Schwarz Waveform Relaxation for A Class of Non-Dissipative Problems

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In this paper, we introduce the results for the Schwarz waveform relaxation (SWR) algorithm applied to a class of non-dissipative reaction diffusion equations. Both the Dirichlet and Robin transmission conditions (TCs) are considered. For the Dirichlet TC, we consider the algorithm for the nonlinear problem $\partial_t u = \mu\partial_{xx}u + f(u)$, in the case of many subdomains. For the Robin TC, we consider the linear problem $\partial_t u = \mu\partial_{xx}u + au$ with $a \geq 0$. We focus on the analysis of finding the optimal parameter involved in the Robin TC. For small overlap size $l = \mathcal{O}(\Delta x)$ and $\Delta t = \mathcal{O}(\Delta x^r)$ with $r < \frac{4}{3}$, we show that the *equioscillation* principle which works for $a < 0$ does not hold for $a \geq 0$. We show numerical results to support our theoretical conclusions.

1 Introduction

We are concerned with the SWR algorithm to compute solutions $u = u(x,t)$: $(0,L) \times (0,T) \to \mathbb{R}$ of the following problem

$$\partial_t u = \mu\partial_{xx}u + f(u), \quad (x,t) \in (0,L) \times (0,T), \tag{1}$$

where $\mu > 0$ and $f \in \mathbb{C}^1(\mathbb{R})$ denotes a function which in general depends nonlinearly on u. For the case of two subdomains, Gander [1998] proved that the classical SWR algorithm converges linearly on unbounded time intervals, if $f'(u)$ satisfies $f'(u) < \left(\sqrt{\mu}\pi/L\right)^2$. For the case $f'(u) \leq 0$, the analysis

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by Gander and Stuart [1998] and Wu et al. [2012] can be used to prove convergence for the classical SWR algorithm in the case of N subdomains ($N \geq 3$). However, there are no results for the case $N \geq 3$ and $f'(u) < d$ with $d > 0$. In this paper, we show that the classical SWR algorithm is convergent for $N \geq 3$, provided $f'(u) \leq \left(\omega^* \sqrt{\mu}\pi/L\right)^2$, where $\omega^* \in (0,1)$ depends on N.

For the purpose of fast convergence, one should use the Robin TC for the SWR algorithm, instead of the Dirichlet TC. For the linear model problem

$$\partial_t u = \mu \partial_{xx} u + au, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{2}$$

a key step for the convergence analysis is to solve a special min-max problem, whose solution corresponds to the best choice of the parameter p involved in the Robin TC. For $a < 0$, the optimization procedure has been deeply analyzed by Gander and Halpern [2007] in the 1-D case, and by Bennequin et al. [2009] in the 2-D case. Other related work also requires $a < 0$; see, e.g., Halpern [2006]. For the case $a > 0$, the existing research always employs a variable transform, like $v(x,t) = e^{-\sigma t} u(x,t)$, and then the original equation is transformed to $\partial_t v = \mu \partial_{xx} v + (a - \sigma) v$ with negative coefficient, $a - \sigma < 0$. However, this trick is not advisable for practical computing. Roughly speaking, for σ large, we find numerically that even though $\max_j \|v_j^k - v\|_{L^\infty([0,T]\times\Omega_j)}$ is very small, $\max_j \|u_j^k - u\|_{L^\infty([0,T]\times\Omega_j)}$ is still a huge quantity, where j is the subdomain index, v_j^k denotes the k-th iterate of the optimized SWR algorithm applied to the transformed problem and u_j^k is obtained from the inverse transform $u_j^k = e^{\sigma t} v_j^k$ (see Fig.3).

The parameter obtained for the linear problem (2) serves the optimized SWR algorithm for the nonlinear problem (1), by using the *'linearization'* idea introduced by Caetano et al. [2010]. For the nonlinear problem (1) with $f'(u) \geq 0$, we first need to know the optimal parameter for (2) with $a \geq 0$ and to the best of our knowledge there are no results up to now. Here, we introduce our analysis of finding the best parameter for the Robin TC in the case $a \geq 0$. We show that, for overlap size $l = \mathcal{O}(\Delta x)$ small and $\Delta t = \mathcal{O}(\Delta x^r)$, the equioscillation principle established recently by Bennequin et al. [2009] still holds, when $r \geq \frac{4}{3}$. But for $r < \frac{4}{3}$, this principle does not hold.

2 Main Results

In this section, we present the main results about the classical and optimized SWR algorithms. The concrete proof of the four propositions are given in our forthcoming paper (Wu [2014]).

2.1 Dirichlet Transmission Condition

The nonlinear IVP consists of the governing equation (1) and the initial and boundary conditions

$$u(x,0) = u_0(x), x \in [0,L]; \quad u(0,t) = g_1(t), u(L,t) = g_2(t), t > 0. \tag{3}$$

The domain $[0,L]$ is decomposed into N subdomains: $\Omega_j = [\alpha_j L, \beta_j L], j = 1,2,\ldots,N$, where $\alpha_1 = 0, \beta_N = 1$ and $0 < \alpha_{j+1} < \beta_j < 1$ for $j = 1,2,\ldots,N-1$. We assume that $\beta_j < \alpha_{j+2}$ so that all the subdomains overlap but domains which are not adjacent do not overlap. Then, the N-subdomain SWR algorithm with Dirichlet TC for the IVP (1) and (3) is

$$\begin{cases} \frac{\partial u_j^k(x,t)}{\partial t} = \mu \frac{\partial^2 u_j^k(x,t)}{\partial x^2} + f(u_j^k(x,t)), & (x,t) \in \Omega_j \times \mathbb{R}^+, \\ u_j^k(\alpha_j L, t) = u_{j-1}^{k-1}(\alpha_j L, t), u_j^k(\beta_j L, t) = u_{j+1}^{k-1}(\beta_j L, t), & t \in \mathbb{R}^+, \end{cases}$$

where k is the iteration index, $u_j^k(x,0) = u_0(x)$ for $x \in \Omega_j$, $\alpha_0 = \beta_0 = 0$, $\alpha_{N+1} = \beta_{N+1} = 1$, $u_0^k = g_1(t)$ and $u_{N+1}^k = g_2(t)$ for all $k \geq 0$. We assume that the overlapping domains and the subdomains are all of the same sizes.

Proposition 1. *Let l be the overlap size, N be the number of subdomains, $\phi = \frac{l}{L}\pi$ and $\varphi = \frac{L-l}{NL}\pi$. Then, suppose the function f in* (1) *satisfies $f'(u) \leq \left(\frac{\sqrt{\mu}\pi}{L}\omega^*\right)^2$ ($\forall u \in \mathbb{R}$), the classical SWR algorithm with $N \geq 2$ is convergent. Here, $\omega^* \in (0,1)$ is the unique solution of $\mathbf{r}(\omega) = 1$ and $\mathbf{r}(\omega)$ is defined by*

$$\mathbf{r}(\omega) = \frac{\min\{1, N-2\}\sin^2(\phi\omega) + \sin^2(\varphi\omega) + 2\cos\left(\frac{\pi}{N}\right)\sin(\phi\omega)\sin(\varphi\omega)}{\sin^2((\phi+\varphi)\omega)}.$$

2.2 Robin Transmission Condition

For the initial value problem (2) with $a > 0$, we decompose the spatial domain $\mathbb{R}$ into two subdomains $\Omega_1 = (-\infty, l]$ and $\Omega_2 = [0, +\infty)$, where $l \geq 0$. The SWR algorithm with Robin TC is given by

$$\begin{cases} \partial_t u_j^k = \mu \partial_{xx} u_j^k + a u_j^k, \quad x \in \Omega_j, \\ (\partial_x + (-1)^{3-j}p) u_j^k((2-j)l, t) = (\partial_x + (-1)^{3-j}p) u_{3-j}^{k-1} j((2-j)l, t), \end{cases}$$

where $j = 1,2$, $u_j^k(x,0) = u_0(x)$, k is the iteration index and p is a free parameter. Based on Laplace transform and maximum principle of analytic functions, we obtain the following results.

Proposition 2 (Overlapping case $l > 0$). *Let $l > 0$ and $a \geq 0$. Then, the best performance of the SWR algorithm with Robin TC is obtained for*

$p = p_{opt} = \frac{q_{opt}}{2l}$. *The argument* q_{opt} *is solution of the min-max problem*

$$\min_{q>0} \max_{y\in[y_0,y_1]} \mathcal{R}(y,q), \text{ with } \mathcal{R}(y,q) = \frac{(y-q)^2+y^2+z_0^2}{(y+q)^2+y^2+z_0^2}e^{-y}, z_0 = 2l\sqrt{\frac{a}{\mu}}. \quad (4)$$

where $y_j = 2l\sqrt{\left(\sqrt{a^2+(\pi/[j\Delta t+(1-j)T])^2}-a\right)/(2\mu)}$, $j = 0, 1$. *Define* $q_{\min} = \sqrt{2y_0^2+z_0^2}$, $q_{\max} = \sqrt{2y_1^2+z_0^2}$, $\mathcal{R}^\dagger(q) = \max\{\mathcal{R}(y_0,q), \mathcal{R}(y_1,q)\}$, $\tilde{q}_{\min} = \max\left\{q_1(z_0), q_{\min}, \frac{q_{\min}^2}{2}\right\}$, $\tilde{q}_{\max} = \min\left\{q_2(z_0), q_{\max}, \frac{q_{\max}^2}{2}\right\}$ *and*

$$q^\dagger = \begin{cases} q_{\min}, & \text{if } \mathcal{R}(y_0,q_{\min}) \geq \mathcal{R}(y_1,q_{\min}), \\ q_{\max}, & \text{if } \mathcal{R}(y_1,q_{\max}) \geq \mathcal{R}(y_0,q_{\max}), \\ q_0^\dagger, & \text{otherwise}, \end{cases} \quad (5)$$

where $q_0^\dagger \in (q_{\min}, q_{\max})$ *is the unique root of* $\mathcal{R}(y_0,q) = \mathcal{R}(y_1,q)$, *and* $q_1(z_0)$ *and* $q_2(z_0)$ *are two different positive roots of the cubic polynomial* $S(q,z_0) = q^3+4q^2-2q(2-z_0^2)+8z_0^2$ *for* $z_0 \in (0, z_0^*)$ *with* $z_0^* = 0.31920496942508$. *Then, the solution of the min-max problem* (4) *is given by*

$$q_{opt} = \begin{cases} q_{\max}, & \text{if } \mathcal{R}^\dagger(\tilde{q}_{\max}) < \bar{\mathcal{R}}(\tilde{q}_{\max}), \\ q_0^*, & \text{if } \mathcal{R}^\dagger(\tilde{q}_{\max}) \geq \bar{\mathcal{R}}(\tilde{q}_{\max}), \end{cases} \quad (6)$$

provided $z_0 < z_0^*$, $\tilde{q}_{\min} < \tilde{q}_{\max}$, $q^\dagger \in [\tilde{q}_{\min}, \tilde{q}_{\max}]$ *and* $\mathcal{R}^\dagger(q^\dagger) < \bar{\mathcal{R}}(q^\dagger)$, *where* $q_0^* \in [q^\dagger, q_{\max}]$ *is the unique root of* $\bar{\mathcal{R}}(q) = \mathcal{R}(y_0,q)$; *otherwise* $q_{opt} = q^\dagger$. *Here,* $\bar{\mathcal{R}}(q) = \mathcal{R}(\bar{y}(q),q)$ *and* $\bar{y}(q) = \sqrt{\frac{2q-z_0^2+\sqrt{-qS(q,z_0)}}{2}}$.

Proposition 3 (Non-overlapping case: $l = 0$). *For* $l = 0$ *and* $a \geq 0$, *the best parameter* p_{opt} *for the Robin TC is given by*

$$p_{opt} = \begin{cases} \sqrt{z_{\min}^2+a_0}, & \text{if } \mathcal{R}_0(z_{\min}, \sqrt{z_{\min}^2+a_0}) \geq \mathcal{R}_0(z_{\max}, \sqrt{z_{\min}^2+a_0}), \\ \sqrt{z_{\max}^2+a_0}, & \text{if } \mathcal{R}_0(z_{\max}, \sqrt{z_{\max}^2+a_0}) \geq \mathcal{R}_0(z_{\min}, \sqrt{z_{\max}^2+a_0}), \\ p_0^*, & \text{otherwise}, \end{cases} \quad (7)$$

where $a_0 = \frac{a}{\mu}$ *and* p_0^* *is the unique root of* $\mathcal{R}_0(z_{\min},p) = \mathcal{R}_0(z_{\max},p)$.

Proposition 4 (Asymptotic properties). *Let* $\Delta t = C\Delta x^r$ *with some positive constants* C *and* r. *Then, for* Δx *small and fixed length of the time interval, the convergence factor* ρ_{Robin} *of the SWR algorithms with Robin TC satisfies the following asymptotic properties:*

$$l = 0 : \rho_{Robin} \approx 1-\mathcal{O}(\Delta t^{\frac{1}{4}}); \; l = C_l\Delta x : \rho_{Robin} \approx \begin{cases} 1-\mathcal{O}(\Delta x^{\frac{r}{4}}), & \text{if } r \leq \frac{4}{3}, \\ 1-\mathcal{O}(\Delta x^{\frac{1}{3}}), & \text{otherwise}. \end{cases}$$

For T sufficiently large and fixed Δx, we have the asymptotic properties:

$$\rho_{Robin} \approx \begin{cases} 1-\mathcal{O}(T^{-1}), & \text{if } l \geq 0 \text{ and } a > 0, \\ 1-\mathcal{O}(T^{-\frac{1}{6}}), & \text{if } l > 0 \text{ and } a = 0, \\ 1-\mathcal{O}(T^{-\frac{1}{4}}), & \text{if } l = 0 \text{ and } a = 0. \end{cases}$$

Remark 1. For the initial value problem (2) with $a < 0$, the min-max problem concerning the best choice of the parameter is

$$\min_{q>0} \max_{y\in[y_0,y_1]} \mathcal{R}(y,q), \text{ with } \mathcal{R}(y,q) = \frac{(y-q)^2 + y^2 - y_0^2}{(y+q)^2 + y^2 - y_0^2} e^{-y},$$

where $y_0 = 2l\sqrt{\left(\sqrt{a^2+(\pi/T)^2} - a\right)/(2\mu)}$. We see that, this min-max problem is different from the one given by (4). For $a < 0$, Δx small and $\Delta t = \mathcal{O}(\Delta x^r)$, the solution q_{opt} is determined by the *equioscillation* principle (Gander and Halpern [2007]); an illustration is shown in Fig. 1 on the left. However, this principle does not always hold for the case $a \geq 0$; in particular, we have shown that for $\Delta t = \mathcal{O}(\Delta x^r)$ with $r < \frac{4}{3}$, it does not hold (Wu [2014]). A concrete example is shown in Fig. 1 on the right, where we see that, based on the optimal parameter q_{opt}, the local maximum of the objective function $\mathcal{R}$ defined by (4) is smaller than $\mathcal{R}(y_0, q_{opt})$.

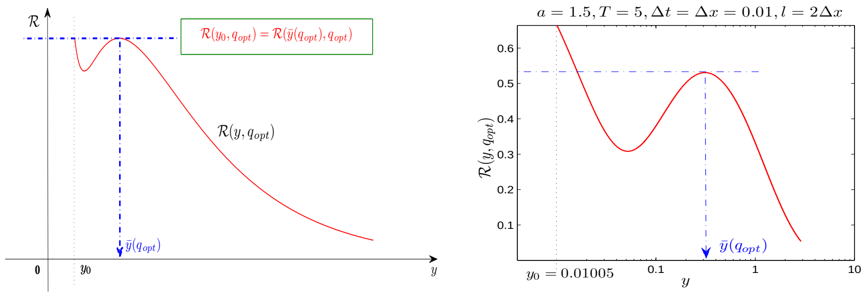

Fig. 1 Left: illustration of the *equioscillation* principle for the case $a < 0$. Right: an example for $a \geq 0$ and $\Delta t = \mathcal{O}(\Delta x^r)$ with $r < \frac{4}{3}$, where the *equioscillation* does not hold.

3 Numerical results

We consider the following linear problem with homogeneous initial and boundary conditions:

$$u_t = u_{xx} + au + t^2 \sin(xt), \;\; (x,t) \in (0,4) \times (0,T), \tag{8}$$

The Laplace operator ∂_{xx} is treated by the centered finite difference scheme and then the derived system of ODEs is solved by the backward Euler method.

Example 1 (Dirichlet transmission condition). For (8), we choose $a > 0$ and $T = 60$. Let $\varDelta t = 0.02$, $\varDelta x = 0.01$ and $l = 2\varDelta x$ (overlap size). Then, from Proposition 1 we know that the allowed maximal a is 0.5814 for $N = 4$ and 0.4312 for $N = 16$. In Figure 2, we show the measured error corresponding to several choices of a. By "error" we denote here the discrete L_∞ norm in time and space of the difference between the converged solution and the iterate. We see that when a tends to its allowed upper bound, the SWR algorithm converges slowly.

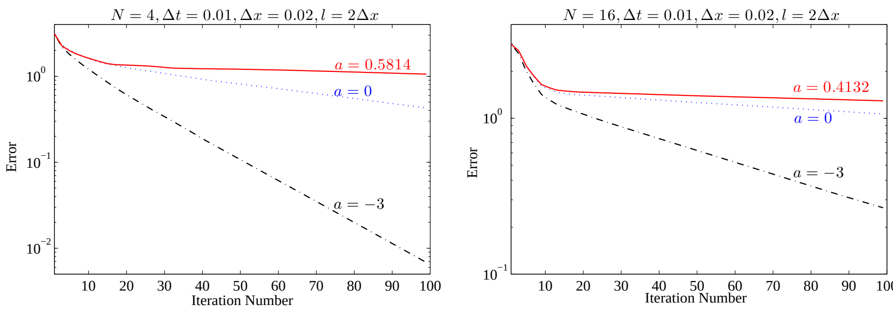

Fig. 2 Measured error of the classical SWR algorithm for different choices of a.

Example 2 (Robin transmission condition). We now choose $T = 5$ for (8), and $\varDelta t = \varDelta x = \frac{1}{2^5}$ for the discretization parameters. For $a > 0$, by employing a changed variables $v(x,t) = e^{-\sigma t}u(x,t)$ the linear problem (8) can be transformed to $\partial_t v = \partial_{xx}v + (a - \sigma)v + e^{-\sigma t}t^2 \sin(xt)$ with homogeneous initial and boundary conditions. Then, by choosing a large σ we will have $a - \sigma < 0$. The SWR algorithm with negative coefficient $a - \sigma$ can converge very fast, while the error $\max_j \left\| e^{\sigma t} v_j^k - u_j \right\|_{\infty,\infty}$ diminishes slowly. By letting $l = 5\varDelta x$ and $a = 1.5$, we illustrate this point in Figure 3 for $\sigma = 2$ (left) and $\sigma = 3.5$ (right).

We next investigated how close the parameter p_{opt} given by Proposition 1 is to the best possible one for the numerical code. In Fig. 4 on the left (resp. right), we computed the error after 5 (resp. 7) iterations by using various p for the algorithm in the case of $N = 2$ (resp. $N = 16$) subdomains. We see that the theoretically optimal choice p_{opt} predicts the optimal numerical choice very well. The asymptotic behavior of the optimized SWR algorithm is shown in Fig. 5, and we see that the results coincide with Proposition 4.

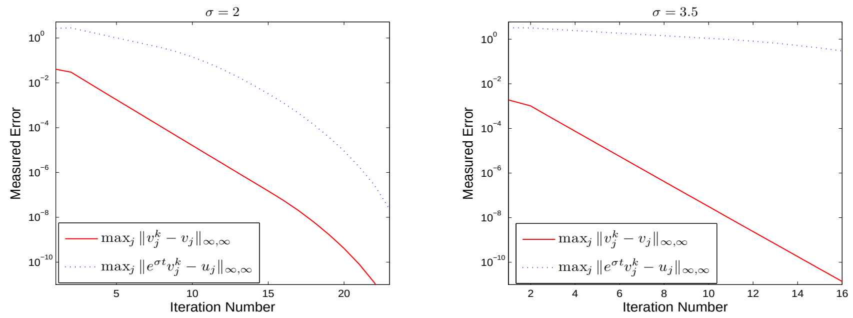

Fig. 3 Measured diminishing rate of $\max_j \|v_j^k - v_j\|_{\infty,\infty}$ and $\max_j \|e^{\sigma t} v_j^k - u_j\|_{\infty,\infty}$, with two choices of σ: $\sigma = 2$ (left) and $\sigma = 3.5$ (right).

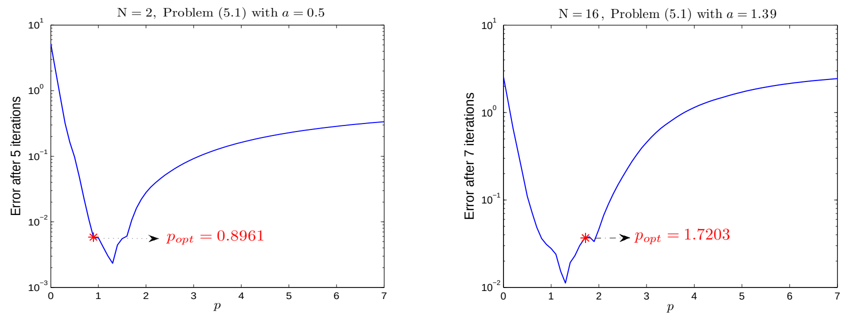

Fig. 4 Comparison of the numerical and analytical optimal parameter. Left: 2 subdomains and $l = 5\Delta x$. Right: 16 subdomains and $l = 4\Delta x$.

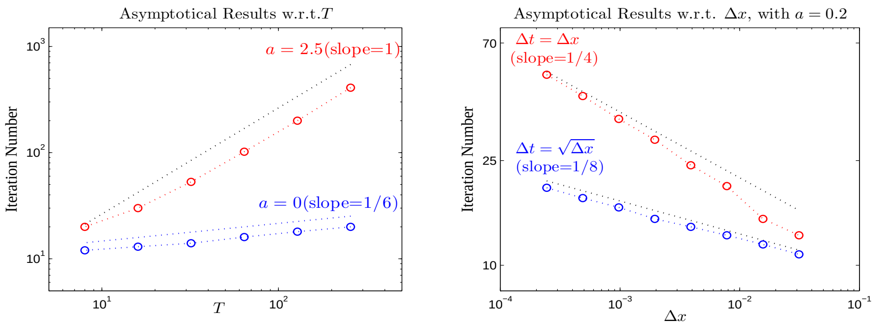

Fig. 5 Asymptotic behavior of the optimized SWR algorithm in the 2 subdomain case.

4 Conclusions

The behavior of Schwarz waveform relaxation (SWR) is well understood for stable time-dependent PDEs. Less is known when the PDEs are not stable.

We have introduced in this paper several results concerning the convergence behavior of the SWR algorithm for a class of these unstable problems. The results for the Dirichlet transmission condition can be regarded as an extension of the work by Gander [1998], Gander and Stuart [1998] and Wu et al. [2012]. The results for the Robin transmission condition are extensions of the work by Gander and Halpern [2007], Bennequin et al. [2009]. The detailed proofs are given in our forthcoming paper (Wu [2014]).

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