An optimized Schwarz algorithm for a discontinuous Galerkin method

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1 Introduction

It has been shown in [4] that block Jacobi iterates of a discretization obtained from hybridizable discontinuous Galerkin methods (HDG) can be viewed as non-overlapping Schwarz methods with Robin transmission condition. The Robin parameter is exactly the penalty parameter μ of the HDG method. There is a stability constraint on the penalty parameter and the usual choice of μ results in slow convergence of the Schwarz method. In this paper we show how to overcome this problem without changing μ. To fix ideas, we consider the model problem

$$\begin{aligned} (\eta - \Delta)u &= f \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded polygon, $0 \leq \eta \leq \eta_0$ and $f \in \mathrm{L}^2(\Omega)$. We then consider a hybridiazble interior penalty (IPH) discretization and develop domain decomposition algorithms to solve the resulting linear system efficiently. For the sake of brevity we consider the two-subdomain case in this paper.

Our paper is organized as follows: in Section 2 we describe the IPH method. We introduce a Schur complement system for the IPH discretization and review some of its properties in Section 3. In Section 4 we introduce two iterative methods for the Schur complement and present their convergence behavior. Finally we present some numerical experiments in Section 5.

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2 Hybridizable Interior Penalty method

IPH was first introduced in [2] and later studied as a member of the class of hybridizable DG methods in [1]. We first establish some notation and then define the IPH method in two different but equivalent forms. Let $\mathcal{T}_h = \{K\}$ be a shape-regular and quasi-uniform triangulation of the domain Ω. Let h_K be the diameter of an element of the triangulation and $h = \max_{K\in\mathcal{T}_h} h_K$. If e is an edge of an element, we denote by h_e the length of that edge.

We denote by $\mathcal{E}^0$ the set of interior edges, by $\mathcal{E}^\partial$ the set of boundary edges and all edges by $\mathcal{E} := \mathcal{E}^\partial \cup \mathcal{E}^0$. We introduce the broken Sobolev space $\mathrm{H}^l(\mathcal{T}_h) := \prod_{K\in\mathcal{T}_h} \mathrm{H}^l(K)$ where $\mathrm{H}^l(K)$ is the Sobolev space in $K \in \mathcal{T}_h$ and l is a positive integer. Therefore the element boundary traces of functions in $\mathrm{H}^l(\mathcal{T}_h)$ belong to $\mathrm{T}(\mathcal{E}) = \prod_{K\in\mathcal{T}_h} \mathrm{L}^2(\partial K)$, where $q \in \mathrm{T}(\mathcal{E})$ can be double-valued on $\mathcal{E}^0$, and is single-valued on $\mathcal{E}^\partial$.

We also define two trace operators: let $q \in \mathrm{T}(\mathcal{E})$ and $\boldsymbol{\sigma} \in [\mathrm{T}(\mathcal{E})]^2$. On $e = \partial K_1 \cap \partial K_2$ we then define average $\{\!\{\cdot\}\!\}$ and jump $[\![\cdot]\!]$ operators by

$$\begin{aligned} \{\!\{q\}\!\} &= \tfrac{1}{2}(q_1 + q_2), \quad & [\![q]\!] &= q_1\,\mathbf{n}_1 + q_2\,\mathbf{n}_2, \\ \{\!\{\boldsymbol{\sigma}\}\!\} &= \tfrac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2), \quad & [\![\boldsymbol{\sigma}]\!] &= \boldsymbol{\sigma}_1\cdot\mathbf{n}_1 + \boldsymbol{\sigma}_2\cdot\mathbf{n}_2, \end{aligned} \tag{2}$$

where $\mathbf{n}_i$ is the unit outward normal of K_i on e, $q_i := q|_{\partial K_i\cap e}$ and $\boldsymbol{\sigma}_i := \boldsymbol{\sigma}|_{\partial K_i\cap e}$. On $\partial\Omega$ we set the average and jump operators to be $\{\!\{\boldsymbol{\sigma}\}\!\} = \boldsymbol{\sigma}$ and $[\![q]\!] = q\,\mathbf{n}$ respectively. Note that we do not need to specify $\{\!\{q\}\!\}$ and $[\![\boldsymbol{\sigma}]\!]$ on $e \in \mathcal{E}^\partial$ because it is not needed in the formulation.

We define a finite-dimensional broken space on $\mathcal{T}_h$ for the discrete approximation $V_h := \left\{v \in \mathrm{L}^2(\Omega) : v|_K \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\right\}$, where $\mathbb{P}_k(K)$ is the space of polynomials of degree $\le k$ in the simplex $K \in \mathcal{T}_h$.

For the sake of simplicity we denote the volume and surface integrals by $(a,b)_K = \int_K a\,b$ for $K \in \mathcal{T}_h$ and $\langle a,b\rangle_e = \int_e a\,b$ for $e \in \mathcal{E}$.

We now present IPH method in *primal* and *hybridizable* form. Let $u, v \in \mathrm{H}^2(\mathcal{T}_h)$, then the IPH bilinear form of the model problem (1) is defined as

$$\begin{aligned} a(u,v) := {} & \eta(u,v)_{\mathcal{T}_h} + (\nabla u, \nabla v)_{\mathcal{T}_h} - \langle \{\!\{\nabla u\}\!\}, [\![v]\!]\rangle_{\mathcal{E}} - \langle \{\!\{\nabla v\}\!\}, [\![u]\!]\rangle_{\mathcal{E}} \\ & + \left\langle \tfrac{\mu}{2}[\![u]\!], [\![v]\!]\right\rangle_{\mathcal{E}} - \left\langle \tfrac{1}{2\mu}[\![\nabla u]\!], [\![\nabla v]\!]\right\rangle_{\mathcal{E}^0}, \end{aligned} \tag{3}$$

where $\mu \in \mathrm{L}^2(\mathcal{E})$ is the penalty parameter. For a constant $\alpha > 0$ we set $\mu|_e = \alpha h_e^{-1}$. We should mention that this scaling cannot be weakened due to stability constraints. The IPH bilinear form is different from the classical IP only in the last term, i.e. the last term is not present in the IP bilinear form. For a formal derivation of the bilinear form (3) see [6, Section 1.2.2].

The IPH bilinear form is coercive over V_h provided $\alpha > 0$ and sufficiently large, that is we can show

$$a(v,v) \ge c\|v\|_{\mathtt{DG}}^2, \quad \forall v \in V_h,$$

where $0 < c < 1$ is a constant independent of h. Here the energy norm is defined as

$$\|v\|_{\mathtt{DG}}^2 := \eta\|v\|_{\mathcal{T}_h}^2 + \|\nabla v\|_{\mathcal{T}_h}^2 + \sum_{e\in\mathcal{E}} \mu_e \|[\![v]\!]\|_e^2, \quad \forall v \in V_h. \tag{4}$$

The discrete problem can be stated as: find $u_h \in V_h$ such that

$$a(u_h, v) = (f, v)_{\mathcal{T}_h}, \quad \forall v \in V_h. \tag{5}$$

Since $a(\cdot,\cdot)$ is coercive over V_h, we can conclude that there exists a unique discrete solution. Furthermore we can show that IPH has optimal approximation properties provided $\alpha > 0$ is sufficiently large; see [6].

We show now that one can write IPH in a hybridized form, such that static condensation with respect to a single-valued unknown is possible. This is not the case for most DG methods, e.g. classical IP. Let us decompose the domain into two non-overlapping subdomains Ω_1 and Ω_2. Denoting the interface by $\Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2$, we assume $\Gamma \subset \mathcal{E}^0$, i.e. the cut does not go through any element of the triangulation. This results in a natural partitioning of $\mathcal{T}_h$ into $\mathcal{T}_1$ and $\mathcal{T}_2$; for an example see Figure 1 (right).

This naturally allows us to introduce local spaces on Ω_1 and Ω_2 by

$$V_{h,i} := \left\{v \in \mathrm{L}^2(\Omega_i) : v|_{K\in\mathcal{T}_i} \in \mathbb{P}_1(K)\right\}, \text{ for } i = 1,2. \tag{6}$$

Note that this domain decomposition setting implies $V_h = V_{h,1} \oplus V_{h,2}$. We define on the interface the space of broken single-valued functions by

$$\Lambda_h := \left\{\varphi \in \mathrm{L}^2(\Gamma) : \varphi|_{e\in\Gamma} \in \mathbb{P}_1(e)\right\}. \tag{7}$$

For the sake of simplicity we denote the restriction of $v \in V_h$ on $V_{h,i}$ by v_i. Observe that the trace of $v_i \in V_{h,i}$ on Γ belongs to Λ_h.

Let $(u,\lambda),(v,\varphi) \in V_h \times \Lambda_h$ and consider the symmetric bilinear form

$$\tilde{a}((u,\lambda),(v,\varphi)) := \tilde{a}_\Gamma(\lambda,\varphi) + \sum_{i=1}^{2} \tilde{a}_i(u_i,v_i) + \tilde{a}_{i\Gamma}(v_i,\lambda) + \tilde{a}_{i\Gamma}(u_i,\varphi), \tag{8}$$

where $\tilde{a}_\Gamma(\lambda,\varphi) := 2\langle \mu\,\lambda, \varphi\rangle_\Gamma$, $\tilde{a}_{i\Gamma}(v_i,\varphi) := \left\langle \frac{\partial v_i}{\partial \mathbf{n_i}} - \mu v_i, \varphi\right\rangle_\Gamma$ and

$$\begin{aligned}\tilde{a}_i(u_i,v_i) := {}& \eta(u_i,v_i)_{\mathcal{T}_i} + (\nabla u_i, \nabla v_i)_{\mathcal{T}_i} - \langle \{\!\{\nabla u_i\}\!\}, [\![v_i]\!]\rangle_{\mathcal{E}_i^0} - \langle \{\!\{\nabla v_i\}\!\}, [\![u_i]\!]\rangle_{\mathcal{E}_i^0} \\ & + \left\langle \tfrac{\mu}{2}[\![u_i]\!], [\![v_i]\!]\right\rangle_{\mathcal{E}_i^0} - \left\langle \tfrac{1}{2\mu}[\![\nabla u_i]\!], [\![\nabla v_i]\!]\right\rangle_{\mathcal{E}_i^0} \\ & - \left\langle \tfrac{\partial u_i}{\partial \mathbf{n_i}}, v_i\right\rangle_{\partial\Omega_i} - \left\langle \tfrac{\partial v_i}{\partial \mathbf{n_i}}, u_i\right\rangle_{\partial\Omega_i} + \langle \mu\, u_i, v_i\rangle_{\partial\Omega_i}.\end{aligned} \tag{9}$$

The bilinear form $\tilde{a}(\cdot,\cdot)$ is also coercive at the discrete level if $\alpha > 0$, independent of h and sufficiently large:

$$\tilde{a}((v,\varphi),(v,\varphi)) \geq c\,\|(v,\varphi)\|^2_{\mathtt{HDG}} \quad \forall (v,\varphi) \in V_h \times \Lambda_h, \tag{10}$$

where c is independent of h and the HDG-norm is defined by

$$\|(v,\varphi)\|^2_{\mathtt{HDG}} := \sum_{i=1}^{2} \eta\|v_i\|^2_{\mathcal{T}_i} + \|\nabla v_i\|^2_{\mathcal{T}_i} + \mu\|[\![v_i]\!]\|^2_{\mathcal{E}_i\setminus\Gamma} + \mu\|v_i-\varphi\|^2_{\Gamma}. \tag{11}$$

Consider the following discrete problem: find $(u_h,\lambda_h) \in V_h \times \Lambda_h$ such that

$$\tilde{a}((u_h,\lambda_h),(v,\varphi)) = (f,v)_{\mathcal{T}_h}, \quad \forall (v,\varphi) \in V_h \times \Lambda_h, \tag{12}$$

which has a unique solution since $\tilde{a}(\cdot,\cdot)$ is coercive on $V_h \times \Lambda_h$. One can eliminate the interface variable, λ_h, and obtain a variational problem in terms of u_h only. It turns out that this coincides with the variational problem (5); for a proof see [6].

Remark 1. By definition of the bilinear forms, each subproblem is imposing λ_h weakly as Dirichlet data along Γ through a Nitsche penalization. This is an IPH discretization of the continuous problem $(\eta-\Delta)w = f$ in Ω_i and $w = \lambda_h$ on Γ.

3 Schur complement system

We choose nodal basis functions for $\mathbb{P}_1(K)$ and denote the space of coefficient vectors with respect to nodal basis functions of V_h by V. If $u_h \in V_h$ we denote by $\boldsymbol{u} \in V$ its corresponding coefficient vector. The variational problem in (5) is equivalent to the linear system $A\boldsymbol{u} = \boldsymbol{f}$. A is called the stiffness matrix. We decompose $\boldsymbol{u}$ into $\{\boldsymbol{u}_1,\boldsymbol{u}_2\}$ where $\boldsymbol{u}_i$ corresponds to coefficients of nodal basis functions in Ω_i. Then we can arrange the entries of A and rewrite the linear system as

$$\begin{bmatrix} A_1 & A_{21} \\ A_{21} & A_2 \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{pmatrix}. \tag{13}$$

We use nodal basis functions for Λ_h and denote by $\boldsymbol{\lambda}$ the corresponding coefficient vector for $\lambda_h \in \Lambda_h$. Then the variational form (12) can be written as

$$\begin{bmatrix} \tilde{A}_1 & & \tilde{A}_{1\Gamma} \\ & \tilde{A}_2 & \tilde{A}_{2\Gamma} \\ \tilde{A}_{\Gamma 1} & \tilde{A}_{\Gamma 2} & \tilde{A}_{\Gamma} \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{0} \end{pmatrix}, \tag{14}$$

where $\tilde{A}_{\Gamma i} = \tilde{A}_{i\Gamma}^{\top}$. Note that the advantage of this formulation over (13) is that subdomains are communicating through $\boldsymbol{\lambda}$ and we can form a Schur complement for a *single-valued* function, λ_h. To do so we define $\tilde{B}_i := \tilde{A}_{\Gamma i}\tilde{A}_i^{-1}\tilde{A}_{i\Gamma}$ and $\boldsymbol{g}_\Gamma := \sum_{i=1}^{2} \tilde{A}_{\Gamma i}\tilde{A}_i^{-1}\boldsymbol{f}_i$. Then the Schur complement system reads

$$\tilde{S}_{\Gamma}\boldsymbol{\lambda} := \Big(\tilde{A}_{\Gamma} - \sum_{i=1}^{2}\tilde{B}_i\Big)\boldsymbol{\lambda} = \boldsymbol{g}_{\Gamma}. \tag{15}$$

We define $u_i := \mathcal{H}_i(\lambda_h)$ to be the discrete harmonic extension of $\lambda_h \in \Lambda_h$ into subdomain Ω_i, i.e. u_i satisfies $\tilde{A}_i\boldsymbol{u}_i + \tilde{A}_{i\Gamma}\boldsymbol{\lambda} = 0$; that is we impose λ_h as Dirichlet data (weakly) on Γ and solve inside Ω_i. The following result shows that an application of $\tilde{B}_i\boldsymbol{\lambda}$ can be viewed as finding the harmonic extension, $u_i := \mathcal{H}_i(\lambda_h)$, and then evaluating a "Robin-like trace" on the interface.

Proposition 1. *Let $\lambda_h \in \Lambda_h$ and define its harmonic extension by $u_i := \mathcal{H}_i(\lambda_h)$. Then $\boldsymbol{\varphi}^\top\tilde{B}_i\boldsymbol{\lambda} = \left\langle \mu u_i - \frac{\partial u_i}{\partial \mathbf{n_i}}, \varphi\right\rangle_{\Gamma}$ for all $\varphi \in \Lambda_h$.*

Proof. Let $u_i := \mathcal{H}_i(\lambda_h)$. Then by definition of $\tilde{B}_i$ and $\tilde{a}_{i\Gamma}(\cdot,\cdot)$ we have $\boldsymbol{\varphi}^\top\tilde{B}_i\boldsymbol{\lambda} = \boldsymbol{\varphi}^\top\tilde{A}_{\Gamma i}\tilde{A}_i^{-1}\tilde{A}_{i\Gamma}\boldsymbol{\lambda} = -\boldsymbol{\varphi}^\top\tilde{A}_{\Gamma i}\boldsymbol{u}_i = \left\langle \mu u_i - \frac{\partial u_i}{\partial \mathbf{n_i}}, \varphi\right\rangle_{\Gamma}$, for all $\varphi \in \Lambda_h$, which completes the proof, since $\tilde{A}_{\Gamma i} = \tilde{A}_{i\Gamma}^\top$. □

One can estimate the eigenvalues of $\{\tilde{B}_i\}$. They are useful in proving convergence of Schwarz methods later on. The proofs are technical and beyond the scope of this short paper. They can be found in [5].

Lemma 1. *$\tilde{B}_i$ is s.p.d. and there exists $\alpha > 0$, sufficiently large, such that*

$$c_B\,\mu\,\|\varphi\|_{\Gamma}^2 \le \boldsymbol{\varphi}^\top\tilde{B}_i\boldsymbol{\varphi} \le \Big(1 - C_B\frac{h}{\alpha}\Big)\,\mu\,\|\varphi\|_{\Gamma}^2,$$

where $0 < c_B < 1$ and $C_B > 0$. Both constants are independent of h. Moreover $\tilde{A}_{\Gamma} - 2\tilde{B}_i$ is s.p.d. for $i = 1, 2$.

4 Schwarz methods for the Schur complement system

One approach in solving the linear system (13) is to use the block Jacobi method:

$$M\mathbf{u}^{(n+1)} = N\mathbf{u}^{(n)} + \mathbf{f}, \quad M = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}, N = M - A. \tag{16}$$

Instead in this section we derive two Schwarz algorithms to solve the Schur complement system where the first one is equivalent to (16) and slow while the second one has much faster convergence.

Let us relax the constraint that λ_h is single-valued. Let $\lambda_{h,1}, \lambda_{h,2} \in \Lambda_h$. Assume $\lambda_{h,2}$ is known; that is we know $u_2 \in V_{h,2}$. Then we can split the Schur complement system (15) and solve for $\lambda_{h,1}$, through $(\tilde{A}_{\Gamma} - \tilde{B}_1)\boldsymbol{\lambda}_1 = \tilde{B}_2\boldsymbol{\lambda}_2 + \boldsymbol{g}_{\Gamma}$. Lemma 1 ensures that $(\tilde{A}_{\Gamma} - \tilde{B}_1)$ is invertible and we can obtain $\lambda_{h,1}$. This suggests an iterative method to find λ_h.

Algorithm 1 (block Jacobi) *Let $\lambda_{h,1}^{(0)}, \lambda_{h,2}^{(0)} \in \Lambda_h$ be two arbitrary initial guesses. Then for $n = 1, 2, \dots$ solve (17) for $\{\lambda_{h,i}^{(n)}\}$.*

$$\begin{aligned}(\tilde{A}_\Gamma - \tilde{B}_1)\boldsymbol{\lambda}_1^{(n)} &= \tilde{B}_2\boldsymbol{\lambda}_2^{(n-1)} + \boldsymbol{g}_\Gamma,\\ (\tilde{A}_\Gamma - \tilde{B}_2)\boldsymbol{\lambda}_2^{(n)} &= \tilde{B}_1\boldsymbol{\lambda}_1^{(n-1)} + \boldsymbol{g}_\Gamma.\end{aligned} \tag{17}$$

Note that at convergence we have $\tilde{A}_\Gamma(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) = 0$ which implies $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_2 = \tilde{S}_\Gamma^{-1}\boldsymbol{g}_\Gamma$ since $\tilde{A}_\Gamma$ is s.p.d. We show now that Algorithm 1 is equivalent to the block Jacobi iteration (16). It suffices to prove this for $f = 0$ ($\boldsymbol{g}_\Gamma = 0$).

Proposition 2. *Let $\lambda_{h,1}^{(0)}, \lambda_{h,2}^{(0)}$ be two random initial guesses. Set the initial guess of the block Jacobi iteration (16) to be $u_i^{(0)} = \mathcal{H}_i(\lambda_{h,i}^{(0)})$. Then $u_i^{(n)} = \mathcal{H}_i(\lambda_{h,i}^{(n)})$ for all $n > 0$, i.e. both methods produce the same iterates.*

Proof. We start by subdomain Ω_1. Set $w_{h,i}^{(n)} = \mathcal{H}_i(\lambda_{h,i}^{(n)})$. By Proposition 1, we have $\boldsymbol{\varphi}^\top \tilde{B}_i\boldsymbol{\lambda}_i^{(n)} = \left\langle \mu w_{h,i}^{(n)} - \partial_{\mathbf{n}_i} w_{h,i}^{(n)}, \varphi\right\rangle_\Gamma$ for all $\varphi \in \Lambda_h$. Then the first equation in iteration (17) implies $\lambda_{h,1}^{(n)} = \left(\frac{1}{2} - \frac{1}{2\mu}\partial_{\mathbf{n}_1}\right)w_{h,1}^{(n)} + \left(\frac{1}{2} - \frac{1}{2\mu}\partial_{\mathbf{n}_2}\right)w_{h,2}^{(n-1)}$. Recall that $w_{h,1}^{(n)}$ is the harmonic extension, hence it satisfies $\tilde{a}_i(w_{h,1}^{(n)}, v_1) + \tilde{a}_{i\Gamma}(v_1, \lambda_{h,1}^{(n)}) = 0$ for all $v_1 \in V_{h,1}$. Now we substitute $\lambda_{h,1}^{(n)}$ in terms of $w_{h,1}^{(n)}$ and $w_{h,2}^{(n-1)}$. We arrive at exactly the first row of block Jacobi (16), i.e. $A_1\boldsymbol{w}_1^{(n)} + A_{12}\boldsymbol{w}_2^{(n-1)} = 0$. The proof for Ω_2 is similar. □

Convergence of the block Jacobi (16) or equivalently Algorithm 1 can be proved with the contraction factor $\rho_h \leq 1 - O(h)$. For details we refer the reader to [5].

The slow convergence of this algorithm is due to the fact that the transmission condition is of Robin type with Robin parameter $\mu = \alpha h^{-1}$; see [4]. According to optimized Schwarz theory the best choice is $\mu = O(h^{-1/2})$; see [3]. We would like to emphasize that for IPH, one cannot change the scaling of μ because of coercivity and approximation property constraints.

The remedy is to split the Schur complement differently. We know from Lemma 1 that $\tilde{A}_\Gamma - 2B_i$ is s.p.d. Therefore assuming $\boldsymbol{\lambda}_2$ is known we can multiply the Schur complement by $(1+\hat{p})$ where $\hat{p}$ is a constant and solve for $\boldsymbol{\lambda}_1$ such that

$$(\tilde{A}_\Gamma - (1+\hat{p})B_1)\boldsymbol{\lambda}_1 = -(\hat{p}\tilde{A}_\Gamma - (1+\hat{p})\tilde{B}_2)\boldsymbol{\lambda}_2 + (1+\hat{p})\boldsymbol{g}_\Gamma.$$

If $0 \leq \hat{p} < 1$ then the left hand side is still s.p.d. We use $\hat{p}$ to obtain a fast converging solver. Note that for $\hat{p} = 0$ we have Algorithm 1.

Algorithm 2 (optimized Schwarz) *Let $\lambda_{h,1}^{(0)}, \lambda_{h,2}^{(0)} \in \Lambda_h$ be two arbitrary initial guesses and $0 \leq \hat{p} < 1$ be a constant. Then for $n = 1, 2, \dots$ solve (18) for $\{\lambda_{h,i}^{(n)}\}$.*

$$\begin{aligned}(\tilde{A}_\Gamma - (1+\hat{p})\tilde{B}_1)\boldsymbol{\lambda}_1^{(n)} &= -(\hat{p}\tilde{A}_\Gamma - (1+\hat{p})\tilde{B}_2)\boldsymbol{\lambda}_2^{(n-1)} + (1+\hat{p})\boldsymbol{g}_\Gamma,\\ (\tilde{A}_\Gamma - (1+\hat{p})\tilde{B}_2)\boldsymbol{\lambda}_2^{(n)} &= -(\hat{p}\tilde{A}_\Gamma - (1+\hat{p})\tilde{B}_1)\boldsymbol{\lambda}_1^{(n-1)} + (1+\hat{p})\boldsymbol{g}_\Gamma.\end{aligned} \tag{18}$$

At convergence we have $(1-\hat{p})\tilde{A}_\Gamma(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) = 0$ which implies $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_2 = \tilde{S}_\Gamma^{-1}\boldsymbol{g}_\Gamma$ if $\hat{p} \neq 1$. An application of Proposition 1 and Remark 1 shows Algorithm 2 has a modified Robin parameter which we summarize in the next proposition.

Proposition 3. *Algorithm 2 is the discrete version of the non-overlapping optimized Schwarz method*

$$\begin{aligned}\mathcal{L}u_1^{(n+1)} &= f \quad &&\text{in } \Omega_1, \quad \mathcal{L}u_2^{(n+1)} = f \quad &&\text{in } \Omega_2,\\ \mathcal{B}_1 u_1^{(n+1)} &= \mathcal{B}_1 u_2^{(n)} \quad &&\text{on } \Gamma, \quad \mathcal{B}_2 u_2^{(n+1)} = \mathcal{B}_2 u_1^{(n)} \quad &&\text{on } \Gamma,\end{aligned}$$

where $\mathcal{L} := (\eta - \Delta)$*,* $\mathcal{B}_i := \hat{\mu} + \partial_{\mathbf{n}_i}$ *and Robin parameter* $\hat{\mu} := \frac{1-\hat{p}}{1+\hat{p}}\mu$*.*

A heuristic approach in obtaining optimal $\hat{p}$ is to set the *modified* Robin parameter to $\hat{\mu} = O(h^{-1/2})$ and solve for $\hat{p}$. This results in $\hat{p} = \frac{1-\sqrt{h}}{1+\sqrt{h}} < 1$. A rigorous proof at the discrete level in [5] gives same scaling and with this choice of $\hat{p}$ the contraction factor of Algorithm 2 is bounded by $\rho_h \leq 1 - O(\sqrt{h})$.

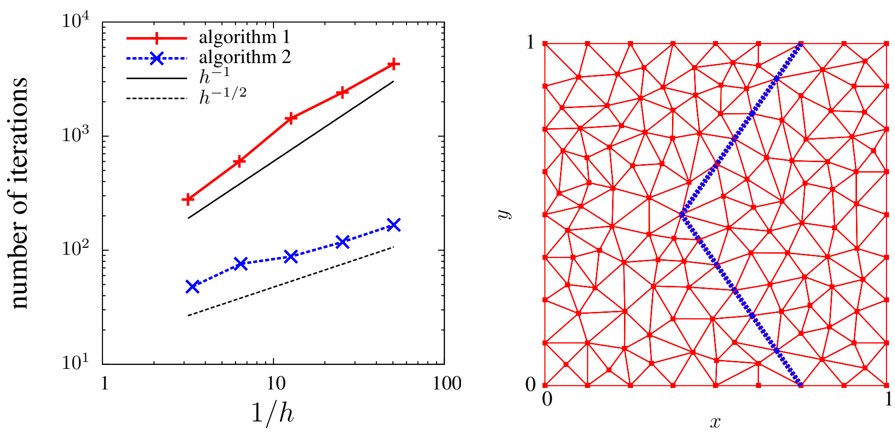

Fig. 1 Convergence of the Schwarz algorithms (left), domain decomposition (right).

5 Numerical experiments

We consider $(\eta - \Delta)u = f$ in Ω and $u = 0$ on $\partial\Omega$ where we set $\eta = 1$, $\Omega = (0,1)^2$ and f such that the exact solution is $u(x,y) = \sin(\pi x)\sin(2\pi x +$

$\frac{\pi}{4})\sin(2\pi y)$ in Ω. We set the penalty parameter to $\mu = 10h_e^{-1}$. We choose a non-straight interface as in Figure 1 (right). We measure the number of iterations necessary to reduce the error $\|u_h - u_h^{(n)}\|_0$ to 10^{-10} on a sequence of (quasi-uniform) unstructured meshes while the interface is *fixed*. As for the initial guess, we set DOFs of the initial guess using a random number generator; in `Matlab` given by `rand(N_DOF)`.

In Figure 1 (left) we observe for Algorithm 1 that the number of iterations grows like $O(h^{-1})$. This is equivalent to $\rho_h \leq 1 - O(h)$. For Algorithm 2 with the optimal value of $\hat{p}$ we see that it grows like $O(h^{-1/2})$ hence $\rho_h \leq 1 - O(\sqrt{h})$. This is in agreement with the results in Section 4. For more extensive numerical experiments see [5].

6 Conclusions

It has been shown in [4] that for some DG methods one can obtain a fast converging solver by just *modifying* the penalty parameter while for some other it is not possible, e.g. IPH. We showed that it is possible to define an iterative method, Algorithm 2, for IPH such that we obtain fast convergence without changing the penalty parameter. We are now studying a multi-subdomain version of Algorithm 2 and the case of higher polynomial degree, $k > 1$.

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