Preconditioning techniques for mass matrices arising from isogeometric analysis

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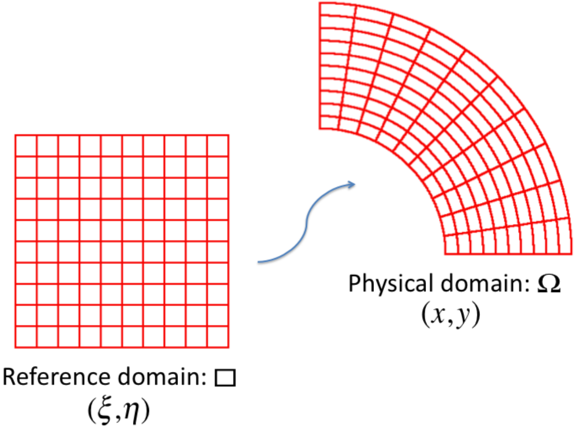
Domain Decomposition 22, Sep 2013, Lugano

Isogeometric analysis [1]



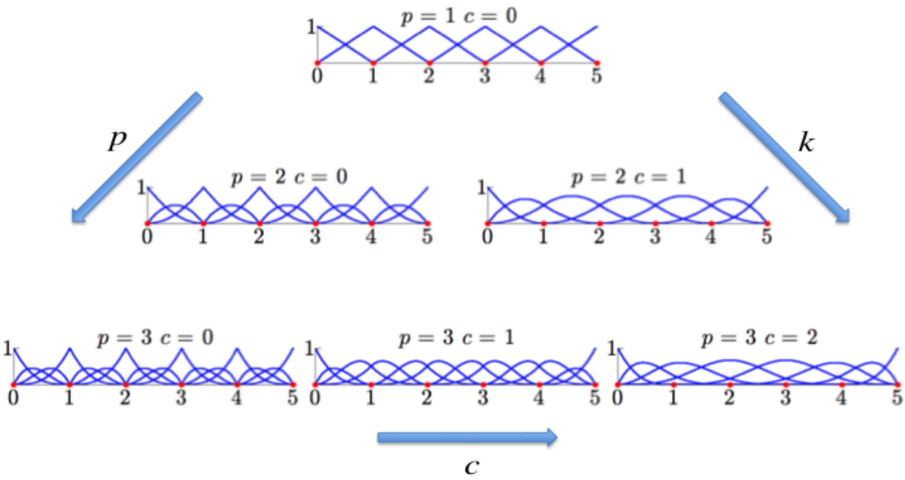
IGA applications:

- Shell modeling
- Turbulence modeling
- Fluid-structure interaction
- Phase-field modeling
- Structure vibrations
- Shape optimization
- Biomedical engineering
- Electromagnetism

$\vdots$

[1] Hughes, Cottrell, and Bazilevs, CMAME, 2005

Basis functions



Mass matrix

Basis functions:

$B = B^\eta \otimes B^\xi$, listed as a column vector.

$$
\begin{aligned}
M^J &= \int_\square BB^T J d_\square \\
&= \int_\square (B^\eta \otimes B^\xi)(B^\eta \otimes B^\xi)^T J d_\square \\
&= \int_\square (B^\eta (B^\eta)^T) \otimes (B^\xi (B^\xi)^T) J d_\square
\end{aligned}
$$

Candidate Preconditioner: Inverse of M

$$
M = \int_\square BB^T d_\square
$$

- 'Close'? The skeleton of M^J is kept
- Cheap? M is the Kronecker product of two 1D matrices

Kronecker product

Definition of Kronecker product

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}$$

- Mixed-product:

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

- Inverse:

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

- Transpose:

$$(A \otimes B)^T = A^T \otimes B^T$$

- Associative:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

Kronecker product structure in M

Tensor product basis functions:

$B = B^\eta \otimes B^\xi$, listed as a column vector.

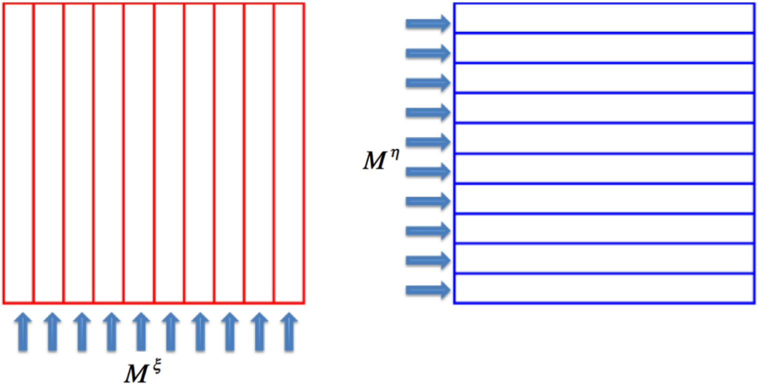
Derivation:

$$
\begin{array}{llcll}
1) & & M & = & \int_\square BB^T d_\square \\
2) & \text{Substitution} & \longrightarrow & = & \int_\square (B^\eta \otimes B^\xi)(B^\eta \otimes B^\xi)^T d_\square \\
3) & \text{Mixed-product} & \longrightarrow & = & \int_\square (B^\eta (B^\eta)^T) \otimes (B^\xi (B^\xi)^T) d_\square \\
4) & \text{Rectangular} & \longrightarrow & = & \left(\int_\eta B^\eta (B^\eta)^T d_\eta \right) \otimes \left(\int_\xi B^\xi (B^\xi)^T d_\xi \right) \\
 & & & & \qquad\quad \Downarrow \qquad\qquad\qquad\qquad\quad \Downarrow \\
5) & \text{Definition} & \longrightarrow & = & \qquad\quad {\color{blue}M^\eta} \qquad\quad \otimes \qquad\quad {\color{red}M^\xi}
\end{array}
$$

Efficient algorithm [2]

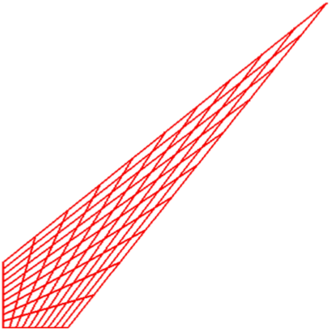
Linear system:

$$Mb = \mathcal{F}$$

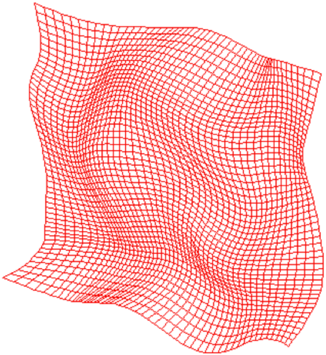


[2] Pereyra and Scherer, 1973, Math. Comp.

Testing domains



Stretched rectangle



Perturbed rectangle

Iteration steps for convergence [3]

h-scaling: $p = 4, c = 3$

N_{1D}	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Stretched	31	32	32	32	33	33	33	33
(Lumped Mass)	168	418	516	413	320	254	206	151
Perturbed	29	34	39	40	42	44	44	45
(Lumped Mass)	289	512	463	380	306	245	200	152

k-scaling: $N_{1D} = 2^9$, $c = p - 1$

p	1	2	3	4	5	6	7	8
Stretched	33	33	33	33	33	33	33	33
(Lumped Mass)	25	54	107	206	373	650	682	728
Perturbed	44	45	44	44	44	44	44	44
(Lumped Mass)	26	55	105	200	346	459	505	599

[3] 0 starting points. Stopping criteria: 10^{-12}

Iteration steps for convergence (Continued)

p-scaling: $N_{1D} = 2^7$, $c = 0$

p	1	2	3	4	5	6	7	8
Stretched	33	33	33	33	33	33	33	33
(Lumped mass)	31	86	200	431	766	798	932	828
Stretched	42	43	44	44	45	45	45	45
(Lumped mass)	31	86	196	406	748	742	930	1334

c-scaling: $N_{1D} = 2^7$, $p = 8$

c	7	6	5	4	3	2	1	0
Stretched	33	33	33	33	33	33	33	33
(Lumped mass)	4040	7664	2983	3990	4757	3449	6059	828
Stretched	42	44	44	44	44	45	45	45
(Lumped mass)	4246	7357	2312	4050	4327	3103	4884	1334

Does it pay off?

Computational cost:

Applying Lumped Mass preconditioner	:	$1 \cdot N$
Applying Direction Splitting preconditioner	:	$2(2p+1) \cdot N$
Multiplication:	:	$(2p+1)^2 \cdot N$

Additional cost ($c = p - 1$):

p	1	2	3	4	5	6	7	8
Percentage (%)	50	35	26	21	17	15	13	11

Acceleration technique 1: better preconditioner?

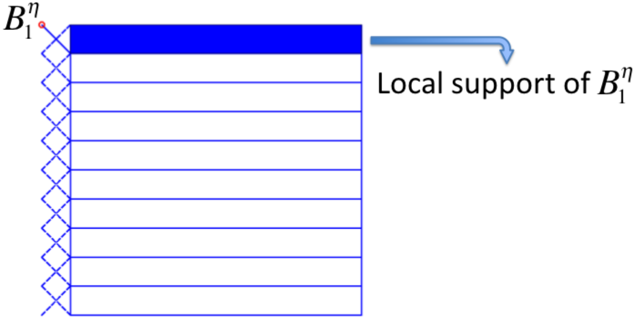
General idea: Local support of the basis functions

$$\int_{\square} (B_i^\eta B_k^\eta)(B_j^\xi B_\ell^\xi) J d_\square = \int_{\square} (B_i^\eta B_k^\eta)(B_j^\xi B_\ell^\xi) J_{\eta|_i} d_\square$$

$$\approx \left(\int_\eta B_i^\eta B_k^\eta d_\eta \right) \left(\frac{1}{L_i^\eta} \int_\square B_j^\xi B_\ell^\xi J_{\eta|_i} d_\square \right)$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

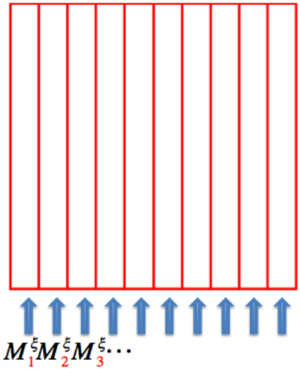
$$M^J \qquad M^\eta \qquad M_i^\xi$$

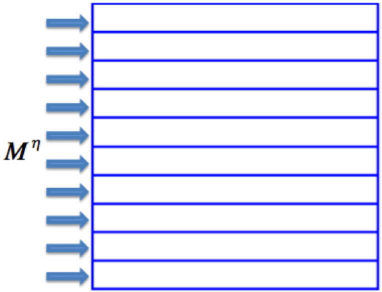


Efficient algorithm

Matrix form:

$$M^J \approx \begin{bmatrix} M_1^\xi & & \\ & \ddots & \\ & & M_{N_\eta}^\xi \end{bmatrix} \begin{bmatrix} M^\eta \otimes I^\xi \end{bmatrix}$$





Iteration steps for convergence

h-scaling: $p = 4$, $c = 3$

N_{1D}	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Stretched	10	8	6	6	5	4	4	3
(M^{-1})	31	32	32	32	33	33	33	33
Perturbed	21	16	12	10	8	7	6	5
(M^{-1})	29	34	39	40	42	44	44	45

k-scaling: $N_{1D} = 2^9$, $c = p - 1$

p	1	2	3	4	5	6	7	8
Stretched	3	4	4	4	4	4	4	4
(M^{-1})	33	33	33	33	33	33	33	33
Perturbed	5	5	6	6	6	7	7	7
(M^{-1})	44	45	44	44	44	44	44	44

Acceleration technique 2: hybrid

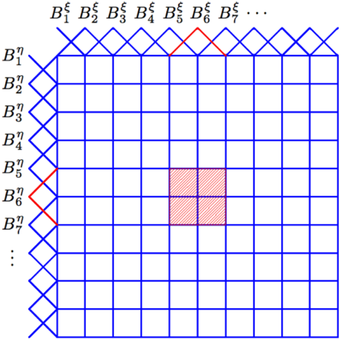
General idea: Local support of the basis functions

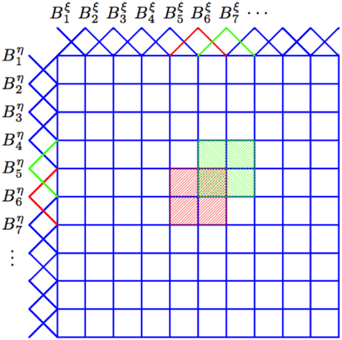
$$\int_{\square}(B_i^\eta B_j^\xi)(B_k^\eta B_\ell^\xi)J d_{\square} = \int_{\square}(B_i^\eta B_j^\xi)(B_k^\eta B_\ell^\xi)J_{(\eta|_i,\xi|_j)} d_{\square}$$

$$\approx \bar{J}_{ij}\int_{\square}(B_i^\eta B_j^\xi)(B_k^\eta B_\ell^\xi) d_{\square}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad \Downarrow$$

$$M^J \qquad\qquad D^J \qquad M^\eta \otimes M^\xi$$





Iteration steps for convergence

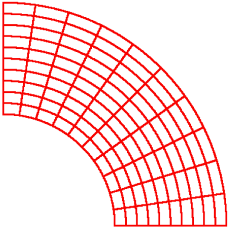
h-scaling: $p = 4, c = 3$

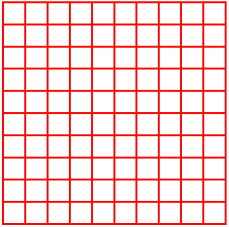
N_{1D}	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Stretched	11	9	7	6	5	4	4	3
(Local Support)	10	8	6	6	5	4	4	4
Perturbed	26	20	15	12	10	8	7	6
(Local Support)	21	16	12	10	8	7	6	5

k-scaling: $N_{1D} = 2^9, c = p - 1$

p	1	2	3	4	5	6	7	8
Stretched	3	4	4	4	4	4	4	4
(Local Support)	3	4	4	4	4	4	4	4
Perturbed	5	6	7	7	7	8	8	8
(Local Support)	5	5	6	6	6	7	7	7

Acceleration technique 3: better starting point?





Desired linear system:
$M^J b^J = \mathcal{F}^J$

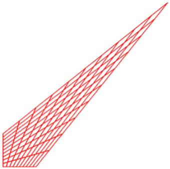
Approximate linear system:
$Mb = \mathcal{F}$

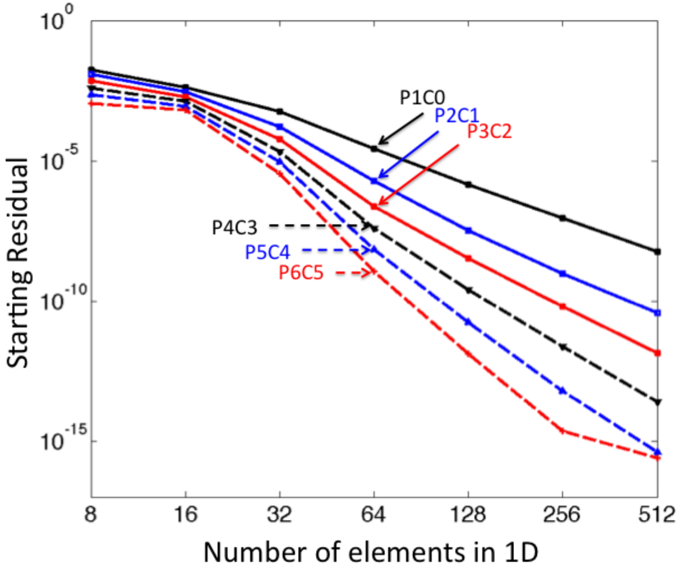
Corresponding minimization problems:

$$\arg\min_{\{b_i^J\}} \int_{\square} (f - \sum_{i=1}^{N} b_i^J B_i)^2 J d\square \quad \longleftrightarrow \quad M^J b^J = \mathcal{F}^J$$

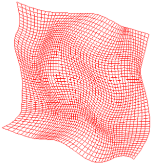
$$\arg\min_{\{b_i\}} \int_{\square} (f - \sum_{i=1}^{N} b_i B_i)^2 d\square \quad \longleftrightarrow \quad Mb = \mathcal{F}$$

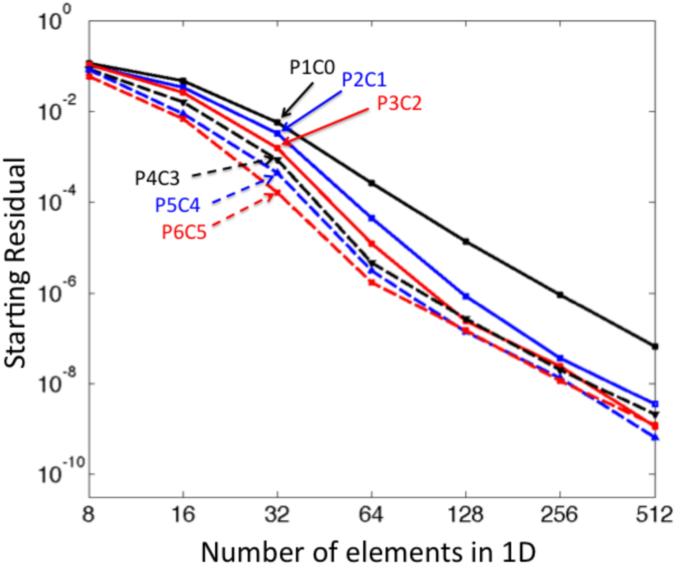
Streched rectangle





Perturbed rectangle





Iteration steps for convergence [4]

I.S.P. vs Z.S.P.: $p = 4, c = 3$

M^{-1}	N_{1D}	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Stretched	I.S.P.	28	20	8	4	2	1	0	0
	Z.S.P.	31	32	32	32	33	33	33	33
Perturbed	I.S.P.	27	31	29	23	19	16	12	8
	Z.S.P.	29	34	39	40	42	44	44	45

I.S.P. vs Z.S.P.: $p = 4, c = 3$

Local Support	N_{1D}	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Stretched	I.S.P.	8	5	4	2	1	1	0	0
	Z.S.P.	10	8	6	6	5	4	4	4
Perturbed	I.S.P.	19	15	9	6	4	3	2	1
	Z.S.P.	21	16	12	10	8	7	6	5

[4] I.S.P.: improved starting points; Z.S.P.: zero starting points.

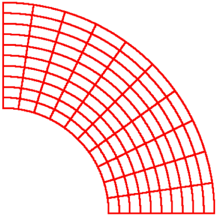
NURBS?

Linear system:

$$\left(\int_{\square} BB^T \frac{1}{W_N^2} d_{\square}\right) b = \left(\int_{\square} fB \frac{1}{W_N} d_{\square}\right)$$

Corresponding minimization problem:

$$\underset{\{b_i^w\}}{\arg\min} \int_{\square} (f\, W_N - \sum_{i=1}^{N} b_i^w B_i)^2 \frac{1}{W_N^2} d_{\square}.$$



$$\begin{aligned} J &= J^\eta \cdot J^\xi \\ J^\eta &= \sqrt{2}(1+\eta) \\ J^\xi &= \left(1 + (-2+\sqrt{2})\xi + (2-\sqrt{2})\xi^2\right)^{-1} \\ W_N &= 1 + (-2+\sqrt{2})\xi + (2-\sqrt{2})\xi^2 \end{aligned}$$

Question:

Where is the domain decomposition?

Question:

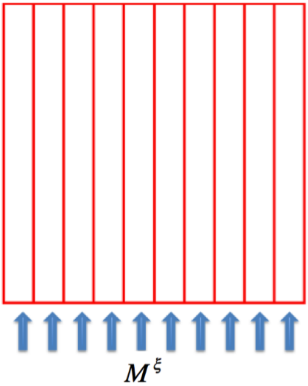
Where is the domain decomposition?

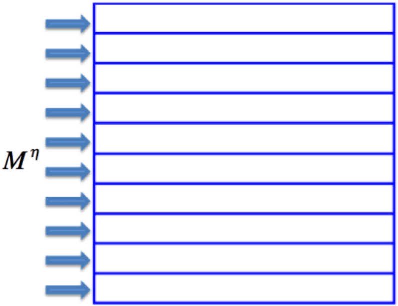
Answer:

In the future work.

Reasons:

- Global communication:

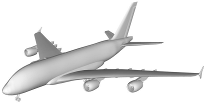




Reasons:

- Really complicated geometries:

1



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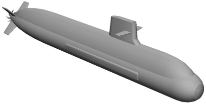

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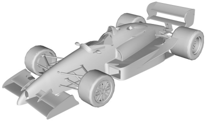

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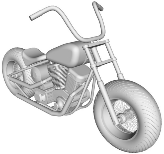

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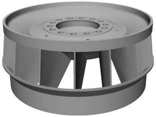

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1 - 6: http://www.3dcadbrowser.com/

Thank you :)