Additive Average Schwarz Method for a Crouzeix-Raviart Finite Volume Element Discretization of Elliptic Problems

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1 Introduction

In this paper we introduce an additive Schwarz method for a Crouzeix-Raviart Finite Volume Element (CRFVE) discretization of a second order elliptic problem with discontinuous coefficients, where the discontinuities are inside subdomains and across and along subdomain boundaries. For recent work addressing domain decomposition methods for such problems, cf. Spillane et al. [2014], Galvis and Efendiev [2010] and references therein. Depending on the distribution of the coefficient in the model problem, the parameters describing the GMRES convergence rate of the proposed method depend linearly or quadratically on the mesh parameters H/h.

The CRFVE method was first introduced by Chatzipantelidis [1999] and investigated further in Rui and Bi [2008].

Additive Schwarz Methods (ASM) for solving elliptic problems discretized by the finite element method have been studied thoroughly, cf. Smith et al. [1996], Toselli and Widlund [2005], but ASMs for conforming FVE discretization have only been consider in Chou and Huang [2003], Zhang [2006]. For the CR finite element discretization, there exists several results for second order elliptic problems; cf. Sarkis [1997], Rahman et al. [2005], Brenner [1996], Marcinkowski [1999]. In the CRFVE case, ASMs have not been studied.

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2 Discrete Problem

We consider the following elliptic boundary value problem

$$
\begin{aligned}
-\nabla \cdot (\alpha(x)\nabla u) &= f \qquad && \text{in } \Omega, \\
u &= 0 \qquad && \text{on } \partial\Omega,
\end{aligned} \tag{1}
$$

where Ω is a bounded convex domain in $\mathbb{R}^2$ and $f \in L^2(\Omega)$. The coefficient $\alpha(x) > a_0 > 0$ has the property $\alpha \in W^{1,\infty}(D_j)$ with respect to a nonoverlapping partitioning of Ω into open, connected Lipschitz polytopes $\mathcal{D} := \{D_j : j = 1, \ldots, n\}$ such that $\bar{\Omega} = \bigcup_{j=1}^{n} \bar{D}_j$. We assume that the restriction of the coefficient α to D_j has the property $|\alpha|_{1,\infty,D_j} \leq C$ for all $j = 1, \ldots, n$, i.e., we assume that locally the coefficient is smooth and not too much varying. For simplicity of presentation we require that $\alpha \geq 1$. This last property can always be achieved by scaling (1).

3 The CRFVE method

In this section we present the Crouzeix-Raviart finite element (CRFE) and finite volume (CRFVE) discretizations of a model second order elliptic problem with discontinuous coefficients inside and across prescribed substructures boundaries.

We assume that there exists another nonoverlapping partitioning of Ω into open, connected Lipschitz polytopes Ω_i such that $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i$. We also assume that these subdomains form a coarse triangulation of the domain which is shape regular as in Brenner and Sung [1999]. We define the sets of Crouzeix-Raviart (CR) nodal points Ω_h^{CR}, $\partial\Omega_h^{CR}$, Ω_{ih}^{CR} and $\partial\Omega_{ih}^{CR}$ as the midpoints of edges of elements in T_h corresponding to Ω, $\partial\Omega$, Ω_i and $\partial\Omega_i$, respectively.

Now we introduce a quasi-uniform triangulation $\mathcal{T}_h$ of Ω consisting of closed triangle elements such that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$. Let h_K be the diameter of K and define $h = \max_{K \in \mathcal{T}_h} h_K$ as the largest diameter of the triangles $K \in \mathcal{T}_h$. We assume that the triangulation is defined in such way that ∂K's are aligned with ∂D_j's. This implies that the coefficient $\alpha(x)$ has the property that $\alpha \in W^{1,\infty}(K)$ for all $K \in \mathcal{T}_h$.

Using this triangulation $\mathcal{T}_h$ we may now introduce a dual mesh $\mathcal{T}_h^*$ with elements called control volumes. Let z_K be an interior point of $K \in \mathcal{T}_h$, we connect it with straight lines to the vertices of K such that K is partitioned into three subtriangles, K_e for each edge $e \in \partial K \cap \Omega$ interior to Ω. Denote this new finer triangulation of Ω by $\widetilde{\mathcal{T}_h}$. With each edge e we associate a corresponding control volume b_e consisting of the two subtriangles of $\widetilde{\mathcal{T}_h}$ which have e as an common edge, cf Figure 1. Define $\mathcal{T}_h^* = \{b_e : e \in \mathrm{E}_h^{\mathrm{in}}\}$ to be the set of all such control volumes, where E_h^{in} is the set of all interior edges of the elements in T_h.

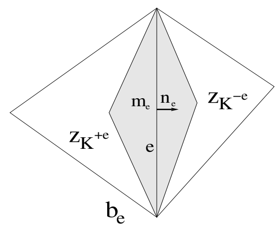


Fig. 1: The control volume b_e for an edge e which is the common edge to the triangles K^{+e} and K^{-e}. Here m_e is the midpoint of e, n_e normal unit vector to e, $z_{K^{+e}}$ and z_{K-e} are the interior points of the the triangles K^{+e} and K^{-e} which share the edge e.

Let V_h be the nonconforming CR finite element space defined on the triangulation $\mathcal{T}_h$,

$$V_h = V_h(\Omega) := \{u \in L^2(\Omega) : v_{|K} \in P_1, \quad K \in T_h \quad v(m) = 0 \quad m \in \partial\Omega_h^{CR}\},$$

and let V_h^* be its dual control volume space

$$V_h^* = V_h^*(\Omega) := \{u \in L^2(\Omega) : v_{|b_e} \in P_0, \quad b_e \in T_h^* \quad v(m) = 0 \quad m \in \partial\Omega_h^{CR}\}.$$

Obviously, $V_h^* = \text{span}\{\chi_e(x) : e \in \mathrm{E}_h^{\text{in}}\}$, where $\{\chi_e\}$ are the characteristic functions of the control volumes $\{b_e\}$. Now, let $I_h^* : V_h \to V_h^*$ be the standard interpolation operator, i.e.

$$I_h^* u = \sum_{e \in \mathrm{E}_h^{in}} u(m_e)\chi_e.$$

We may then define the CRFVE approximation u_h of (1) as the solution to the following problem: Find $u_h \in V_h$ such that

$$a_h(u_h, I_h^* v) = (f, I_h^* v), \qquad v \in V_h \tag{2}$$

where the bilinear form is defined as

$$a_h(u, v) = -\sum_{e \in \mathrm{E}_h^{in}} v(m_e) \int_{\partial b_e} \alpha(x) \nabla u \cdot \mathbf{n}\, ds \qquad u \in V_h, v \in V_h^*.$$

where $\mathbf{n}$ is the outward unit normal vector of the control volume b_e.

The corresponding CR finite element bilinear form is defined as: $a(u, v) = \sum_{K \in T_h} \int_K \alpha(x) \nabla u \cdot \nabla v\, dx$, and we define the energy norm induced by $a(\cdot, \cdot)$ as

$$\| \cdot \|_a = \sqrt{a(\cdot, \cdot)}. \tag{3}$$

4 The GMRES Method

The linear system of equations which arises from problem (2) is in general nonsymmetric. We may solve such a system using a preconditioned GMRES method; cf. Saad and Schultz [1986] and Eisenstat et al. [1983]. This method has proven to be quite powerful for a large class of nonsymmetric problems. The theory originally developed for $L^2(\Omega)$ in Eisenstat et al. [1983] can easily be extended to an arbitrary Hilbert space; see Cai and Widlund [1989].

In this paper, we use GMRES to solve the linear system of equations

$$Tu_h = g, \tag{4}$$

where T is a nonsymmetric, nonsingular operator, $g \in V_h$ is the right hand side and $u_h \in V_h$ is the solution vector. The formulation of T will be given in the next section.

The main idea of the GMRES method is to solve a least square problem in each iteration, i.e. at step m we approximate the exact solution $u_h = T^{-1}g$ by a vector $u_m \in \mathcal{K}_m$ which minimizes the a-norm (energy norm) of the residual, cf. (3), where $\mathcal{K}_m$ is the m-th Krylov subspace defined as $\mathcal{K}_m = \operatorname{span}\left\{r_0, Tr_0, \cdots T^{m-1}r_0\right\}$ and $r_0 = g - Tu_0$. In other words, z_m solves

$$\min_{z \in \mathcal{K}_m} \|g - T(u_0 + z)\|_a.$$

Thus, the m-th iterate is $u_m = u_0 + z_m$.

The convergence rate of the GMRES method is usually expressed in terms of the following two parameters

$$c_p = \inf_{u \neq 0} \frac{a(Tu, u)}{\|u\|_a^2} \text{ and } C_p = \sup_{u \neq 0} \frac{\|Tu\|_a}{\|u\|_a}.$$

The decrease of the norm of the residual in a single step is described in the next theorem.

Theorem 1 (Eisenstat-Elman,Schultz). *If $c_p > 0$, then the GMRES method converges and after m steps, the norm of the residual is bounded by*

$$\|r_m\|_a \leq \left(1 - \frac{c_p^2}{C_p^2}\right)^{m/2} \|r_0\|_a, \tag{5}$$

where $r_m = g - Tu_m$.

5 An Additive Average Method

In this section we introduce the additive Schwarz method for the discrete problem (2) and provide bounds on the convergence rate, both for the cases of symmetric and nonsymmetric preconditioners.

5.1 Decomposition of $V_h(\Omega)$

We decompose the original space into

$$V_h(\Omega) = V_0(\Omega) + V_1(\Omega) + \cdots + V_N(\Omega), \tag{6}$$

where for $i = 1, \ldots, N$ we have defined $V_i(\Omega)$ as the restriction of $V_h(\Omega)$ to Ω_i with functions vanishing on $\partial\Omega_{ih}^{CR}$ and as well as on the other subdomains. The coarse space $V_0(\Omega)$ is defined as the range of the interpolation operator I_A. For $u \in V_h(\Omega)$, we let $I_A u \in V_h(\Omega)$ be defined as

$$I_A u := \begin{cases} u(x), & x \in \partial\Omega_{ih}^{CR} \\ \bar{u}_i, & x \in \Omega_{ih}^{CR} \end{cases} \qquad i = 1, \ldots, N, \tag{7}$$

where

$$\bar{u}_i := \frac{1}{n_i} \sum_{x \in \partial\Omega_{ih}^{\mathrm{CR}}} u(x). \tag{8}$$

Here n_i is the number of nodal points of $\partial\Omega_{ih}^{CR}$.

We also assume that $\mathcal{T}_h(\Omega_i)$ inherits the shape regular and quasi-uniform triangulation for each Ω_i with mesh parameters h_i and $H_i = diam(\Omega_i)$. The layer along $\partial\Omega_i$ consisting of unions of triangles $K \in \mathcal{T}(\Omega_i)$ which touch $\partial\Omega_i$ is denoted as Ω_i^δ. Corresponding to each layer we define the maximum and minimum values of the coefficient α as

$$\overline{\alpha}_i := \sup_{x \in \bar{\Omega}_i^\delta} \alpha(x) \qquad \text{and} \qquad \underline{\alpha}_i := \inf_{x \in \bar{\Omega}_i^\delta} \alpha(x),$$

respectively.

For $i = 0, \ldots, N$ we define the two types of projection like operators $T_i^{(k)} \colon V_h \to V_i(\Omega)$, $k = 1, 2$ as

$$a(T_i^{(1)} u, v) = a_h(u, I_h^* v) \qquad \forall v \in V_i(\Omega), \tag{9}$$

for the symmetric preconditioner, and

$$a_h(T_i^{(2)} u, v) = a_h(u, I_h^* v) \qquad \forall v \in V_i(\Omega), \tag{10}$$

for the non-symmetric preconditioner. Each of these problems have a unique solution. We now introduce

$$T_A^{(k)} := T_0^{(k)} + T_1^{(k)} + \cdots + T_N^{(k)}, \qquad k = 1, 2, \tag{11}$$

which allow us to replace the original problem, respectively for $k = 1$ and $k = 2$, by the equation

$$T_A^{(k)} u = g^{(k)}, \tag{12}$$

where $g^{(k)} = \sum_{i=0}^{N} g_i$ and $g_i^{(k)} = T_i^{(k)} u$. Note that $g_i^{(k)}$ may be computed without knowing the solution u of (2).

Theorem 2. *There exists $h_0 > 0$ such that for all $h < h_0$, $k = 1, 2$, and $u \in V_h$*

$$\|T^{(k)} u\|_a \le C \|u\|_a,$$
$$a(T^{(k)} u, u) \ge c \max_i \frac{\overline{\alpha}_i}{\underline{\alpha}_i} \left(\frac{H_i}{h_i} \right)^{-2} a(u, u),$$

where C, c are positive constants independent of α, $\frac{\overline{\alpha}_i}{\underline{\alpha}_i}$, h_i and H_i for $i = 1, \ldots, N$.

For certain distributions of α we may improve the estimate.

Proposition 1. *There exists $h_0 > 0$ such that for all $h < h_0$, $u \in V_h$ and $\underline{\alpha}_i \le \alpha(x)$ in $\Omega_i \setminus \Omega_i^\delta$*

$$\|T^{(k)} u\|_a \le C \|u\|_a,$$
$$a(T^{(k)} u, u) \ge c \max_i \frac{\overline{\alpha}_i}{\underline{\alpha}_i} \left(\frac{H_i}{h_i} \right)^{-1} a(u, u) \qquad \forall u \in V_h,$$

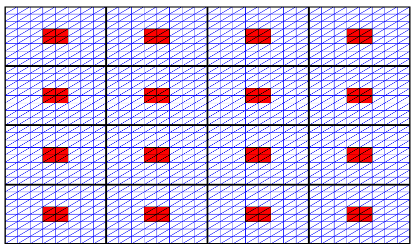
where C, c are positive constants independent of α, $\frac{\overline{\alpha}_i}{\underline{\alpha}_i}$, h_i and H_i for $i = 1, \ldots, N$.

6 Numerical results

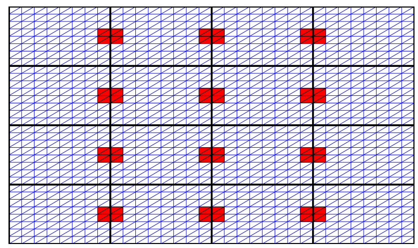
In this section we present some preliminary numerical results for the proposed method with the symmetric preconditioner, i.e. for $k = 1$ in (12). All experiments are done for problem (1) on a unit square domain $\Omega = (0, 1)^2$. The coefficient α is equal to $2 + \sin(100\pi x)\sin(100\pi y)$ except for the areas marked with red where α equals $\alpha_1(2 + \sin(100\pi x)\sin(100\pi y))$. The right hand side is chosen as $f = 1$.

The numerical solution is found by using the Generalized minimal residual method (GMRES). We run the method until the l_2 norm of the residual is reduced by a factor 10^6, i.e., as soon as $\|r_i\|_2/\|r_0\|_2 \leq 10^{-6}$. For each of the problems under consideration the number of iterations until convergence for different values of α_1 are shown in Table 1.

The numerical results from our two examples shows that the performance of the method agrees with the theory. If the inclusions are in the interior of the subdomains the method is completely insensitive to any discontinuities in the coefficient, while if the inclusions are on the subdomain layer the method depends strongly on the jumps in the coefficient.



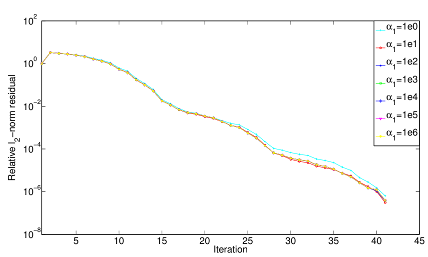
Problem 1.



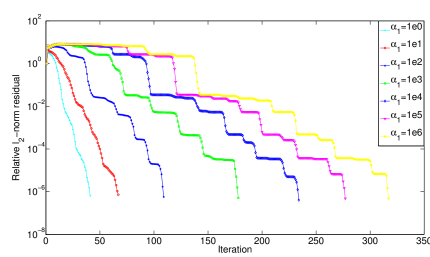
Problem 2.

α_1	$1e0$	$1e1$	$1e2$	$1e3$	$1e4$	$1e5$	$1e6$
Problem 1.	40	40	40	40	40	40	40
Problem 2.	40	66	108	177	233	276	316

Table 1: Number of iterations until convergence for the solution of (1) for different values of α_1 in the distributions of the coefficient α given in Figures 2a and 2b.



Problem 1.



Problem 2.

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