Numerical solution of PDE eigenvalue problems in acoustic field computation

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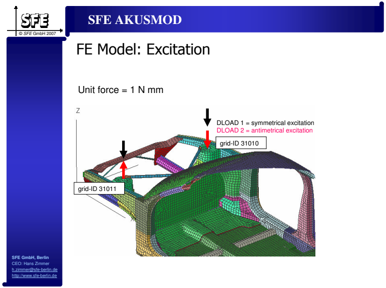
DFG Research Center MATHEON

Mathematics for key technologies

Outline

Industrial Project with company SFE in Berlin



- ▷ SFE has its own parameterized discrete FEM model which allows geometry and topology changes.
- ▷ Goals: Numerical methods for frequency response and numerical methods for large scale structured polynomial eigenvalue problems. Implementation of parallel solver in SFE Concept.

The acoustic field in the car is modeled by the 3-D lossless wave equation (in air). For this we need:

The continuity equation (conservation of mass):

$$\frac{\partial \tilde{\rho}}{\partial t} + \nabla(\tilde{\rho} v) = 0.$$

The Euler equation (Newton's Second Law)

$$\tilde{\rho}(\frac{\partial v}{\partial t} + (v \cdot \nabla) v) = -\nabla \tilde{p}$$

$$\begin{aligned} v &= v(x; y; z; t) \text{ particle velocity,} \\ \tilde{\rho} &= \tilde{\rho}(x; y; z; t) \text{ particle density,} \\ \tilde{p} &= \tilde{p}(x; y; z; t) \text{ pressure.} \end{aligned}$$

Simplifying assumptions

- There is no temperature change.
- The fluid is inviscid (no shear forces).
- No influence of external forces.
- We can make the expansions

$$
\begin{aligned}
\tilde{p} &= p_0 + p(x;y;z;t) \text{ with } p_0 \gg p \; (p_0 = 10^6 p), \\
\tilde{\rho} &= \rho_0 + \rho(x;y;z;t) \text{ with } \rho_0 \gg \rho.
\end{aligned}
$$

- Adiabatic fluid (no heat exchange during compression).
- Ideal gas $\rho = \frac{p}{c^2}$ where c is the speed of sound.
- $(v \cdot \nabla)v$ and $\rho \frac{\partial v}{\partial t}$ are small.

Damping/absorption is realized by adding an additional first order term that includes a material dependent parameter r. After some manipulations we obtain a second order PDE:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + \frac{r}{\rho_0^2 c^2}\frac{\partial p}{\partial t} - \Delta p = 0.$$

Fluid-structure interaction is modeled via boundary conditions that describe the displacements of the structure u.
Discretization via FEM in space yields

$$M_f \ddot{p}_d + D_f \dot{p}_d + K_f p_d + D_{sf} \ddot{u}_d = 0.$$

Here $M_f = M_f^T$ and $K_f = K_f^T$ are positive definite and D_f is symmetric positive semidefinite, D_{sf} describes the coupling.

The (discrete) finite element model for the vibration of the structure (linear materials) is:

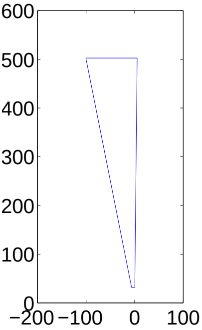
$$M_s \ddot{u}_d + D_s \dot{u}_d + K_s u_d = f_e + f_p.$$

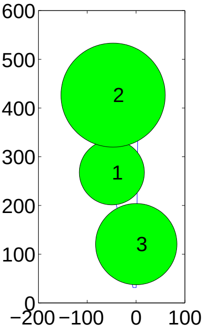
- Here f_e is a (discrete) external load and f_p is the pressure load.
- M_s, D_s, K_s are real symm. pos. semidef. mass/damping/stiffness matrices of the structure.
- M_s is singular and diagonal.
- The stiffness matrix $K_s = K_1(\omega) + \imath K_2$ is complex symmetric and frequency dependent.

Full model: summary

$$\begin{bmatrix} M_s & 0 \\ D_{sf}^T & M_f \end{bmatrix} \begin{bmatrix} \ddot{u}_d \\ \ddot{p}_d \end{bmatrix} + \begin{bmatrix} D_s & 0 \\ 0 & D_f \end{bmatrix} \begin{bmatrix} \dot{u}_d \\ \dot{p}_d \end{bmatrix} + \begin{bmatrix} K_s(\omega) & D_{sf} \\ 0 & K_f \end{bmatrix} \begin{bmatrix} u_d \\ p_d \end{bmatrix} = \begin{bmatrix} f_s \\ 0 \end{bmatrix}.$$

- M_s, M_f, K_f are real symm. pos. semidef. mass/stiffness matrices of structure and air, M_s is singular and diagonal, M_s is a factor $1000 - 10000$ larger than M_f.
- $K_s(\omega) = K_s(\omega)^T = K_1(\omega) + \imath K_2$.
- D_s, D_f are real symmetric damping matrices.
- D_{sf} is real coupling matrix between structure and air.
- Blocks depend on geometry, topology and material parameters.





- Goal: Compute all ev in a typical trapezoidal region.
- Shift-invert block Arnoldi method with many different shifts.
- Many solves with $F(\lambda_i) = \lambda_i^2 M + \lambda_i D + K$ for many λ_i. We used the direct solver (MUMPS).

Shift-invert block Arnoldi

Compute ℓ eigenvalues in a trapezoidal region and orthogonal basis S_ℓ of corresponding subspace spanned by the evecs and generalized evecs to these eigenvalues.
The projected system has the form

$$Q_\ell(\lambda) := \lambda^2 M_\ell + \lambda D_\ell + K_\ell := \lambda^2 S_\ell^T M S_\ell + \lambda S_\ell^T D S_\ell + S_\ell^T K S_\ell$$

Open: Guarantee that we found all eigenvalues in a region.

Shift-invert Arnoldi II

- Mass matrix is singular, this causes convergence problems.
- Large multiplicity at 0, theoretically 6 in real life much higher.
- Codes like ARPACK or Anazazi use a shift-invert 'preconditioner'.

$$\mathcal{A} := \begin{bmatrix} Q(\sigma)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 2\sigma M + D & M \\ -I & 0 \end{bmatrix}.$$

- The solve is done by the direct solver MUMPS
- One forms the block Krylov subspace

$$\mathcal{K}_m(\mathcal{A}, B) := span\{B, \mathcal{A}B, \mathcal{A}^2 B, \ldots, \mathcal{A}^{m-1}B\},$$

  and runs Gram-Schmidt to get S_ℓ.
- B is arbitrary or recycles information.
- $\mathcal{A}^i B$ is never explicitly formed.

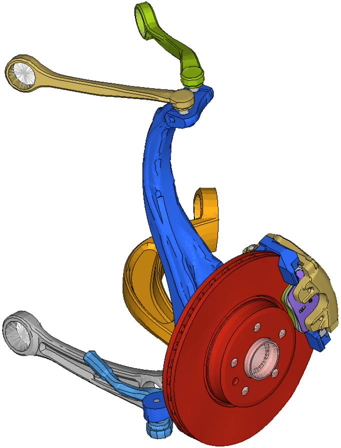
- Real car model. Regular mesh of 35mm, 219432 dofs
- Evs in triangular region bordered by lines $\Im(\lambda) > 20\Re(\lambda)$, $\Im(\lambda) > -20\Re(\lambda)$, and $\Im(\lambda) < f \cdot 2\pi$, for $f \in \{50, 100, 150, 200, 250\}$.
- PC with an Intel Core2 Duo E6850 CPU clocked at 3.0GHz, with 4Gb RAM.
- One shift was addressed at a time using one processor. The block size was 5.

0Hz to	50Hz	100Hz	150Hz	200Hz	250Hz
no of found evss	20	52	129	217	346
no of shifts	1	3	3	5	7
no of iterations	39	139	136	294	437
time (sec)	713	3959	3784	21882	32771

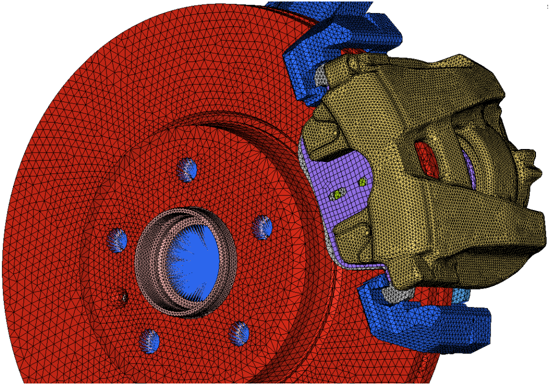
Current project with Audi and Opel and several SMEs (2012-14)
Joint with N. Gräbner, U. von Wagner, TU Berlin, Mechanics and N. Hoffmann, TU Hamburg-Harburg, Mechanics, S. Quraishi, C. Schröder, TU Berlin Mathematics.

Goals:

- Develop mechanics based discrete FE model of brake system with friction contact including circulatory and gyroscopic effects.
- Identification of energy dissipation effects.
- Model and simulate nonlinear effects in brake behavior near squeaking frequency.
- Passive and active remedies to avoid squeaking.



View of the brake model



Classical discrete FE-Model of disk brake.

$$M\ddot{q} + D\dot{q} + Kq = 0,$$

M mass, D damping, K stiffness matrix, all real symmetric.

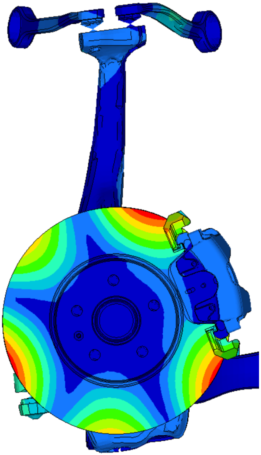
Further terms:

- Rotation frequency Ω.
- Geometric stiffness matrix $K_{geo} = K_{geo}^T$ proportional to Ω^2.
- Gyroscopic matrix $D_G = -D_G^T$ which is proportional to Ω.

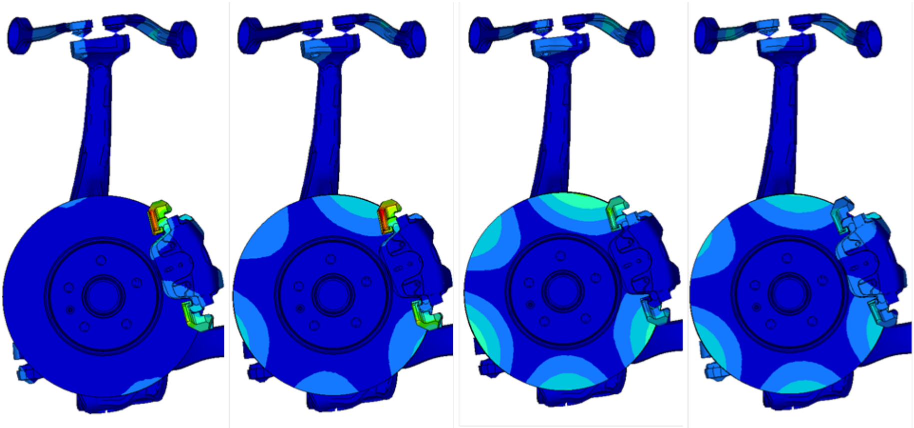
Complete equations of motion

$$M\ddot{q} + (D + \frac{1}{\Omega} D_R + \Omega D_G)\dot{q} + (K + K_R + \Omega^2 K_{geo})q = f.$$

Undamped model without circulatory and gyroscopic forces:
$(\lambda^2 M + K + K_{geo})x = 0.$

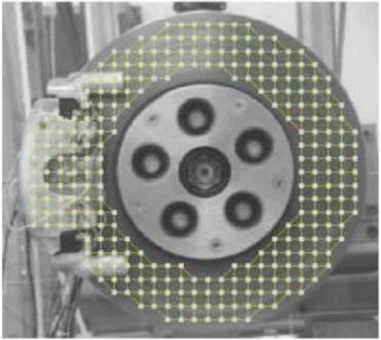


Eigenmodes at 1859 Hz.

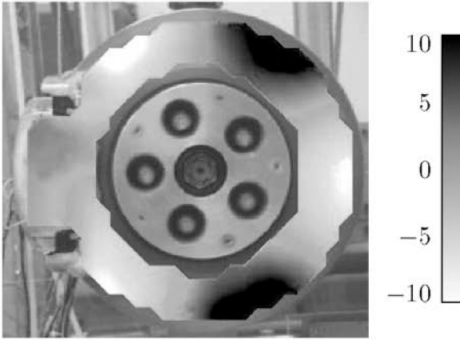


Eigenform at 1873 Hz with positive real part and a phase of 0,45,90 and 135.

Measurement of brake vibrations



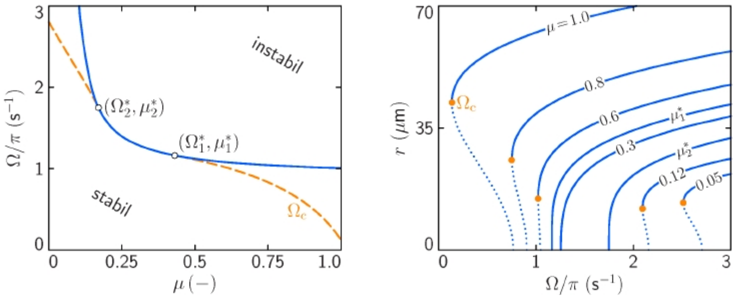
Gitter der Messpunkte



Betriebsschwingform (1750 Hz)

Measurements indicate subcritical Hopf bifurcations, i.e. eigenvalues crossing imaginary axis for certain disk frequencies.

Stability regions, linear vs. nonlinear



Bifurcation diagram linear analysis (blue), nonlinear analysis (red). Coefficient of friction μ via disk frequency Ω.

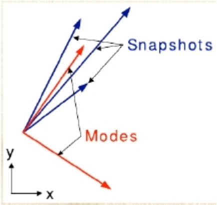
A quick and dirty approach

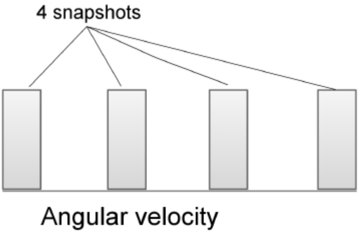
- Solve first order problem

$$\left(\lambda \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} - \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}\right) z = 0$$

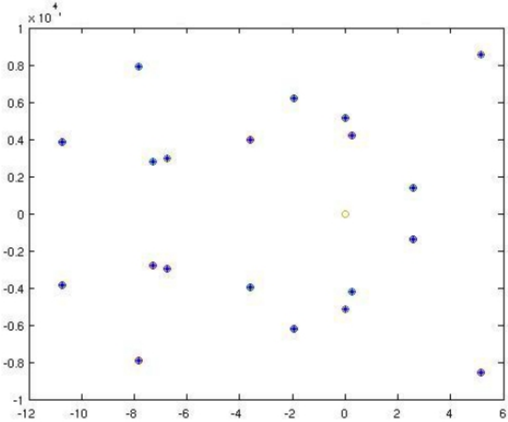
  via shift-and-invert Arnoldi for many frequencies Ω_i and compute space of evecs $X(\Omega_i)$ to smallest eigenvalues.
- Construct $A = [X(\Omega_1), \dots, X(\Omega_s)]$.
- Extract dominant directions by computing partial singular value decomposition $A_k = U_k \Sigma_k V_k^T$ and use dominant singular vectors as projection space.

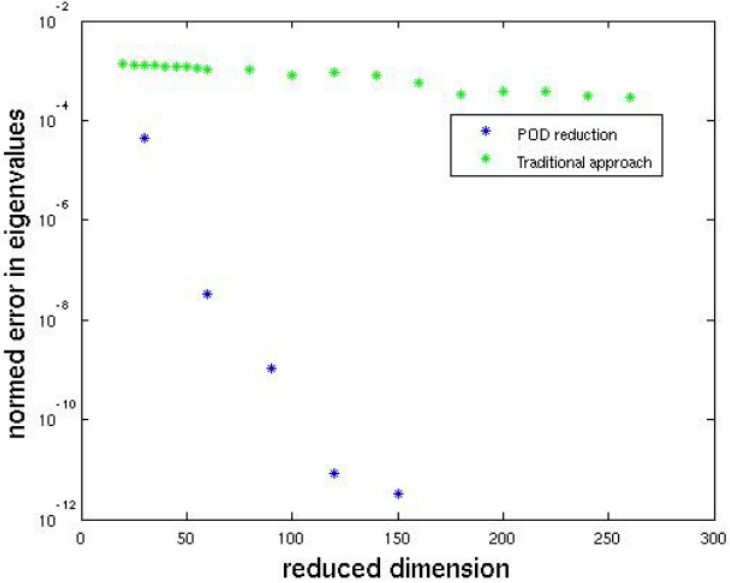
Proper orthogonal decomposition





snapshots for $\omega = 2\pi(1, 40, 80, 120, 160)$. Comparison of eigenvalues for full system *o* with reduced system *x* for $\omega = 100 \times 2\pi$.




normed error in eigenvalues
reduced dimension
POD reduction
Traditional approach
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
0
50
100
150
200
250
300

Lessons learned

- POD is better than traditional approach but still not satisfactory.
- Discrete finite elements and quasi-uniform grids followed by expensive model reduction is really a waste.
- The numerical linear algebra methods that we currently use are not efficient (also those in commercially available codes).
- For eigenvalue problem everything is partially heuristic.
- Can we make that non-heuristic by developing error estimates?
- How about AFEM, the adaptive finite element method ?

Can we disprove the engineers that say that uniform mesh and brute force linear algebra is best.

Outline

- Adaptive Finite Element methods refine the mesh where it is necessary, and coarsen it where the solution is well represented.
- They use a priori and a posteriori error estimators to get information about the discretization error.
- They are well established for PDE boundary value problems.
- But here we want to use them for PDE eigenvalue problems, which is much harder.

Incomplete literature survey:

- Most results and methods only for the self-adjoint elliptic case.
- First results Babuska/Osborn 1989, Strang/Fix 1973.
- a priori estimates Larsson 2001, Knyazev et al. 2006, 2007, 2008.
- a posteriori estimates Verführt 1996, Giani/Graham 2008, Grubisic/Ovall 2009, Carstensen/Gedicke 2008, 2011, 2013, Gedicke Diss. 2013, Garau/Morin/Zuppa 2008, Miedlar 2011, 2012, 2013.
- Nonsymmetric problems: Heuveline/Rannacher 2001, Rannacher 2009, Dahmen et al 2009, Carstensen, Gedicke, M./ Miedlar 2011,.
- Very few applications in real codes Zschiedrich et al 2007/2008.

Model problem: Elliptic PDE evp

Model problem (such as disk brake problem without damping/gyroscopic/circulatory terms)

$$\begin{aligned} \Delta u &= \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Classical FEM discretization (with mesh-width H) leads to generalized discrete evp

$$A_H u_H = \lambda_H B_H u_H$$

Solve $\rightarrow$ Estimate $\rightarrow$ Mark $\rightarrow$ Refine

Weak formulation:
Determine eigenvalue/eigenfunction pair
$(\lambda, u) \in \mathbb{R} \times V := \mathbb{R} \times H_0^1(\Omega; \mathbb{R})$ with $b(u, u) = 1$ and

$$a(u, v) = \lambda b(u, v) \quad \text{for all } v \in V,$$

where the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are defined by

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx, \ \ b(u, v) := \int_\Omega uv \, dx \quad \text{for } u, v \in V.$$

Induced norms $|||\cdot||| := |\cdot|_{H^1(\Omega)}$ on V and $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ on $L^2(\Omega)$.

Discrete eigenvalue problem:
Determine eigenvalue/eigenfunction pair $(\lambda_\ell, u_\ell) \in \mathbb{R} \times V_\ell$ with $b(u_\ell, u_\ell) = 1$ and

$$a(u_\ell, v_\ell) = \lambda_\ell b(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell.$$

Algebraic eigenvalue problem: Use coordinate representation to get the finite-dimensional generalized algebraic evp

$$A_\ell x_\ell = \lambda_\ell B_\ell x_\ell$$

with stiffness matrix $A_\ell = [a(\varphi_i, \varphi_j)]_{i,j=1,\dots,N_\ell}$, mass matrix $B_\ell = [b(\varphi_i, \varphi_j)]_{i,j=1,\dots,N_\ell}$, associated with the nodal basis functions $V_\ell = \{\varphi_1, \dots, \varphi_{N_\ell}\}$.
Discrete eigenvector: $x_\ell =: [x_{\ell,1}, \dots, x_{\ell,N_\ell}]^T$.
Approximated eigenfunction:

$$u_\ell = \sum_{k=1}^{N_\ell} x_{\ell,k}\varphi_k \in V_\ell.$$

Estimate the error a posteriori via

$$|\lambda - \lambda_\ell| + |||u - u_\ell|||^2 \lesssim \eta_\ell^2 := |||u_{\ell-1} - u_\ell|||^2.$$

Here $\lesssim$ denotes an inequality that holds up to a multiplicative constant.
A posteriori error estimators for Laplace eigenvalue problem
Grubisic/Ovall 2009, M./Miedlar 2011, Neymeyr 2002

Mark

- For a triangulation $\mathcal{T}_\ell$ let $\mathbb{N}_\ell$ (resp. $\mathbb{N}_\ell(\Omega)$) denote the set of nodes (resp. interior nodes) and let $\mathbb{E}_\ell$ (resp. $\mathbb{E}_\ell(\Omega)$) denote the set of edges (resp. interior edges).
- For a node $z \in \mathbb{N}_\ell$, we denote by $\mathbb{E}_\ell(z)$ the set of edges in $\mathbb{E}_\ell$ and by ω_z the union of triangles in $\mathcal{T}_\ell$ that share the node z.
- The maximal mesh-size is denoted by $H_\ell := \max_{T\in\mathcal{T}_\ell} \operatorname{diam}(T)$.
- For $E \in \mathbb{E}_\ell(\Omega)$ let $T_+, T_- \in \mathcal{T}_\ell$ be the two neighboring triangles such that $E = T_+ \cap T_-$.
- The jump of the discrete gradient ∇u_ℓ along an inner edge $E \in \mathbb{E}_\ell(\Omega)$ in normal direction ν_E, pointing from T_+ to T_-, is defined by $[\nabla u_\ell] \cdot \nu_E := \left(\nabla u_\ell|_{T_+} - \nabla u_\ell|_{T_-}\right) \cdot \nu_E$.

Employ an edge residual a posteriori error estimator
Duran et al 2003, Carstensen/Gedicke 2008.

$$\eta_\ell^2 := \sum_{E \in \mathbb{E}_\ell(\Omega)} \eta_\ell^2(E) \quad \text{with} \quad \eta_\ell^2(E) := |E| \| [\nabla u_\ell] \cdot \nu_E \|^2_{L^2(E)},$$

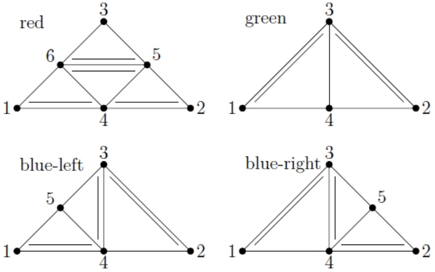
which is reliable and efficient for sufficiently small mesh-size H_0

$$|\!|\!| u - u_\ell |\!|\!| \approx \eta_\ell.$$

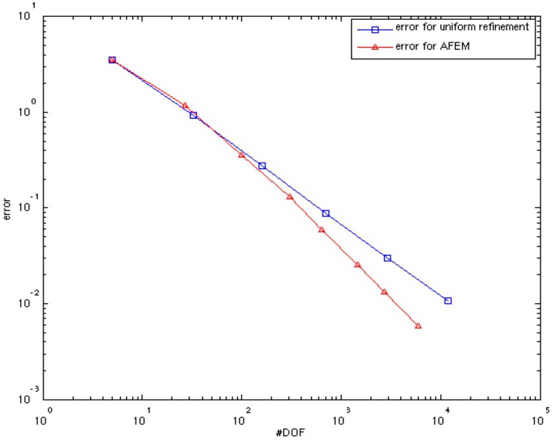
Based on the local refinement indicators $\eta_\ell(E)$, nodes are marked for refinement.
Let $\mathbb{M}_\ell \subseteq \mathbb{N}_\ell(\Omega)$ be the minimal set of refinement nodes such that for $0 < \theta \leq 1$

$$\theta \sum_{z \in \mathbb{N}_\ell(\Omega)} \eta_\ell^2(\mathbb{E}_\ell(z)) \leq \sum_{z \in \mathbb{M}_\ell} \eta_\ell^2(\mathbb{E}_\ell(z)).$$

- For each refinement node $z \in \mathbb{M}_\ell \subseteq \mathbb{N}_\ell(\Omega)$, mark all edges $\mathbb{E}_\ell(z)$ for refinement and then use a closure algorithm.
- The refinement $\mathcal{T}_{\ell+1}$ is computed by the application of one of the following rules where all triangles $T \subseteq \omega_z$, $z \in \mathbb{M}_\ell$, are refined either *red* or *blue*.



Convergence on L-shape domain.



ref. level	#DOF	$\tilde{\lambda}_1$	$\lvert\lambda_1 - \tilde{\lambda}_1\rvert$
1	5	13.1992	3.5595
2	27	10.8173	1.1775
3	99	9.9982	0.3584
4	306	9.7721	0.1323
5	641	9.6982	0.0585
6	1461	9.6652	0.0255
7	2745	9.6528	0.0131
8	5961	9.6455	0.0058

- AFEM works nicely for elliptic self-adjoint evps.
- For the analysis in most AFEM methods it is assumed that the algebraic evp is solved exactly.
- But this requires the largest percentage of the computing time.
- The solution of the algebraic evp is only used to determine where the grid is refined. This is a complete waste of computational work.
- How we can incorporate the solution of the algebraic eigenvalue problem (AEVP) into the adaptation process?

Outline

Solve:

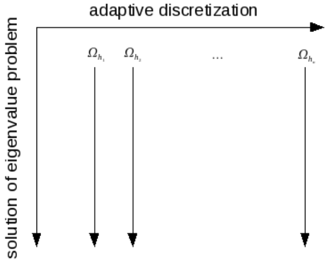
- compute eigenpair $(\tilde{\lambda}_H, \tilde{\mathbf{u}}_H)$ on the coarse mesh,
- use iterative solver, i.e. Krylov subspace method,
- but do not solve very accurately, stop after *k* steps or when tolerance *tol* is reached.

Estimate:

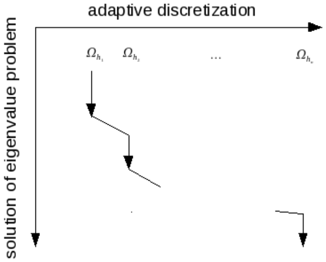
- prolongate $\tilde{\mathbf{u}}_H$ from the coarse mesh $\mathcal{T}_H$ to the uniformly refined mesh $\mathcal{T}_h$,
- Balance residual vector $\hat{\mathbf{r}}_h$ and error estimate Miedlar 2011.

Mark and Refine: mark elements and refine the mesh.

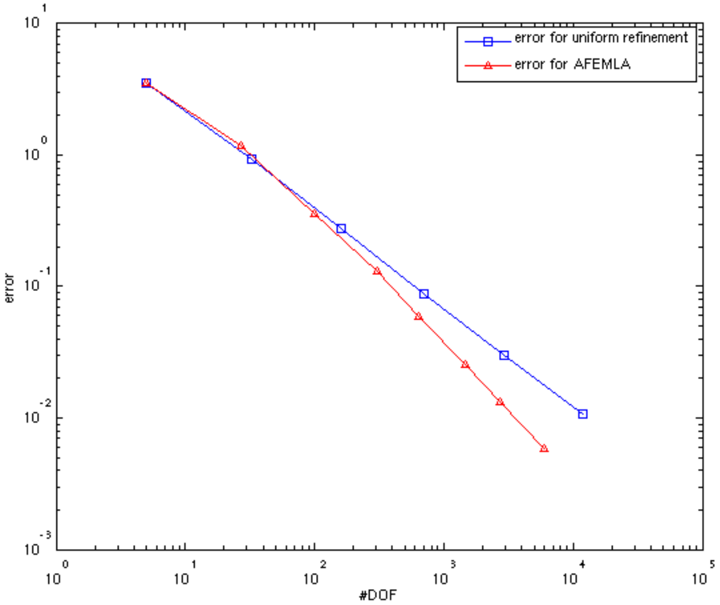
Solve → Estimate → Mark → Refine



Standard AFEM



AFEMLA


error for uniform refinement
error for AFEMLA
error
#DOF
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^0
10^1
10^2
10^3
10^4
10^5

- AFEMLA works nicely for elliptic self-adjoint evps.
- It significantly reduces the computing time.
- Balancing of discretization and LA error possible, Miedlar 2011.
- Convergence if saturation property holds, i.e., $|\lambda_h - \lambda| \leq \beta |\lambda_H - \lambda|$

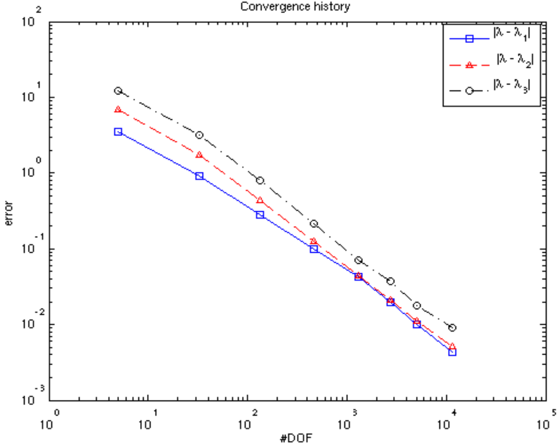
Theorem (Carstensen/Gedicke/M./Miedlar 2013)

Suppose that the initial triangulation $\mathcal{T}_0$ has sufficiently small maximal mesh-size H_0. Then there exists $0 \le \varrho < 1$ such that for all $\ell \in \mathbb{N}_0$ the following inequalities hold

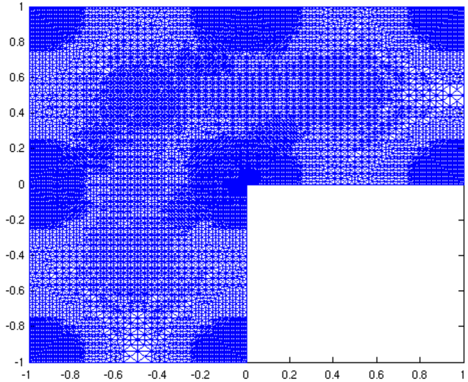
$$|\!|\!| u - u_{\ell+1} |\!|\!|^2 \le \varrho |\!|\!| u - u_\ell |\!|\!|^2 + \lambda_{\ell+1}^3 H_\ell^4;$$
$$|\lambda - \lambda_{\ell+1}| \le \varrho |\lambda - \lambda_\ell| + \lambda_{\ell+1}^3 H_\ell^4.$$

Outline

Conv. first 3 evs, L-shape domain.



Adaptive Mesh, first 3 evs



Errors for three smallest eigenvalues

ref. level	1	2	3	4	5	6	7	8
#DOF	5	33	133	465	1306	2770	4997	11499
$\tilde{\lambda}_1$	13.1992	10.5542	9.9192	9.7376	9.6817	9.6591	9.6496	9.6440
$\tilde{\lambda}_2$	22.0215	16.9097	15.6315	15.3211	15.2421	15.2184	15.2085	15.2024
$\tilde{\lambda}_3$	32.0000	22.9075	20.5262	19.9515	19.8089	19.7760	19.7569	19.7482

ref. level	1	2	3	4	5	6	7	8
$\lvert\lambda_1 - \tilde{\lambda}_1\rvert$	3.5595	0.9144	0.2795	0.0979	0.0420	0.0194	0.0099	0.0043
$\lvert\lambda_2 - \tilde{\lambda}_2\rvert$	6.8242	1.7125	0.4342	0.1239	0.0448	0.0211	0.0112	0.0051
$\lvert\lambda_3 - \tilde{\lambda}_3\rvert$	12.2608	3.1683	0.7870	0.2123	0.0697	0.0367	0.0177	0.0090

A simple non-self-adjoint model problem

Carstensen/Gedicke/M./Miedlar 2009

Convection-diffusion eigenvalue problem:

$$-\Delta u + \gamma \cdot \nabla u = \lambda u \text{ in } \Omega \qquad \text{and} \qquad u = 0 \text{ on } \partial\Omega$$

Discrete weak primal and dual problem:

$$\begin{aligned} a(u_\ell, v_\ell) + c(u_\ell, v_\ell) &= \lambda_\ell b(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell, \\ a(w_\ell, u_\ell^\star) + c(w_\ell, u_\ell^\star) &= \overline{\lambda_\ell^\star} b(w_\ell, u_\ell^\star) \quad \text{for all } w_\ell \in V_\ell. \end{aligned}$$

Generalized algebraic eigenvalue problem:

$$(A_\ell + C_\ell)\mathbf{u}_\ell = \lambda_\ell B_\ell \mathbf{u}_\ell \qquad \text{and} \qquad \mathbf{u}_\ell^\star (A_\ell + C_\ell) = \lambda_\ell^\star \mathbf{u}_\ell^\star B_\ell$$

The eigenvalue with the smallest real part, which is proved to be simple and well separated Evans '00, is considered.

$$\mathcal{H}(t) = (1-t)\mathcal{L}_0 + t\mathcal{L}_1 \quad \text{for } t \in [0,1],$$

where $\mathcal{L}_0 u := -\Delta u$ and $\mathcal{L}_1 u := -\Delta u + \beta \cdot \nabla u$.
Discrete homotopy for the model eigenvalue problem:

$$\mathcal{H}_\ell(t) = (A_\ell + C_\ell)(t) = (1-t)A_\ell + t(A_\ell + C_\ell) = A_\ell + tC_\ell.$$

Homotopy, discretization, approximation and iteration error.

Homotopy error:

$$|\lambda(1) - \lambda(t)| \lesssim (1 - t)\|\gamma\|_{L^\infty(\Omega)}\|u\|_A = \nu,$$

Discretization error:

$$\|\lambda(t) - \lambda_\ell(t)\| \lesssim \sum_{T \in \mathcal{T}_\ell} \left(\eta_\ell^2(T) + \eta_\ell^{\star 2}(T)\right).$$

Approximation error:

$$|\lambda_\ell(t) - \tilde{\lambda}_\ell(t)| + |\lambda_\ell^\star(t) - \tilde{\lambda}_\ell^\star(t)| \le \mu_\ell.$$

Iteration error: The iterative eigensolver should be stopped early.

Lemma

For the model problem, the difference between the iterative eigenvalue $\tilde{\lambda}_\ell(t)$ in the homotopy $\mathcal{H}_\ell(t)$ and the continuous eigenvalue $\lambda(1)$ of the original problem can be estimated a posteriori via

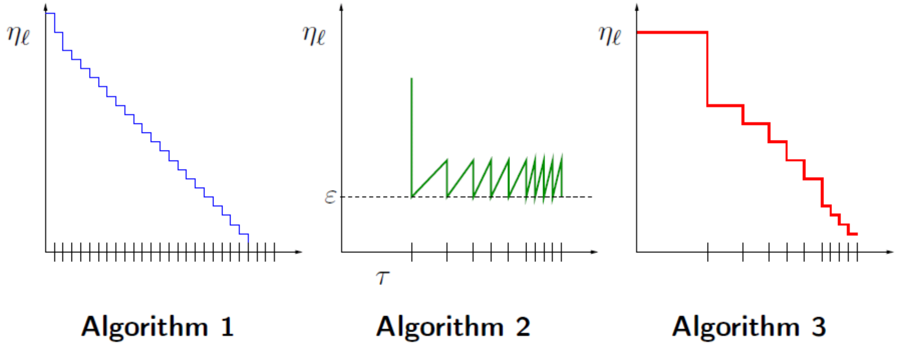
$$\begin{aligned}\|\lambda(1) - \tilde{\lambda}_\ell(t)\| &\lesssim \nu(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}_\ell^\star(t)) + \eta^2(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}_\ell^\star(t)) \\ &+ \mu^2(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}_\ell^\star(t))\end{aligned}$$

in terms of

$$\begin{aligned}&\nu(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}_\ell^\star(t)) := (1-t)\|\gamma\|_\infty \left(|\!|\!| \tilde{u}_\ell(t) |\!|\!| + |\!|\!| \tilde{u}_\ell^\star(t) |\!|\!| \right) \\ + \;& (1-t)\|\gamma\|_\infty \left(\eta(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}_\ell^\star(t)) + \mu(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}_\ell^\star(t)) \right).\end{aligned}$$

Solve $\rightarrow$ Estimate $\rightarrow$ Mark $\rightarrow$ Refine

Algorithm 1: Balances the homotopy, discretization, iteration and approximation errors but uses fixed stepsize in continuation method.
Algorithm 2: Adaptivity in homotopy and in the iteration is achieved by simple stepsize control, no homotopy error is considered.
Algorithm 3: Adaptivity in the homotopy error, the discretization error, the iteration error including step size control.


η_ℓ
η_ℓ
ε
τ
η_ℓ
Algorithm 1
Algorithm 2
Algorithm 3

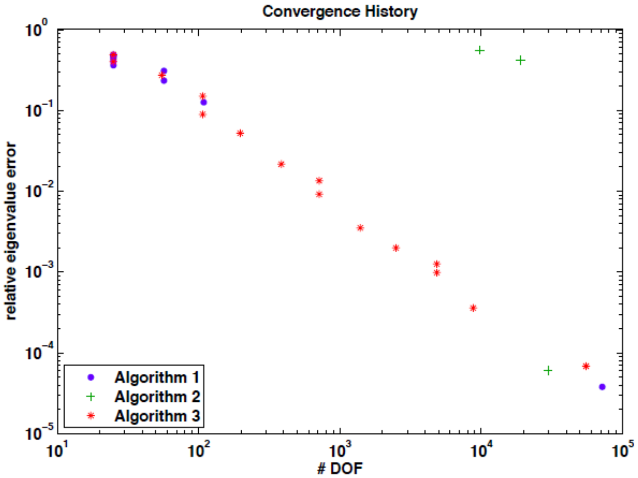


Figure: Conv. history of Algorithm 1, 2 and 3 with respect to #DOF.

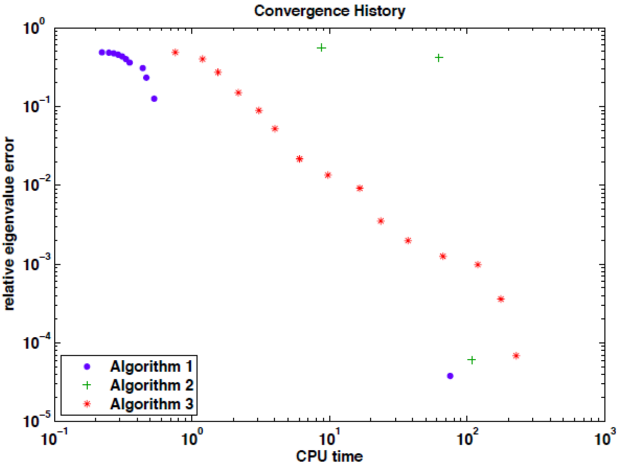


Figure: Conv. history of Algorithm 1, 2 and 3 with respect to CPU time.

$\lambda \approx 44.739208802205724$

t	$\eta_\ell(t)$	$\nu_\ell(t)$	$\mu_\ell(t)$	est. error
0.0000	23.0271	95.6366	0.2701265	118.93382
0.5000	32.6896	51.7112	0.0843690	84.48512
0.7500	11.6020	28.5244	0.4515713	40.57800
0.8750	6.7380	15.4099	0.4711298	22.61912
0.9375	7.8500	7.9782	0.0272551	15.85547
0.9688	3.2088	4.0697	0.2891100	7.56762
0.9844	1.2060	2.0673	0.4278706	3.70119
0.9922	0.4560	1.0380	0.0004539	1.49451
0.9961	0.4602	0.5202	0.0029006	0.98322
0.9980	0.1864	0.2608	0.0012530	0.44843
0.9990	0.0707	0.1305	0.0204610	0.22162
0.9995	0.0282	0.0653	0.0003639	0.09386
0.9998	0.0282	0.0326	0.0001766	0.06105
0.9999	0.0106	0.0163	0.0001521	0.02703
1.0000	0.0007	0.0000	0.0000243	0.00073

t	$\tilde{\lambda}_\ell(t)$	$\frac{\lvert\lambda_\ell(1)-\tilde{\lambda}_\ell(t)\rvert}{\lvert\lambda_\ell(1)\rvert}$	#DOF	CPU time
0.0000	22.86578	0.48891	25	0.76
0.5000	26.73866	0.40234	25	1.20
0.7500	32.54928	0.27247	55	1.55
0.8750	38.00079	0.15062	107	2.18
0.9375	40.73818	0.08943	107	3.07
0.9688	42.39339	0.05243	197	4.01
0.9844	43.77023	0.02166	385	6.06
0.9922	44.13547	0.01349	715	9.74
0.9961	44.32847	0.00918	715	16.59
0.9980	44.58151	0.00352	1398	23.57
0.9990	44.65025	0.00199	2494	37.14
0.9995	44.68298	0.00126	4848	66.70
0.9998	44.69522	0.00098	4848	119.47
0.9999	44.72311	0.00036	8785	175.75
1.0000	44.73615	0.00007	55235	226.87

Conclusions

- Eigenvalue methods are important in practice for stability analysis and model reduction.
- Using a fine mesh and then doing model reduction usually works fine, but hardly any error estimates exist.
- Discrete finite elements and quasi-uniform grids are a waste.
- The current numerical linear algebra methods (also those in commercially available codes) are not satisfactory. AFEMLA is an alternative, it gives error bounds for reduced order model.
- Extension of backward error analysis to infinite dimensional case Miedlar 2011/2013
- A posteriori error estimates for hp-finite elements for non-self-adjoint PDE evps Giani/Grubisic/Miedlar/Ovall 2013

- Development of good preconditioned iterative solvers for $F(\lambda_i) = \lambda_i^2 M + \lambda_i D + K$ for many λ_i.
- Adaptive FEM for non-self adjoint eigenvalue problems need to be developed and made industrially available.
- Complex and multiple eigenvalues, no theory, no methods.
- A posteriori and a priori error estimates for non-self-adjont acoustic problems.
- Real world solvers.

Thank you very much
for your attention.

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