1(\Omega_j)} + \|(u_i)_i - (u_i)_j\|^2_{L^2(F_{ij})}) \\
&+\gamma^2_{kij}(h_j|(u_i)_i|^2_{H^1(\Omega_i)} + h_j|(u_k)_k|^2_{H^1(\Omega_k)} + \|(u_k)_k - (u_k)_j\|^2_{L^2(F_{jk})})\}. \qquad (34)
\end{aligned}
$$

Substituting (32), (33) and (34) into (31), we obtain

$$
\frac{\rho_i\delta}{l_{ij}h_{ij}}I_{\partial F_{ij}} \le C(1+\log\frac{H}{h})\{\frac{h_i+h_i}{h_{ij}}(\tilde{d}_i(u_i,u_i) + \tilde{d}_j(u_j,u_j)) + \sum_{E_{ijk}\subset\partial F_{ij}}\frac{h_{jk}}{h_{ij}}\tilde{d}_k(u_k,u_k)\}.
$$

It remains to estimate II in (24). Using a L^2-projection property, we have

$$
\begin{aligned}
II &\le Ch_i|(v_i)_j|^2_{H^{1/2}(F_{ji})} \le C\{h_i|\theta_{F_{ji}}(v_i)_j|^2_{H^{1/2}(F_{ji})} + h_i|\theta_{\partial F_{ji}}(v_i)_j|^2_{H^{1/2}(F_{ji})}\} \\
&\equiv C\{II_{F_{ji}} + II_{\partial F_{ji}}\}. \qquad (35)
\end{aligned}
$$

Using similar arguments as above, we obtain

$$II_{F_{ji}} \leq Ch_i(1+\log\frac{H_j}{h_j})^2\gamma_{ji}^2|(u_i)_j-(u_j)_j|^2_{H^{1/2}(F_{ji})} \tag{36}$$
$$\leq C(1+\log\frac{H_j}{h_j})^2\gamma_{ji}^2\{h_i|(u_i)_i|^2_{H^1(\Omega_i)}+h_i|(u_j)_j|^2_{H^1(\Omega_j)}+\frac{h_i}{h_j}\|(u_i)_i-(u_i)_j\|^2_{L^2(F_{ij})}\},$$

$$II_{\partial F_{ji}} \leq C\frac{h_i}{h_j}\|\theta_{\partial F_{ji}}(v_i)_j\|^2_{L^2(F_{ji})} \leq Ch_i\sum_{E_{jik}\subset\partial F_{ji}}\|(v_i)_j\|^2_{L^2(E_{jik})}, \tag{37}$$

and

$$h_i\|(v_i)_j\|^2_{L^2(E_{jik})} \leq C(1+\log\frac{H_j}{h_j})\{(\gamma_{jik}+\gamma_{kij})* \tag{38}$$
$$(h_i|(u_i)_i|^2_{H^1(\Omega_i)}+h_i|(u_j)_j|^2_{H^1(\Omega_j)}+\frac{h_i}{h_j}\|(u_i)_j-(u_i)_i\|^2_{L^2(F_{ji})})$$
$$+\gamma_{kij}(h_i|(u_k)_k|^2_{H^1(\Omega_k)}+h_i|(u_j)_j|^2_{H^1(\Omega_j)}+\frac{h_i}{h_j}\|(u_k)_j-(u_k)_k\|^2_{L^2(F_{jk})})\}.$$

Substituting (38) into (37) and adding (36), see (35), we obtain

$$\frac{\rho_i\delta}{l_{ij}h_{ij}}II \leq C(1+\log\frac{H_j}{h_j})(\frac{h_i}{h_{ij}}\tilde{d}_i(u_i,u_i)+\frac{h_i}{h_{ij}}\tilde{d}_j(u_j,u_j)+\sum_{E_{ijk}\subset\partial F_{ij}}\frac{h_{jk}}{h_{ij}}\frac{h_i}{h_j}\tilde{d}_k(u_k,u_k)\}.$$

The proof is complete.

Remark 1. The proof of Lemma 1 also works with minor modifications when $\bar{F}_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is an union of faces, also, for FETI-DP with corner and average face constraints only, or with corner and edge constraints only.

Acknowledgements The first author has been partially supported by the Polish NSC grant 2011/01/B/ST1/01179.

References

1. Dryja, M.: On discontinuous Galerkin methods for elliptic problems with discontinuous coefficients. Comput. Methods Appl. Math. **3**(1), 76–85 (electronic) (2003). Dedicated to Raytcho Lazarov
2. Dryja, M., Galvis, J., Sarkis, M.: A FETI-DP preconditioner for a composite finite element and discontinuous Galerkin method. SIAM J. Numer. Anal. **51**(1), 400–422 (2013). DOI 10.1137/100796571. URL http://dx.doi.org/10.1137/100796571
3. Toselli, A., Widlund, O.: Domain decomposition methods—algorithms and theory, *Springer Series in Computational Mathematics*, vol. 34. Springer-Verlag, Berlin (2005)