Augmented Lagrangian Domain Decomposition Method for Bonded Structures

J. Koko[1] and T. Sassi[2]

1 Introduction

Domain decomposition methods are subject to a greater interest, due to obvious implication for parallel computing. Non-overlapping methods are particularly well suited for coupled problems through an interface as bonded structures (e.g., Geymonat et al. [1998]) air/water flows (e.g., Bresch and koko [2006]), two-body contact problems (e.g., Haslinger et al. [2014], Koko [2008b]), etc. For these coupled problems, the domain decomposition methods applied in a natural way, since the sub-domains are already defined.

Two types of domain decomposition methods exist for bonded structures: Lagrangian (dual) methods (Bresch and Koko [2004]) and least-square methods (Geymonat et al. [1998], Koko, J. [2002]). In Lagrangian methods, the objective functional is the energy functional and the constraint is the solution jump across the interface. In least-square methods, the original problem is reformulated as a constrained minimization problem for which the objective functional controls the solution jump across the interface. The constraints are the partial differential equations stated in each sub-domain with suitable boundary conditions. In a comparative study, Koko [2008a] shows that the least-square methods solve twice as many linear systems than the dual methods. But both methods fail if one of the subdomains allows rigid-body motions.

The paper is organized as follows. In Section 2 we present the simplified model of bonded structures. The Uzawa block relaxation domain decomposition algorithm is described in Section 3. Some numerical experiments are carried out in Section 4.

[1]LIMOS, Université Blaise Pascal – CNRS UMR 6158, F-63000 Clermont-Ferrand, France,e-mail: koko@isima.fr .[2] LMNO, Université de Caen – CNRS UMR, F-14032 Caen, France e-mail: sassi@univ-caen.fr

2 Model problem

We adopt the model problem described in Koko [2008a]. To simplify, we present a model problem with two subdomains. A generalization to more than two subdomains is straightforward.

Consider a system of two isotropic elastic bodies each of which occupies, in the reference configuration, a bounded domain Ω_i in $\mathbb{R}^2$ $(i = 1, 2)$. Both elastic bodies are bonded along their common boundary S, assumed to be a nonempty surface of positive measure. Hooke's law is assumed for each elastic body, i.e.

$$\sigma^i_{\alpha\beta}(u_i) = 2\mu_i\varepsilon_{\alpha\beta}(u_i) + \lambda_i \mathrm{tr}(\varepsilon(u_i))\mathbb{I}_2, \quad \alpha, \beta = 1, 2,$$

where $\varepsilon(u_i) = (\nabla u_i + \nabla u_i^T)/2$, $\lambda_i \geq 0$ and $\mu_i > 0$ denote Lamé constants. Let u_i be the displacement field of the body Ω_i. We set $u = (u_1, u_2)$ the displacement field of the bonded structure and $[u] = (u_1 - u_2)_{|S}$ the relative tangential displacement along S. The simplified model of bonded structures we study in this paper can be formulated as follows

$$-\mathrm{div}\sigma^i(u_i) = f_i \ \text{ in } \Omega_i, \tag{1}$$

$$u_i = 0 \ \text{ on } \Gamma_i = \partial\Omega_i \setminus S, \tag{2}$$

$$\sigma^i(u_i) \cdot n_i = (-1)^i K[u] \ \text{ on } S, \tag{3}$$

where n_i is the unit outward normal to Ω_i, and K is the second order bonding tensor assumed to be symmetric and coercive with bounded coefficients. Equation (3) is the transmission condition for u_1 and u_2. The domain decomposition algorithms are (generally) parallel iterative procedures on (1)-(2) that tend to satisfy the transmission condition (3).

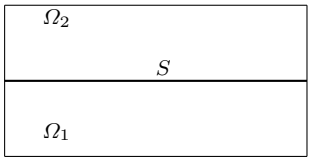

Fig. 1 Bonded structure : Ω_1 and Ω_2 the sub-domains (adherents), S the interface (thin adhesive layer)

Let us introduce the subspaces $V_i = \left\{v \in H^1(\Omega_i);\ v = 0 \text{ on } \Gamma_i\right\}$, $V = V_1 \times V_2$ and the notations, for $u_i, v_i \in V_i$

$$a_i(u_i, v_i) = \int_{\Omega_i} \sigma^i(u_i)\varepsilon(v_i)\,\mathrm{d}x, \tag{4}$$

$$(u_i, v_i)_{\Omega_i} = \int_{\Omega_i} u_i v_i\,\mathrm{d}x \quad \text{and} \quad (u_i, v_i)_{\Gamma_i} = \int_{\Gamma_i} u_i v_i\,\mathrm{d}\Gamma_i. \tag{5}$$

With the above notations, the total potential energy of the simplified model of a bonded structure we study is

$$F(v) = J(v) + \frac{1}{2}(K[v],[v])_S \quad \forall (v_1, v_2) \in V_1 \times V_2 \tag{6}$$

where

$$J(v) = \frac{1}{2}\sum_{i=1}^{2} a_i(v_i, v_i) - \sum_{i=1}^{2}(f_i, v_i)_{\Omega_i}.$$

The bonded structure problem can now be formulated as the following minimization problem.

Find $u \in V$ such that

$$F(u) \leq F(v), \quad \forall v \in V. \tag{7}$$

The functional J is convex and coercive (see, e.g., Ciarlet [1988]) on V. Since K is symmetric and coercive, it follows that F is convex and coercive on V. Consequently, the minimization problem (7) has a unique solution.

In the method proposed by Bresch and Koko Bresch and Koko [2004], the objective functional is the energy functional and the constraint is the solution jump across the interface. With the use of the Lagrangian functional, the resulting domain decomposition algorithm is of Uzawa type, precisely its conjugate gradient version. In the method proposed in Geymonat et al. [1998], Koko, J. [2002], the original problem is reformulated as a constrained minimization problem for which the objective functional controls the solution jump across the interface. The constraints are the partial differential equations stated in each sub-domain with suitable boundary conditions. Both methods fail if one of the subdomain allows rigid-body motions.

3 Augmented Lagrangian domain decomposition

Let us introduce the auxiliary interface unknowns $q_i = v_{i|S}$ so that the energy funcional (6) becomes

$$F(v, q) = J(v) + \frac{1}{2}(K[q],[q])_S \quad \forall (v, q) \in V \times H,$$

where $H = L^2(S)^2$. We then replace the unconstrained minimization problem (7) by the following (equivalent) constrained minimization problem

Find $(u, p) \in V \times H$ such that

$$F(u, p) \leq F(v, q) \quad \forall (v, q) \in V \times H, \tag{8}$$

$$u_i = p_i \quad \text{on } S, \quad i = 1, 2. \tag{9}$$

With (8)-(9), we associate the augmented Lagrangian functional

$$\mathcal{L}_r(v,q;\mu) = F(v,q) + \sum_{i=1}^{2}\Big((\mu_i, v_i - q_i)_S + \frac{r}{2} \parallel v_i - q_i \parallel^2_{L^2(S)}\Big); \qquad (10)$$

where $r > 0$ is the penalty parameter. The saddle-point problem for the augmented Lagrangian functional is

Find $(u,p,\lambda) \in V \times H \times H$ such that:

$$\mathcal{L}_r(u,p,\mu) \leq \mathcal{L}_r(u,p,\lambda) \leq \mathcal{L}_r(v,q,\lambda) \quad \forall (v,q,\mu) \in V \times H \times H \qquad (11)$$

The functional $\mathcal{L}_r$ is Gâteaux-differentiable on $V \times H \times H$, then the solution of (11) is characterized by the saddle-point (Euler-Lagrange) equations of the primal and dual problems as follows

Find $(u,p,\lambda) \in V \times H \times H$ such that

$$\frac{\partial \mathcal{L}_r}{\partial u}(u,p,\lambda)\cdot v = 0, \quad \forall v \in V, \qquad (12)$$

$$\frac{\partial \mathcal{L}_r}{\partial p}(u,p,\lambda)\cdot q = 0, \quad \forall q \in H \qquad (13)$$

$$\frac{\partial \mathcal{L}_r}{\partial \lambda}(u,p,\lambda)\cdot \mu = 0, \quad \forall \mu \in H. \qquad (14)$$

Subdomain problems in u are uncoupled if the multipliers λ and the coordination variable p are known. We can use this property through a Uzawa algorithm associated with a block relaxation method.

Uzawa block/relaxation methods have been used in nonlinear mechanics for operator-splitting methods (see e.g. Glowinski and Le Tallec [1989]). The idea is to minimize successively in u and p, in block Gauss-Seidel fashion. Applying a Uzawa block relaxation method to (12)-(14) we obtain the following algorithm, assuming p^0 and λ^0

$$u^{k+1} = \arg\min_v \mathcal{L}_r(v, p^k, \lambda^k), \qquad (15)$$

$$p^{k+1} = \arg\min_q \mathcal{L}_r(u^{k+1}, q, \lambda^k), \qquad (16)$$

$$\lambda^{k+1} = \lambda^k + r(u^{k+1} - p^{k+1}). \qquad (17)$$

The minimization subproblem (15) is equivalent to the uncoupled subdomain problems

$$a_i(u_i^{k+1}, v_i) + r(u_i^{k+1}, v_i)_S = f_i(v_i) + (rp_i^k - \lambda_i^k, v), \quad \forall v_i \in V_i, \quad i = 1,2 \qquad (18)$$

while (16) leads to the point-wise interface subproblem

$$(K + r\mathbb{I})p_1^{k+1} - Kp_2^{k+1} = \lambda_1^k + ru_1^{k+1}, \tag{19}$$

$$-Kp_1^{k+1} + (K + r\mathbb{I})p_2^{k+1} = \lambda_2^k + ru_2^{k+1}. \tag{20}$$

Gathering the results above, we obtain the Uzawa block relaxation method presented in Algorithm 1. We iterate until the relative error on (u^k, p^k) becomes sufficiently small.

Remark 1. The problem (18) always has a unique solution even without the Dirichlet condition (2). This property is useful for solving problems allowing rigid body motions.

Remark 2. Algorithm 1 is equivalent to the operator-splitting standard algorithm ALG 2 described in, e.g., [Glowinski and Le Tallec, 1989, ch. 3], applied to the minimization problem (8)-(9). Since F is convex and coercive and the constraints (9) are linear, the convergence of Algorithm 1 is guaranteed by, e.g., [Glowinski and Le Tallec, 1989, theorem 3.4].

Algorithm 1 Uzawa block relaxation algorithm for a bonded structure

Initialization. p^0, λ^0 and $r > 0$ are given
Iteration $k \geq 0$. Compute successively u^{k+1}, p^{k+1} and λ^{k+1} as follows

1. Compute $u_i^{k+1} \in V_i$ such that

$$a_i(u_i^{k+1}, v_i) + r(u_i^{k+1}, v_i)_S = f_i(v_i) + (rp_i^k - \lambda_i^k, v_i)_S, \quad \forall v_i \in V_i, \quad i = 1, 2.$$

2. Compute $(p_1^{k+1}, p_2^{k+1}) \in H$ such that

$$(K + r\mathbb{I})p_1^{k+1} - Kp_2^{k+1} = \lambda_1^k + ru_1^{k+1}$$
$$-Kp_1^{k+1} + (K + r\mathbb{I})p_2^{k+1} = \lambda_2^k + ru_2^{k+1}$$

3. Update Lagrange multipliers: $\lambda_i^{k+1} = \lambda_i^k + r(u_{i|S}^{k+1} - p_i^{k+1})$, $i = 1, 2$.

The discrete version of Algorithm 1 is straightforward using the finite element method (or the finite difference scheme). The only condition is the meshes compatibility on S. Assuming that Ω_{ih} is a triangulation of Ω_i, the meshes are compatible on S in the sense that $\bar{\Omega}_{1h} \cap S = \bar{\Omega}_{2h} \cap S$.

The uncoupled elasticity subproblems (18) lead to linear systems with symmetric positive definite matrices. Since these matrices do not change during the iterative process, a Cholesky factorization can be performed once and for all in the initialization step. Then forward/backward substitutions are performed in the rest of the iterative process. If a preconditioned iterative solver is used for solving (18), an incomplete factorization is performed once and for all in the initialization step.

The (linear) interface subproblem (19)-(20) is solved point-wise. At each point we have to invert a small size matrix (4×4 in 2D or 6×6 in 3D).

We can therefore use a semi-analytical solution for (19)-(20). Indeed, direct Gaussian elimination yields to

$$p_1^{k+1} = (K_r^2 - \mathbb{I})^{-1} K^{-1} \left(K_r(\lambda_1^k + r u_1^{k+1}) + \lambda_2^k + r u_2^{k+1} \right)$$
$$p_2^{k+1} = K_r p_1^{k+1} - K^{-1}(\lambda_1^k + r u_1^{k+1}),$$

where we have set $K_r = \mathbb{I} + rK^{-1}$. The size of K is 2×2 in 2D and 3×3 in 3D and in many applications, K is a diagonal matrix.

4 Numerical experiments

Algorithm 1 was implemented in MATLAB 7 on a Linux workstation with 2.67 GHz clock frequency and 12 GB RAM. The test problem used is designed to illustrate the numerical behavior of the algorithm more than to model actual bonded structures. Setting $z = (u, p)$, the stopping criterion is

$$\| z^k - z^{k-1} \|_{L^2} < 10^{-6} \| z^k \|_{L^2} . \tag{21}$$

We are interested in the bonded structure of Figure 2, made from three isotropic (linear) elastic bodies. The subdomains are $\Omega_1 = (0,\ 20) \times (5,\ 10) \cup (0,\ 20) \times (-10,\ 5)$ and $\Omega_2 = (0,\ 60) \times (-5,\ 5)$. The interface is therefore $S = (0,\ 20) \times \{5\} \cup (0,\ 20) \times \{-5\}$. The material constants of the adherents are $E_1 = 5 \times 10^4$ MPA, $\nu_1 = 0.3$, $E_2 = 2.5 \times 10^4$ MPA and $\nu_2 = 0.3$.

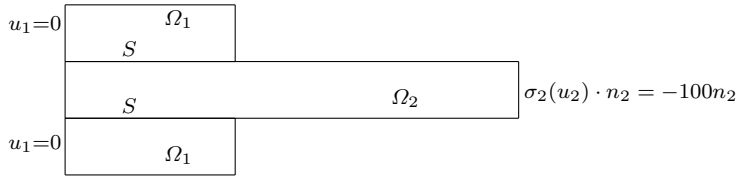

Fig. 2 Geometry of the bonded structure, $n_2 = (1, 0)^T$.

The material constants of the adhesive layer are $E^* = 1800$MPA, $\nu^* = 0.35$, $\tilde{\nu} = 2(1 - \nu^*)/(1 - 2\nu^*)$

$$K = \frac{E^*}{2(1 + \nu^*)} \text{diag}\,(\tilde{\nu},\ 1)$$

Remark 3. Since the subproblem over Ω_2 allows rigid body motions, pure Lagrangian (Bresch and Koko [2004], Koko [2008a]) and least square methods (Geymonat et al. [1998], Koko [2008a], Koko, J. [2002]) are not applicable.

The bonded structure is first modeled by a uniform mesh consisting of 2×121 nodes and 2×224 triangles (for Ω_1) and 723 nodes and 1344 triangles

for Ω_2,with 2×15 nodes on S. We use piecewise linear finite element spaces. Applying Algorithm 1 with the penalty parameter $r = 1500$, (21) is satisfied after 27 iterations. Figure 3 shows the Von Mises effective stress distribution inside the tree-body system.

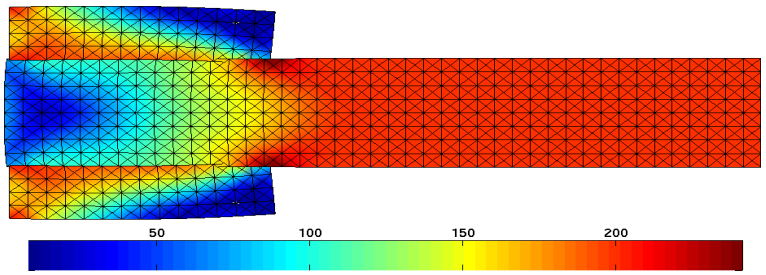

Fig. 3 Deformed configuration and Von Mises effective stress (magnification factor 20).

Augmented Lagrangian type algorithms are very sensitive to the choice of the penalty (or augmentation) parameter r. Figure 4 shows the number of iterations versus the penalty parameter r. The optimal penalty value is $r \approx 1300$. Choosing smaller or larger values for r increases the number of iterations without improving the final result.

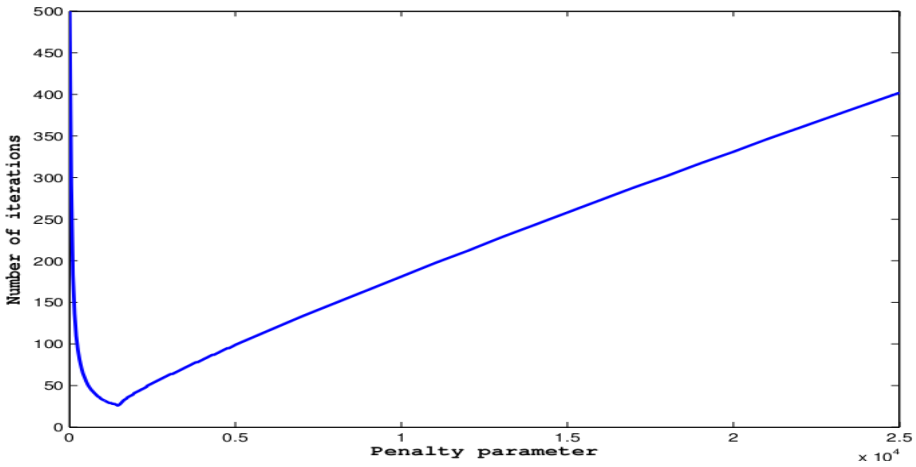

Fig. 4 Number of iterations versus penalty parameter

To study the scalability of our algorithm, we report in Table 1 the iteration count for different interface mesh sizes. We can notice that the iteration count is virtually independent of interface mesh size, for the chosen number of subdomains.

Number of interface nodes	2×11	2×21	2×41	2×81	2×161
Number of iterations	27	27	27	27	27

Table 1 Number of iterations versus interface mesh size with $r = 1500$, chosen independently of the mesh size.

5 Conclusion

We have studied a Uzawa block relaxation method for bonded structures. The method is easy to implement and numerical experiments show that the number of iterations is virtually independent of the mesh size for a fixed number of adherents. Even though the domain decomposition method proposed converges for any $r > 0$, choosing automatically the "optimal" value of the penalty parameter is still an open question.

References

D. Bresch and J. Koko. An optimization-based domain decomposition method for nolinear wall laws in coupled systems. *Math. Models Methods Appl. Sci.*, 14:1085–1101, 2004.

D. Bresch and J. koko. Operator-splitting and Lagrange multiplier domain decomposition methods for numerical simulation of two coupled Navier-Stokes fluids. *Int. J. Appl. Math. Comput. Sci.*, 16:101–113, 2006.

P.-G. Ciarlet. *Mathematical Elasticity I: Three-Dimensional Elasticity.* North-Holland, Amsterdam, 1988.

G. Geymonat, F. Krasucki, D. Marini, and M. Vidrascu. A domain decomposition method for bonded structures. *Math. Models Methods Appl. Sci.*, 8:1387–1402, 1998.

R. Glowinski and P. Le Tallec. *Augmented Lagrangian and Operator-splitting Methods in Nonlinear Mechanics.* SIAM, Philadelphia, 1989.

J. Haslinger, R. Kucera, J. Riton, and T. Sassi. A domain decomposition method for two-body contact problems with Tresca friction. *Adv. Comput. Math.*, 40:65–90, 2014.

J. Koko. Convergence analysis of optimization-based domain decomposition methods for a bonded structure. *Appl. Numer. Math.*, 58:69–87, 2008a.

J. Koko. Uzawa block relaxation domain decomposition method for the two-body contact problem with Tresca friction. *Comput. Methods. Appl. Mech. Engrg.*, 198:420–431, 2008b.

Koko, J. An optimization based domain decomposition method for a bonded structure. *Math. Models Methods Appl. Sci.*, 12:857–870, 2002.