Auxiliary space multigrid method for elliptic problems with highly varying coefficients

Johannes Kraus[1] and Maria Lymbery[2]

1 Introduction

The robust preconditioning of linear systems of algebraic equations arising from discretizations of partial differential equations (PDE) is a fastly developing area of scientific research. In many applications these systems are very large, sparse and therefore it is vital to construct (quasi-)optimal iterative methods that converge independently of problem parameters.

The most established techniques to accomplish this objective are domain decomposition (DD), see, e.g., Toselli and Widlund [2005], Mathew [2008], and multigrid (MG)/algebraic multilevel iteration (AMLI) methods, see, e.g., Hackbusch [2003], Trottenberg et al. [2001], Vassilevski [2008].

As demonstrated by Klawonn et al. [2002], Toselli and Widlund [2005], Graham et al. [2007], two-level DD methods can be proven to be robust for scalar elliptic PDE with varying coefficient if the variations of the coefficient inside the coarse grid cells are assumed to be bounded. A key tool in the classical analysis of overlapping DD methods is the Poincaré inequality or its weighted analog as for problems with highly varying coefficients. It is well-known that the weighted Poincaré inequality holds only under certain conditions, e.g., in case of quasi-monotonic coefficients, see Sarkis [1994]. The concept of quasi-monotonic coefficients has been further developed in Pechstein and Scheichl [2008] for the convergence analysis of finite element tearing and interconnecting (FETI) methods.

Recently the robustness of DD methods has also been achieved for problems with general coefficient variations using coarse spaces that are constructed by solving local generalized eigenvalue problems, see, e.g., Efendiev et al. [2012], Galvis and Efendiev [2010], Spillane et al. [2014].

RICAM, Altenberger Str. 69, 4040 Linz, Austria johannes.kraus@oeaw.ac.at · IICT, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Bl. 25A, 1113 Sofia, Bulgaria mariq@parallel.bas.bg

In view of computational complexity, MG methods have been known to be most efficient as they have demonstrated optimality with respect to the problem size, see Hackbusch [2003], Vassilevski [2008]. Their design, however, needs careful adaptation for problems with large variations in the physical problem parameters. The AMLI framework contributes in achieving this goal, e.g. by providing more general polynomial acceleration techniques or Krylov cycles, see Axelsson and Vassilevski [1989, 1990, 1994], Kraus et al. [2012].

The idea of integrating domain decomposition techniques into multigrid methods can be found as early as in Kuznetsov [1989]. The method that is presented in the following combines DD and MG techniques with those from auxiliary space preconditioning, see Xu [1996]. It is related to substructuring methods like FETI, see Farhat and Roux [1991], and balancing domain decomposition (BDD) methods, see Mandel [1993].

The most advanced of these methods, BDDC (BDD based on constraints), see Dohrmann [2003], and FETI-DP (FETI dual-primal), see Farhat et al. [2001], can be formulated and analyzed in a common algebraic framework, see Mandel and Dohrmann [2003], Mandel et al. [2005], Mandel and Sousedík [2007]. The BDDC method enforces continuity across substructure interfaces by a certain averaging operator. The additional constraints can be interpreted as subspace corrections where coarse basis functions are subject to energy minimization. From this point of view the BDDC method has a high degree of similarity with the present approach.

However, contrary to BDDC, the auxiliary space multigrid (ASMG) method considered here naturally allows overlapping of subdomains and coarse degrees of freedom (DOF) are associated in general not only with the interfaces of subdomains but also with their interior. Moreover, the aim is to define a full multilevel method by recursive application of a two-level method. In contrary to standard (variational) multigrid algorithms coarse-grid correction is replaced by an auxiliary space correction. The coarse-grid operator then appears as the exact Schur complement of the auxiliary matrix and defines an additive approximation of the Schur complement of the original system, see Kraus [2006, 2012].

The purpose of the present paper is to summarize the main steps of the construction of the ASMG method recently proposed in Kraus et al. [2014] on a less technical level (Sections 2 and 4) and further to discuss its spectral properties and robustness with respect to highly varying coefficients (Section 3). The latter issue is also illustrated by numerical tests (Section 5).

2 Auxiliary space two-grid preconditioner

Consider the linear system of algebraic equations

$$A\mathbf{u} = \mathbf{f} \tag{1}$$

obtained after a finite element (FE) discretization of a partial differential equation (PDE) defined over a domain Ω, where A denotes the global stiffness matrix and $\mathbf{f}$ is a given right-hand side vector.

Consider a covering of Ω by n (overlapping) subdomains Ω_i, i.e., $\overline{\Omega} = \bigcup_{i=1}^{n} \overline{\Omega}_i$. Assume that for each subdomain Ω_i there is a symmetric positive semi-definite (SPSD) subdomain matrix A_i and that $A = \sum_{i=1}^{n} R_i^T A_i R_i$ where R_i restricts a global vector $\mathbf{v} \in V = \mathbb{R}^N$ to the local space $V_i = \mathbb{R}^{n_i}$ related to Ω_i. In practice the matrices A_i are assembled from scaled element matrices where the scaling factors account for the overlap of the subdomains. The DOF are split into two groups, coarse and fine, and the matrices A and A_i are partitioned accordingly into two-by-two blocks, where the lower right blocks (with index 22) are associated with coarse DOF, i.e.,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad A_i = \begin{bmatrix} A_{i:11} & A_{i:12} \\ A_{i:21} & A_{i:22} \end{bmatrix}, \quad i = 1, \ldots, n.$$

Introduce the following auxiliary domain decomposition matrix

$$\widetilde{A} = \begin{bmatrix} A_{1:11} & & & & A_{1:12}R_{1:2} \\ & A_{2:11} & & & A_{2:12}R_{2:2} \\ & & \ddots & & \vdots \\ & & & A_{n:11} & A_{n:12}R_{n:2} \\ R_{1:2}^T A_{1:21} & R_{2:2}^T A_{2:21} & \ldots & R_{n:2}^T A_{n:21} & \sum_{i=1}^{n} R_{i:2}^T A_{i:22} R_{i:2} \end{bmatrix}. \tag{2}$$

Denote $\widetilde{A}_{11} = \operatorname{diag}\{A_{1:11}, \ldots, A_{n:11}\}$, $\widetilde{A}_{22} = A_{22} = \sum_{i=1}^{n} R_{i:2}^T A_{i:22} R_{i:2}$, i.e., $\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}$. The matrices $A \in \mathbb{R}^{N \times N}$ and $\widetilde{A} \in \mathbb{R}^{\widetilde{N} \times \widetilde{N}}$ are related via

$$A = R\widetilde{A}R^T \text{ where } R = \begin{bmatrix} R_1 & 0 \\ 0 & I_2 \end{bmatrix}, \; R_1 = \begin{bmatrix} R_{1:1}^T & \ldots & R_{n,1}^T \end{bmatrix}, \; A_{11} = R_1 \widetilde{A}_{11} R_1^T.$$

Definition 1. (Kraus [2012]) The additive Schur complement approximation (ASCA) of $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is defined as the Schur complement Q of $\widetilde{A}$:

$$Q := \widetilde{A}_{22} - \widetilde{A}_{21}\widetilde{A}_{11}^{-1}\widetilde{A}_{12} = \sum_{i=1}^{n} R_{i:2}^T (A_{i:22} - A_{i:21}A_{i:11}^{-1}A_{i:12}) R_{i:2} \tag{3}$$

Next define a surjective mapping $\mathit{\Pi}_{\widetilde{D}} : \widetilde{V} \to V$ by

$$\mathit{\Pi}_{\widetilde{D}} = (R\widetilde{D}R^T)^{-1}R\widetilde{D}, \tag{4}$$

where $\widetilde{V} = \mathbb{R}^{\widetilde{N}}$ and $\widetilde{D}$ is a two-by-two block-diagonal SPD matrix.

The proposed auxiliary space two-grid preconditioner is defined by

$$B^{-1} := \overline{M}^{-1} + (I - M^{-T}A)C^{-1}(I - AM^{-1}) \tag{5}$$

where the operator M in (5) denotes an A-norm convergent smoother, i.e. $\|I - M^{-1}A\|_A \leq 1$, and $\overline{M} = M(M + M^T - A)^{-1}M^T$ is the corresponding symmetrized smoother. The matrix C defines a fictitious (auxiliary) space preconditioner approximating A and is given by

$$C^{-1} = \Pi_{\widetilde{D}} \widetilde{A}^{-1} \Pi_{\widetilde{D}}^T. \tag{6}$$

Denote $\Pi = (I - M^{-T}A)\Pi_{\widetilde{D}} = (I - M^{-T}A)(R\widetilde{D}R^T)^{-1}R\widetilde{D}$, then the preconditioner (5) can also be presented as

$$B^{-1} = \overline{M}^{-1} + \Pi \widetilde{A}^{-1} \Pi^T. \tag{7}$$

The proposed auxiliary space two-grid method differs from the classical two-grid methods in the replacement of the coarse grid correction step by a subspace correction with iteration matrix $I - C^{-1}A$.

3 Spectral properties and robustness

As it has been shown in Kraus et al. [2014] the condition number of the two-grid preconditioner defined in (7) satisfies the estimate

$$\kappa(B^{-1}A) \leq (\bar{c} + c_\Pi)(\underline{c} + \eta)/\underline{c},$$

where $\rho_A = \lambda_{\max}(A)$, c_Π is the constant in the estimate $\|\Pi\tilde{\mathbf{v}}\|_A^2 \leq c_\Pi \|\tilde{\mathbf{v}}\|_{\widetilde{A}}^2$ for all $\tilde{\mathbf{v}} \in \widetilde{V}$, and the constants $\bar{c}$, $\underline{c}$ and η characterize the smoother, i.e.,

$$\underline{c}\langle \mathbf{v}, \mathbf{v}\rangle \leq \rho_A \langle \overline{M}^{-1}\mathbf{v}, \mathbf{v}\rangle \leq \bar{c}\langle \mathbf{v}, \mathbf{v}\rangle \quad \text{and} \quad \|M^{-T}A\mathbf{v}\|^2 \leq \frac{\eta}{\rho_A}\|\mathbf{v}\|_A^2.$$

Moreover, the ASCA defined in (3) is spectrally equivalent to S, i.e. $Q \simeq S$:

Theorem 1. *(Kraus et al. [2014]) Denote $\pi_{\widetilde{D}} = R^T \Pi_{\widetilde{D}}$ where $\Pi_{\widetilde{D}}$ is defined as in (4) and $\widetilde{D}$ is an arbitrary two-by-two block-diagonal SPD matrix for the same fine-coarse partitioning of DOF as used in the construction of $\widetilde{A}$. Then $\langle A^{-1}\mathbf{u}, \mathbf{u}\rangle \leq \langle \Pi_{\widetilde{D}} \widetilde{A}^{-1} \Pi_{\widetilde{D}}^T \mathbf{u}, \mathbf{u}\rangle \leq c\, \langle A^{-1}\mathbf{u}, \mathbf{u}\rangle \ \ \forall \mathbf{u} \in V$ where $c := \|\pi_{\widetilde{D}}\|_{\widetilde{A}}^2$. Hence,*

$$\frac{1}{c}\langle S\mathbf{v}_2, \mathbf{v}_2\rangle \leq \langle Q\mathbf{v}_2, \mathbf{v}_2\rangle \leq \langle S\mathbf{v}_2, \mathbf{v}_2\rangle \quad \forall \mathbf{v}_2. \tag{8}$$

The upper bound in (8) is sharp, the lower bound is sharp for $\widetilde{D} = \begin{bmatrix} \widetilde{A}_{11} & 0 \\ 0 & I \end{bmatrix}$.

To verify that $\langle S\mathbf{v}_2, \mathbf{v}_2\rangle \leq c\langle Q\mathbf{v}_2, \mathbf{v}_2\rangle$ is robust with respect to an arbitrary variation of an elementwise constant coefficient $\alpha(\boldsymbol{x}) = \alpha_e$ for all $\boldsymbol{x} \in e$ and all elements e, see (15), one has to consider all possible distributions of $\{\alpha_e\}$ on the finest mesh. However, in the following we will show that the worst condition number (largest values of c) is obtained for a certain binary distribution of $\{\alpha_e\}$ so it suffices to study distributions of this type.

Let n_e denote the number of elements e and consider first an arbitrary distribution $\{\alpha_e\}$ of a piecewise constant coefficient where $\alpha_e \in (0, 1]$ for all e. Further, let A denote the global stiffness matrix corresponding to this distribution. Then there exists a set of binary distributions $\{\mathcal{C}_i : i = 1, 2, \ldots, n_e\}$ with $\mathcal{C}_i = \{\alpha_{e_j} : j = 1, 2, \ldots, n_e,\ \alpha_{e_j} = \beta_{e_i}$ if $j = i$ and $\alpha_{e_j} = \delta$ else$\}$ for some constants $0 < \delta \leq \beta_{e_i} \leq 1$ such that $A = \sum_{i=1}^{n_e} A_i$ where A_i is the global stiffness matrix corresponding to the distribution $\mathcal{C}_i$. It is easy to see that if A is SPD then A_i is SPD for all i. Now, let S_i denote the exact Schur complement of A_i and S be the Schur complement of A. Moreover, let Q_i denote the ASCA corresponding to A_i, i.e., $Q_i \simeq S_i$ where Q_i is the exact Schur complement of $\widetilde{A}_i$, cf. (2).

Lemma 1. *Using the above notation, assume that*

$$\frac{1}{c_j}\langle S_j\mathbf{v}_2, \mathbf{v}_2\rangle \leq \langle Q_j\mathbf{v}_2, \mathbf{v}_2\rangle \leq \langle S_j\mathbf{v}_2, \mathbf{v}_2\rangle \quad \forall \mathbf{v}_2 \text{ and } j = 1, \ldots, n_e. \tag{9}$$

Further, denote $c_{\max} = \max_{i\in\{1,\ldots,n_e\}}\{c_i\}$. *Then the following relations hold:*

$$\frac{1}{c_{\max}}\langle S\mathbf{v}_2, \mathbf{v}_2\rangle \leq \langle Q\mathbf{v}_2, \mathbf{v}_2\rangle \leq \langle S\mathbf{v}_2, \mathbf{v}_2\rangle \quad \forall \mathbf{v}_2. \tag{10}$$

Proof. The right inequality in (10) follows directly from the energy minimization property of Schur complements. In order to prove the left inequality we assume that (10) is wrong. Then there exists a vector $\mathbf{v}_2 \neq \mathbf{0}$ such that $\mathbf{v}_2^T S\mathbf{v}_2 \geq \bar{c}\,\mathbf{v}_2^T Q\mathbf{v}_2 > c_{\max}\mathbf{v}_2^T Q\mathbf{v}_2$, the left inequality of which can also be written in the form $\min_{\mathbf{v}_1} \begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix}^T A \begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix} \geq \bar{c}\,\mathbf{v}_2^T Q\mathbf{v}_2$, or, equivalently as $\min_{\mathbf{v}_1} \begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix}^T \left(\sum_{j=1}^{n_e} A_j\right) \begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix} \geq \bar{c}\,\min_{\tilde{\mathbf{v}}_1} \begin{pmatrix}\tilde{\mathbf{v}}_1\\ \mathbf{v}_2\end{pmatrix}^T \left(\sum_{j=1}^{n_e} \tilde{A}_j\right) \begin{pmatrix}\tilde{\mathbf{v}}_1\\ \mathbf{v}_2\end{pmatrix}$. From the latter inequality it follows that

$$\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix}^T \left(\sum_{j=1}^{n_e} A_j\right) \begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix} \geq \bar{c} \sum_{j=1}^{n_e} \min_{\tilde{\mathbf{v}}_1} \begin{pmatrix}\tilde{\mathbf{v}}_1\\ \mathbf{v}_2\end{pmatrix}^T \tilde{A}_j \begin{pmatrix}\tilde{\mathbf{v}}_1\\ \mathbf{v}_2\end{pmatrix} \quad \forall \mathbf{v}_1,$$

which is equivalent to

$$\sum_{j=1}^{n_e} \begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix}^T A_j \begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix} \geq \bar{c} \sum_{j=1}^{n_e} \mathbf{v}_2^T Q_j\mathbf{v}_2 \quad \forall \mathbf{v}_1. \tag{11}$$

Then, since all matrices A_j and Q_j are SPSD, it follows from (11) that there exists at least one index $j_0 \in \{1, 2, \ldots, n_e\}$ such that

$$\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}^T A_{j_0} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} \geq \bar{c}\, \mathbf{v}_2^T Q_{j_0} \mathbf{v}_2 \quad \forall \mathbf{v}_1.$$

Hence $\mathbf{v}_2^T S_{j_0} \mathbf{v}_2 = \min_{\mathbf{v}_1} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}^T A_{j_0} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} \geq \bar{c}\, \mathbf{v}_2^T Q_{j_0} \mathbf{v}_2$ which is in contradiction to (9) since $\bar{c} > c_{\max}$.

A crucial step in the application of the two-level preconditioner is the realization of the operator $\Pi_{\widetilde{D}}$. We propose two different variants that correspond to the following choices of $\widetilde{D}$:

[I] $\widetilde{D} = \mathrm{diag}(\widetilde{\mathrm{A}})$;

[II] $\widetilde{D} = \mathrm{blockdiag}(\widetilde{\mathrm{A}})$. The diagonal blocks are determined by the groups of fine DOF related to different macro structures whereas $\widetilde{D} = \mathrm{diag}(\widetilde{\mathrm{A}})$ in rows corresponding to coarse DOF.

In variant [I] the matrix $R\widetilde{D}R^T$ is diagonal, which makes the application of $\Pi_{\widetilde{D}}$ notably simple and cost-efficient. In case of variant [II] the action of $(R\widetilde{D}R^T)^{-1}$ can be implemented via an inner iterative method such as a preconditioned conjugate gradient (PCG) method, which then for reasons of efficiency requires a uniform preconditioner. A possible candidate is the one-level additive Schwarz (AS) preconditioner which however has to be adapted in order to be robust with respect to coefficient jumps. For this reason we study the scaled one-level AS preconditioner B_{AS} defined via

$$B_{\mathrm{AS}}^{-1} = SR\widetilde{S}^{-1}(\widetilde{S}\widetilde{D}\widetilde{S})^{-1}\widetilde{S}^{-1}R^T S \tag{12}$$

which can be applied to the scaled system with the matrix

$$D_s = SDS = SR\widetilde{D}R^T S,$$

where $S = [\mathrm{diag}(\mathrm{A})]^{-1/2}$, if the result is then rescaled. Let us further denote

$$\widetilde{D}_s = \widetilde{S}\widetilde{D}\widetilde{S} \;\text{ and }\; R_s = SR\widetilde{S}^{-1} \text{ where } \widetilde{S} = [\mathrm{diag}(\widetilde{\mathrm{A}})]^{-1/2}.$$

Then the following lemma holds:

Lemma 2. *The condition number of the preconditioned system using the scaled one-level AS preconditioner satisfies the estimate*

$$\kappa(B_{\mathrm{AS}}^{-1}D_s) \leq \kappa(\widetilde{D}_s). \tag{13}$$

Proof. First we show that $\lambda_{\min}(B_{\mathrm{AS}}^{-1}D_s) \geq 1$. Note that $D_s = R_s\widetilde{D}_s R_s^T$ and

$$R_s R_s^T = SR\widetilde{S}^{-1}\widetilde{S}^{-1}R^T S = [\mathrm{diag}(A)]^{-1/2} R\, [\mathrm{diag}(\widetilde{A})] R^T [\mathrm{diag}(A)]^{-1/2} = I.$$

Consider next the matrix

$$\begin{bmatrix} R_s\widetilde{D}_s R_s^T & I \\ I & R_s\widetilde{D}_s^{-1}R_s^T \end{bmatrix} = \begin{bmatrix} R_s & 0 \\ 0 & R_s \end{bmatrix} \begin{bmatrix} \widetilde{D}_s & I \\ I & \widetilde{D}_s^{-1} \end{bmatrix} \begin{bmatrix} R_s^T & 0 \\ 0 & R_s^T \end{bmatrix}$$

which is SPSD with an SPD pivot block $D_s = R_s\widetilde{D}_s R_s^T$. Consequently, its Schur complement is an SPSD matrix, i.e.

$$R_s\widetilde{D}_s^{-1}R_s^T - (R_s\widetilde{D}_s R_s^T)^{-1} \geq 0$$

which proves that $\lambda_{\min}(B_{\mathrm{AS}}^{-1}D_s) \geq 1$.

On the other hand we have

$$\begin{aligned} \lambda_{\max}(B_{\mathrm{AS}}^{-1}D_s) &= \lambda_{\max}(R_s\widetilde{D}_s^{-1}R_sD_s) \\ &= \lambda_{\max}(D_s^{1/2}R_s\widetilde{D}_s^{-1}R_sD_s^{1/2}) \\ &= \lambda_{\max}(\widetilde{D}_s^{-1/2}R_s^TD_sR_s\widetilde{D}_s^{-1/2}) \\ &\leq \lambda_{\max}(\widetilde{D}_s^{-1})\lambda_{\max}(R_s^TR_s\widetilde{D}_sR_s^TR_s) \\ &\leq \lambda_{\max}(\widetilde{D}_s^{-1})\lambda_{\max}(\widetilde{D}_s)\lambda_{\max}(R_s^TR_s) = \kappa(\widetilde{D}_s) \end{aligned}$$

which completes the proof.

Remark 1. For conforming FEM discretization of the second order scalar elliptic PDE it is not difficult to show that $\kappa(\widetilde{D}_s)$ is uniformly bounded with respect to jumps of an elementwise constant coefficient. Furthermore, $\widetilde{D}_s$ is block-diagonal with small-sized blocks and thus $\kappa(\widetilde{D}_s)$ is easily computable.

4 Auxiliary space multigrid method

Consider the exact block factorization of the sequence of auxiliary stiffness matrices $\widetilde{A}^k$, where the superscript $k = 0, 1, \ldots, \ell-1$ indicates the coarsening level:

$$\widetilde{A}^{(k)^{-1}} = \widetilde{L}^{(k)^T}\widetilde{D}^{(k)}\widetilde{L}^{(k)}, \quad A^{(k+1)} := Q^{(k)},$$

$$\widetilde{L}^{(k)} = \begin{bmatrix} I & & \\ -\widetilde{A}_{21}^{(k)}\ \widetilde{A}_{11}^{(k)^{-1}} & I \end{bmatrix}, \qquad \widetilde{D}^{(k)} = \begin{bmatrix} \widetilde{A}_{11}^{(k)^{-1}} & \\ & Q^{(k)^{-1}} \end{bmatrix}.$$

Let the algebraic multilevel iteration (AMLI)-cycle auxiliary space multigrid (ASMG) preconditioner $B^{(k)}$ be defined by (see Kraus et al. [2014]):

$$\begin{aligned} B^{(k)^{-1}} &:= \overline{M}^{(k)^{-1}} \\ &\quad +(I - M^{(k)^{-T}}A^{(k)})\Pi^{(k)}\ \widetilde{L}^{(k)^T}\overline{D}^{(k)}\widetilde{L}^{(k)}\Pi^{(k)^T}(I - A^{(k)}M^{(k)^{-1}}), \\ \overline{D}^{(k)} &:= \begin{bmatrix} \widetilde{A}_{11}^{(k)^{-1}} & \\ & B_\nu^{(k+1)} \end{bmatrix}, \qquad B_\nu^{(\ell)} := A^{(\ell)^{-1}}. \end{aligned}$$

In the nonlinear AMLI-cycle $B_\nu^{(k+1)} = B_\nu^{(k+1)}[\cdot]$ is a nonlinear mapping realized by ν iterations of a Krylov subspace method (e.g. the generalized conjugate gradient (GCG) method), thus employing the coarse level preconditioner $B^{(k+1)}$. In Kraus [2002] the convergence of the multiplicative nonlinear AMLI has been first analyzed, while Notay and Vassilevski [2008], Vassilevski [2008], Hu et al. [2013] have provided the multigrid framework along with a comparative analysis.

We want to stress the fact that the presented construction provides a framework for both linear and nonlinear AMLI cycle multigrid as well as classical multigrid methods.

5 Numerical Results

Subject to numerical testing is the scalar elliptic boundary-value problem

$$-\nabla \cdot (\boldsymbol{k}(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x}) \quad in \ \ \Omega, \tag{14a}$$

$$u = 0 \quad on \ \ \Gamma. \tag{14b}$$

Here Ω is a polygonal domain in $\mathbb{R}^2$, f is a given function in $L_2(\Omega)$ and

$$\boldsymbol{k}(\boldsymbol{x}) = \alpha(\boldsymbol{x})I = \alpha_e I. \tag{15}$$

Upon the entire boundary of the domain Dirichlet boundary conditions have been imposed as other boundary conditions would not qualitatively affect the numerical results.

Piecewise bilinear functions have been used in the process of discretization of (14) leading to the linear system of algebraic equations (1). A uniform mesh consisting of $N \times N$ elements (squares) is considered where $N = 2^{\ell+2}$, $\ell = 1, \ldots, 7$, and the covering is assumed to consist of subdomains composed of 8×8 elements that overlap with half of their width or height. The mesh hierarchy is such that the coarsest mesh corresponds to $\ell = 1$ and is composed of $2^{1+2} \times 2^{1+2} = 64$ elements whereas the finest mesh is obtained by performing $\ell - 1 = 1, \ldots, 6$ steps of uniform mesh refinement.

The vector of all zeros was chosen to be the right hand side $\mathbf{f}$ in (1) while the outer iteration was initialized with a random vector. Three representative coefficient configurations are considered (on the respective finest mesh):

[0] log-uniformly distributed coefficient $\alpha_e = 10^{p_{rand}}$ where α_e is constant on each element e and $p_{rand} \in (0, q]$;
[1] inclusions with coefficient $\alpha_\iota = 10^{p_{rand}}$ against a background as in [0] where α_ι is constant on every inclusion ι and $p_{rand} \in (0, q]$, see Fig. 2(a);
[2] stiff inclusions with coefficient $\alpha_\iota = 10^q$ against a background as in [0], see Fig. 2(b).

In Table 1 we compare the condition numbers

$$\kappa(\widetilde{D}_s) = \kappa(S\widetilde{D}S), \quad \kappa(B_{\mathrm{AS}}^{-1}D_s) = \kappa(SR\widetilde{S}^{-2}\widetilde{D}^{-1}\widetilde{S}^{-2}R^TS(SR\widetilde{D}R^TS)),$$

with that of the corresponding unscaled preconditioned system

$$\kappa(R\widetilde{D}^{-1}R^T(R\widetilde{D}R^T))$$

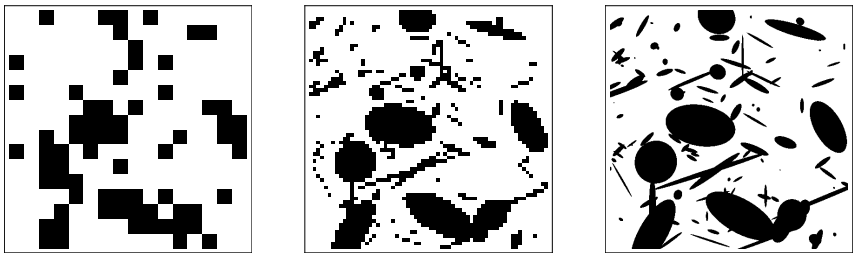
for the coefficient distribution [0] on three different meshes with mesh size $h \in \{1/16, 1/32, 1/64\}$ and varying contrast q. The obtained numerical results are in accordance with Lemma 2; They further show that the scaled one-level additive Schwarz method yields a uniform preconditioner whereas its unscaled analog suffers from high-contrast coefficients.

Next, the numerical performance of the nonlinear (AMLI)-cycle ASMG method (V-cycle and W-cycle) utilizing the preconditioner B_{AS} is tested for:

(P1) Problem (14) with coefficient distributions [1] and variants [I] and [II] of $\mathit{\Pi}_{\widetilde{D}}$. Variant [II] is realized by 10 inner PCG iterations with the scaled one-level AS preconditioner.

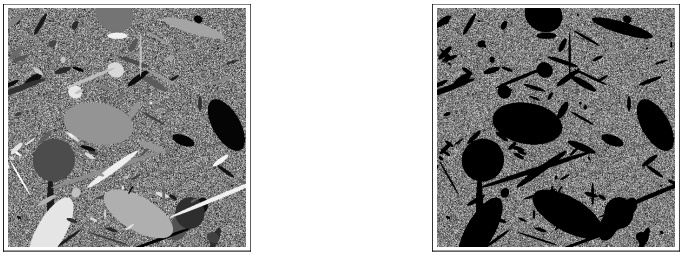
(P2) Same as Problem (P1) but for coefficient distribution 2.

A comparison between variant [I] and variant [II] of the ℓ-level V-cycle and W-cycle is presented in Tables 2–3. Pre- and post-smoothing is performed by



(a) 16×16 mesh (b) 64×64 mesh (c) 512×512 mesh

Fig. 1 Inclusions resolved on different fine scales (meshes)



(a) Coefficient for Problem (P1) on 512×512 mesh (b) Coefficient for Problem (P2) on 512×512 mesh

Fig. 2 Random and stiff inclusions against random background $\alpha_e = 10^{p_{rand}}$

Table 1 Condition numbers of AS-preconditioned systems versus $\kappa(\widetilde{D}_s)$

	unscaled AS method			scaled AS method			$\kappa(\widetilde{D}_s)$		
q \ h	1/16	1/32	1/64	1/16	1/32	1/64	1/16	1/32	1/64
1	9.76×10^1	9.47×10^1	9.35×10^1	1.25	1.26	1.26	4.73	4.73	4.73
2	2.25×10^2	3.69×10^2	5.89×10^2	1.28	1.27	1.29	4.73	4.73	4.73
3	6.93×10^2	2.42×10^3	3.70×10^3	1.29	1.32	1.33	4.73	4.73	4.73
4	1.93×10^4	1.97×10^4	3.77×10^4	1.33	1.33	1.33	4.73	4.73	4.73
5	1.78×10^5	1.87×10^5	2.16×10^5	1.32	1.33	1.33	4.73	4.73	4.73
6	3.07×10^5	1.34×10^6	2.15×10^6	1.33	1.33	1.33	4.73	4.73	4.73

one symmetric point Gauss-Seidel iteration on each level except the coarsest one where all linear systems are solved directly.

Table 2 Number of iterations for residual reduction by 10^6

Problem (P1)																								
	Nonlinear AMLI V-cycle												Nonlinear AMLI W-cycle											
	[I]						[II]						[I]						[II]					
q \ ℓ	2	3	4	5	6	7	2	3	4	5	6	7	2	3	4	5	6	7	2	3	4	5	6	7
1	4	5	6	6	7	8	5	5	6	6	7	8	4	5	5	5	5	5	5	5	5	5	5	5
2	5	5	6	6	7	8	5	5	6	6	7	8	5	5	5	5	5	5	5	5	5	5	5	5
3	5	6	6	7	7	8	5	6	6	7	7	8	5	6	6	6	6	6	5	5	5	5	5	5
4	5	6	7	8	8	9	5	6	7	8	8	8	5	6	6	6	6	6	5	6	6	6	6	6
5	5	7	7	8	9	9	5	6	7	8	8	8	5	6	6	6	7	7	5	6	6	6	6	6
6	5	7	8	9	13	15	5	7	8	8	8	9	5	6	6	7	9	10	5	6	6	6	6	6

Table 3 Number of iterations for residual reduction by 10^6

Problem (P2)																								
	Nonlinear AMLI V-cycle												Nonlinear AMLI W-cycle											
	[I]						[II]						[I]						[II]					
q \ ℓ	2	3	4	5	6	7	2	3	4	5	6	7	2	3	4	5	6	7	2	3	4	5	6	7
1	5	5	6	6	7	8	5	5	6	6	7	8	5	5	5	5	5	5	5	5	5	5	5	5
2	5	5	6	6	7	8	5	5	6	6	7	8	5	5	5	5	5	5	5	5	5	5	5	5
3	5	5	6	6	7	8	5	5	6	6	7	8	5	5	5	6	5	6	5	5	5	5	5	5
4	5	6	6	7	7	8	5	5	6	7	8	8	5	5	6	6	6	6	5	6	5	5	5	6
5	5	6	7	7	9	9	5	6	7	7	8	8	5	6	6	6	6	6	5	6	6	6	6	6
6	5	6	8	8	12	13	5	6	7	8	9	9	5	6	6	6	8	9	5	6	6	6	6	6

The obtained results demonstrate that the choice of $\widetilde{D}$ and consequently of the surjective mapping $\Pi_{\widetilde{D}}$ affect the performance of the nonlinear AMLI-cycle ASMG method crucially. As for variant [I] the number of ASMG iterations required to achieve the prescribed accuracy increases with the contrast, variant [II] shows full robustness.

References

O. Axelsson and P. Vassilevski. Algebraic multilevel preconditioning methods I. *Numer. Math.*, 56(2-3):157–177, 1989.

O. Axelsson and P. Vassilevski. Algebraic multilevel preconditioning methods II. *SIAM J. Numer. Anal.*, 27(6):1569–1590, 1990.

O. Axelsson and P. Vassilevski. Variable-step multilevel preconditioning methods, I: Self-adjoint and positive definite elliptic problems. *Numer. Linear Algebra Appl.*, 1:75–101, 1994.

C.R. Dohrmann. A preconditioner for substructuring based on constrained energy minimization. *SIAM J. Sci. Comput.*, 25(1):246–258, 2003.

Y. Efendiev, J. Galvis, R. Lazarov, and J. Willems. Robust domain decomposition preconditioners for abstract symmetric positive definite bilinear forms. *ESAIM: Mathematical Modelling and Numerical Analysis*, 46(05): 1175–1199, 2012.

C. Farhat and F.X. Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Int. J. Numer. Methods Engrg.*, 32:1205–1227, 1991.

C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, and D. Rixen. FETI-DP: A dual-primal unified FETI method. I. A faster alternative to the two-level FETI method. *Int. J. Numer. Methods Engrg.*, 50(7):1523–1544, 2001.

J. Galvis and Y. Efendiev. Domain decomposition preconditioners for multiscale flows in high-contrast media. *Multiscale Model. Simul.*, 8(4):1461–1483, 2010.

I.G. Graham, P.O. Lechner, and R. Scheichl. Domain decomposition for multiscale PDEs. *Numer. Math.*, 106(4):489–626, 2007.

W. Hackbusch. *Multi-Grid Methods and Applications*. Springer, Berlin Heidelberg, 2003.

X. Hu, P. Vassilevski, and J. Xu. Comparative convergence analysis of nonlinear AMLI-cycle multigrid. *SIAM J. Numer. Anal.*, 51(2):1349–1369, 2013.

A. Klawonn, O. Widlund, and M. Dryja. Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients. *SIAM J. Numer. Anal.*, 40(1):159–179, 2002.

J. Kraus. An algebraic preconditioning method for M-matrices: linear versus non-linear multilevel iteration. *Numer. Linear Algebra Appl.*, 9:599–618, 2002.

J. Kraus. Algebraic multilevel preconditioning of finite element matrices using local Schur complements. *Numer. Linear Algebra Appl.*, 13:49–70, 2006.

J. Kraus. Additive Schur complement approximation and application to multilevel preconditioning. *SIAM J. Sci. Comput.*, 34:A2872–A2895, 2012.

J. Kraus, P. Vassilevski, and L. Zikatanov. Polynomial of best uniform approximation to 1/x and smoothing for two-level methods. *Comput. Meth. Appl. Math.*, 12:448–468, 2012.

J. Kraus, M. Lymbery, and S. Margenov. Auxiliary space multigrid method based on additive Schur complement approximation. *Numer. Linear Algebra Appl.*, 2014. Online (wileyonlinelibrary.com). DOI: 10.1002/nla.1959.

Y. Kuznetsov. Algebraic multigrid domain decomposition methods. *Sov. J. Numer. Anal. Math. Modelling.*, 4(5):351–379, 1989.

J. Mandel. Balancing domain decomposition. *Comm. Numer. Methods Engrg.*, 9(3):233–241, 1993.

J. Mandel and C. R. Dohrmann. Convergence of a balancing domain decomposition by constraints and energy minimization. *Numer. Linear Algebra Appl.*, 10(7):639–659, 2003.

J. Mandel and B. Sousedík. Adaptive selection of face coarse degrees of freedom in the BDDC and FETI-DP iterative substructuring methods. *Comput. Methods Appl. Mech. Engrg.*, 196(8):1389–1399, 2007.

J. Mandel, C. R. Dohrmann, and R. Tezaur. An algebraic theory for primal and dual substructuring methods by constraints. *Appl. Numer. Math.*, 54 (2):167–193, 2005.

T.P.A. Mathew. *Domain Decomposition Methods for the Numerical Solution of Partial Differential Equations*. Springer, Berlin Heidelberg, 2008.

Y. Notay and P. Vassilevski. Recursive Krylov-based multigrid cycles. *Numer. Linear Algebra Appl.*, 15:473–487, 2008.

C. Pechstein and R. Scheichl. Analysis of FETI methods for multiscale PDEs. *Numer. Math.*, 111(2):293–333, 2008.

M. Sarkis. *Schwarz Preconditioners for Elliptic Problems with Discontinuous Coefficients Using Conforming and Non-Conforming Elements*. PhD thesis, Courant Institute, New York University, 1994.

N. Spillane, V. Dolean, P. Hauret, F. Nataf, C. Pechstein, and R. Scheichl. Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps. *Numer. Math.*, 126(4):741–770, 2014.

A. Toselli and O. Widlund. *Domain Decomposition Methods–Algorithms and Theory*. Springer, Berlin Heidelberg, 2005.

U. Trottenberg, C.W. Oosterlee, and A. Schüller. *Multigrid*. Academic Press Inc., San Diego, CA, 2001.

P. Vassilevski. *Multilevel Block Factorization Preconditioners: Matrix-based Analysis and Algorithms for Solving Finite Element Equations*. Springer, New York, 2008.

J. Xu. The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids. *Computing*, 56:215–235, 1996.