Algebraic multigrid for discontinuous Galerkin methods using local transformations

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Abstract In this paper we present an algebraic multigrid for discontinuous Galerkin methods. Coarser grid levels are created by applying a semi-coarsening approach based on an edge-coloring of the matrix-graph. The grid-transfer uses local basis transformations between the polynomial bases of neighboring elements. Along the coarsening process, the implicit block structure of the linear system is preserved. High frequency errors are reduced by applying an overlapping block smoother. The overlapping patches are constructed and locally weighted depending on the problem type. As model problems serve the Poisson and Stokes equations. The multigrid method is implemented in C++ using the DUNE framework.

Key words: algebraic multigrid, discontinuous Galerkin, local transformations

1 Introduction

Discontinuous Galerkin methods are popular discretization methods for partial differential equations for over a decade. For the resulting linear system, the need arises for robust and efficient solvers. A geometric multigrid algorithm which maintains the properties of the discretization along the grid hierarchy has been presented in Johannsen [2005]. The grid transfer is based on an L^2-projection and an overlapping element block smoother is applied on each level. For cases where the construction of a geometric grid hierarchy is not feasible, certain classes of algebraic multigrid methods have been developed. In Ayuso de Dios and Zikatanov [2009], an iterative method has

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been proposed, based on the splitting of the function space into two non-overlapping subspaces. On those spaces, the problem can be solved more efficiently. Another approach has been followed in Prill et al. [2009]. There, an algebraic multigrid method has been presented which uses a smoothed aggregation method to form the coarser grid levels. A combination of both approaches has been developed in Bastian et al. [2012]. The algebraic multigrid being described there uses a projection of the discontinuous space onto the conforming subspace of linear elements. An agglomeration strategy is employed to create the smoother and the coarse grid levels. This strategy drops the block structure of the linear system and loses the information of the discontinuous Galerkin discretization on coarser grid levels. In addition, it is not applicable to the Stokes equation since the inf-sup stability is lost on the first order conforming subspace due to the equal order discretization of velocity and pressure.

The aim of this paper is to develop and evaluate an algebraic multigrid method for discontinuous Galerkin discretizations, which preserves and uses the block structure on each grid level and can be applied to different problems, including the Stokes equation. We follow the general structure of the geometric multigrid of Johannsen [2005] but also take ideas from Bastian et al. [2012] into account. The derivation of the method uses the Poisson equation and includes comments on the differences for the Stokes equation when applicable. The paper is structured as follows: Sect. 2 provides a short introduction to the discretization of the Poisson equation and the resulting linear system. In Sect. 3 the algebraic multigrid algorithm is presented, including the transfer between different grid levels and the smoothing operator. The algorithm is evaluated in Sect. 4 and finally a short conclusion is given.

2 Preliminaries

We describe our method using the discontinuous Galerkin discretization for the Poisson equation, cf. Arnold et al. [2001/02]. Let $T_h(\Omega) := \{\Omega_0, \ldots, \Omega_{N-1}\}$ define a triangulation of the domain Ω with the size parameter $h \in \mathbb{R}$. The broken Sobolev space is defined as $V_h := \{u \in L^2(\Omega) | u_{|\Omega_i} \in P(\Omega_i)\}$ for some polynomial spaces $P(\Omega_i)$. The discontinuous Galerkin formulation of the Poisson equation with homogeneous Dirichlet boundary conditions reads: find $u_h \in V_h$ such that $a_\epsilon(u_h, v) = f(v)$ holds for all $v \in V_h$ (cf. Arnold et al. [2001/02] for a derivation and definition of a_ϵ). The method parameter is denoted by ϵ and the penalty parameter by $\eta \in \mathbb{R}$. For each grid element Ω_i, we assume there is a diffeomorphism $\mu_i : \mathbb{R}^n \to \mathbb{R}^n$ with $\mu_i(\hat{\Omega}) = \Omega_i$, mapping local coordinates on a reference element $\hat{\Omega}$ to global coordinates on Ω. Next we introduce local polynomial basis functions on the reference element:

$$\phi_i : \mathbb{R}^n \to \mathbb{R}, \quad i \in \{0, \ldots, N_b - 1\} \tag{1}$$

In order to simplify the description, we assume the same local basis on all elements. Note that this restriction is not essential. Using the local to global transformations, we define the basis function in global coordinates as $\phi_{i,k} = \phi_i \circ \mu_k^{-1}$. Introducing a representation of u_h and v with respect to the global basis functions in the discontinuous Galerkin formulation yields the linear system for the Poisson equation:

$$\begin{aligned} Ax = b, \qquad & A = (A_{kl})_{k,l} \\ A_{kl} = (a_{ij}^{kl})_{ij} \in \mathbb{R}^{N_b \times N_b}, \qquad & a_{ij}^{kl} = a_\epsilon(\phi_{j,l}, \phi_{i,k}) \end{aligned} \tag{2}$$

Besides for the Poisson equation, we also construct the multigrid method for the Stokes equation. We will not present its discontinuous Galerkin formulation here, but refer to Rivière and Girault [2006]. We block the degrees of freedom for pressure and velocity element wise, which again yields a sparse block linear system.

3 Algorithm

General

The proposed algebraic multigrid method is a method to solve a linear sparse block system $Ax = b$ resulting from a discontinuous Galerkin discretization using only geometric information on the finest grid level. The grid levels are numbered from coarse to fine with $0, \ldots, L$, such that 0 denotes the coarsest grid level. By $N_l \in \mathbb{N}$ we denote the number of elements on level l. We will mark matrices and vectors with the level they are associated with. If the level index is missing, the matrix or vector refers to the finest level, if not stated otherwise. On each grid level but L, we assume there is a prolongation operator P^l mapping a coefficient vector from grid level l to the next finer level $l+1$. The restriction R^l of a vector from level $l+1$ to level l is accomplished using the transposed of the prolongation $R^l := (P^l)^t$. We compute the coarse grid matrices recursively from the finest matrix by applying the Galerkin product $A^{l-1} = R^{l-1} A^l P^{l-1}$. To reduce oscillating error frequencies, we apply the smoother S^l on level l. Both, the prolongation and the smoother are described in the remainder of this section.

Grid transfer

The spaces on coarse levels are constructed recursively as subspaces of the space at the next finer level using a semi coarsening approach. A semi coarsening can be constructed based on a matching in the block matrix graph of the block matrix A. The graph $G(A) = (V(A), E(A))$ consists of the nodes and the edges:

$$\begin{aligned} V(A) &= \{0, \dots, N_L - 1\} \\ E(A) &= \{(i,j) \in V(A) \times V(A) : i < j \wedge A_{ij} \neq 0\} \end{aligned} \tag{3}$$

Since the sparsity pattern of A is symmetric, $G(A)$ is undirected. In the following, when selecting edges for coarsening, we only consider strong edges $E_s(A) \subset E(A)$. We divide the edges into weak and strong ones, using the same criterion as in Bastian et al. [2012]. The strength of an edge $(i,j) \in E$ is defined as

$$\rho((i,j)) := \frac{\|A_{ij}\|\|A_{ji}\|}{\|A_{ii}\|\|A_{jj}\|} \tag{4}$$

An edge is called strong if its strength is greater than β times the maximum strength among its neighbors, for a constant $\beta \in [0,1]$. The selection of disjoint strong edges corresponds to finding a graph matching. A graph matching of strong edges is a subset of $E_s(A)$ such that every node is part of at most one edge.

The transfer between two grid levels is constructed using so called *shift matrices*, consisting of local basis transformations. For a pair of elements, we select the polynomial basis of the first element to be the basis of the combined element and embed the basis of the second element into the one of the first. The shift from l to k for two neighboring elements l and k is defined as

$$\begin{aligned} S_{kl} &:= M_k^{-1} \tilde{S}_{kl} \\ M_k &:= (m_{ij}^k)_{ij} \qquad m_{ij}^k := \langle \phi_{i,k}, \phi_{j,k} \rangle_{L^2(\Omega_k)} \\ \tilde{S}_{kl} &:= (\tilde{s}_{ij}^{kl})_{ij} \qquad \tilde{s}_{ij}^{kl} := \langle \phi_{i,k}, \phi_{j,l} \rangle_{L^2(\Omega_k)} \end{aligned} \tag{5}$$

These local shift matrices can be combined into a global sparse block matrix. Due to the coupling of neighboring elements in the discontinuous Galerkin discretization, the global shift matrix has the same sparsity pattern as the matrix A. The shift matrices on coarser grid levels can be obtained from the next finer level by successive shifting into neighboring elements.

Having selected a set of pairs to be coarsened, we can construct the prolongation matrix which transfers a block coefficient vector from the coarse to the fine level. For an element which has not been selected for coarsening, we keep its basis on the coarse grid and therefore set the associated prolongation block to the identity matrix. For each pair, we keep the basis of the first element and again set the block to the identity matrix. The basis of the other element gets transfered into the basis of the first using the local shift matrices described above. This approach yields the prolongation as a sparse block matrix P^l which can be defined as

$$P^l(\dots, x_e, \dots) := (\dots, [x_e, S_{fe} x_e], \dots) \tag{6}$$

for each selected pair (e,f). We define the corresponding restriction matrix as $R^l := (P^l)^T$. The domain Ω_i^l associated with an element i on level l is defined as the union of all elements on the finest grid level which have been

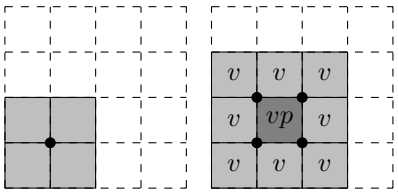

Fig. 1 Overlapping patches for the Poisson equation (left) and Stokes equation (right)

aggregated in element i. Accordingly, the function space V^l on level l is spanned by the bases of the elements of level l.

Smoother

The smoother S^l should reduce oscillating components of the error on the current level. As presented in Johannsen [2005], we use an additive and multiplicative Schwarz method as a smoother. Let $V_i^l \subset V^l$ define a subspace of V^l for each $i \in I^l$ with an index set I^l to be defined later. For each subspace $V_i^l, i \in I^l$, we solve $a^l(u_k^l + c_{k,i}^l, v) = f(v) \quad \forall v \in V_i^l$ for $c_{k,i}^l \in V_i^l$. The *additive Schwarz method* is then given by

$$u_{k+1}^l = u_k^l + \theta^l \sum_{i \in I^l} c_{k,i}^l \tag{7}$$

with a damping parameter $\theta^l \in \mathbb{R}$. In a similar way, the *multiplicative Schwarz method* can be introduced, where the updates are computed and applied successively.

In Johannsen [2005] different types of patches have been evaluated in a geometric multigrid setting. The results indicate that non-overlapping element block patches do not yield a robust smoother. Overlapping vertex based patches, depicted in Fig. 1, show robust smoothing behavior and are therefore used by the smoother in our method. It should be pointed out, that the geometric information about vertices and their connection to elements is only available at the finest level. We need to adopt this information along the coarsening process. This is done, by keeping only those vertices from level $l+1$, which have not become internal vertices between two elements. The connectivity information between the remaining vertices and their adjoining elements on the coarse level can be transfered from the fine level: a vertex on level l is connected to an element i on level l if it was connected to an element on level $l+1$ which has been aggregated into i. The smoother is said to fulfill the *smoothing property*, if $\|A^l(S^l)^\nu\| \leq C\eta(\nu)$ with a function $\eta(\nu) \to 0$ for $\nu \to \infty$.

For the Poisson equation, we set I^l to be the index set of grid vertices on level l in the algebraic sense. V_i^l is the linear subspace spanned by the degrees of freedom associated with an element which is connected to the

grid vertex i. The numerical results in Johannsen [2005] indicate for the one dimensional problem using the NIPG method, that the additive smoother fulfills the smoothing property with $1/\nu$.

For the Stokes equation, in addition to the vertex based patches, we need to take into account the saddle point structure of the problem. We adopt the idea of the Vanka type smoother from Vanka [1986], where, in a staggered grid context, a pressure degree of freedom is combined with all coupling velocity degrees of freedom. In addition we include the vertex based approach in order to construct a robust smoother. Combining both approaches in the context of the discontinuous Galerkin formulations, we set I^l to be the index set of elements on level l. V_i^l is the linear subspace spanned by the degrees of freedom associated with an element which shares a grid vertex with element i (see Fig. 1). Based on experimental results, we apply a different damping factor depending on the position of an element inside the patch. Theoretic results from Schöberl and Zulehner [2003] and numerical experiments indicate that for Stokes SIPG, the additive smoother fulfills the smoothing property with at least $1/\sqrt{\nu}$.

4 Evaluation

We implemented the algebraic multigrid method using the *Distributed and Unified Numerics Environment (DUNE)* (see Bastian et al. [2008]), using the PDELab toolbox (see Bastian et al. [2010]) for the PDE discretization. First, we apply our method to a two dimensional Poisson problem with $\Omega = [0,1]^2$ on a structured grid with rectangular elements, in order to reproduce the results given in Johannsen [2005] for a geometric multigrid method. As local basis functions we use an orthogonalized Q_k basis, with $Q_k := \{(x,y) \mapsto x^{\alpha_x} y^{\alpha_y} : \alpha_x, \alpha_y \in \mathbb{N}, \alpha_x, \alpha_y \leq k\}$. For the following tests, we set $k := 2$. We use a NIPG discretization with different penalties and different sizes of the finest grid level. The penalty ranges from 10^{-3} to 10^6 and the fine grid size is increased by successive uniform refinement starting with a size of 5×5. The convergence rate is measured as

$$\rho := \left(\frac{\|d_{20}\|_2}{\|d_{10}\|_2} \right)^{\frac{1}{10}} \quad , \tag{8}$$

where d_i denotes the defect in iteration i. We apply the additive method with damping $\theta = \frac{1}{2}$ and $\nu = 4$ pre- and post smoothing steps. The multiplicative method is applied with $\theta = 1$ and $\nu = 1$. The results can be seen in the second column of Fig. 2. In this Fig. and in the following, graphs with higher convergence rates correspond to finer grid sizes. It can be observed, that the general convergence behavior of the geometric method can be reproduced, while producing better convergence rates for higher penalties.

In the next test, we apply our method to the SIPG discretization of the Poisson equation. The test parameters are the same as in the previous test. The convergence results can be seen in the third column of Fig. 2. We observe similar convergence behavior as in the NIPG case. The method does not converge for a penalty less than $\sigma_0 < 10$, which corresponds to the theoretic findings in Arnold et al. [2001/02].

Next, we use the method as a preconditioner in a BiCGSTAB solver for a second order NIPG discretization on different unstructured grids. For different values of h, we create triangulations of the unit sphere and unit cube using tetrahedral elements. We use the multiplicative smoother and stop the iteration at a relative defect reduction of 10^{-10}. The results can be seen in Table 1.

Finally, we test for NIPG and SIPG discretizations of the Stokes equation on the unit square. We choose the orthogonalized Q_k basis for the velocity components and an orthogonalized P_{k-1} basis for the pressure, where $P_{k-1} := \{(x, y) \mapsto x^{\alpha_x} y^{\alpha_y} : \alpha_x, \alpha_y \in \mathbb{N}, \alpha_x + \alpha_y \leq k - 1\}$. Again we use a structured grid with rectangular elements, choose $k = 2$ and apply the method with

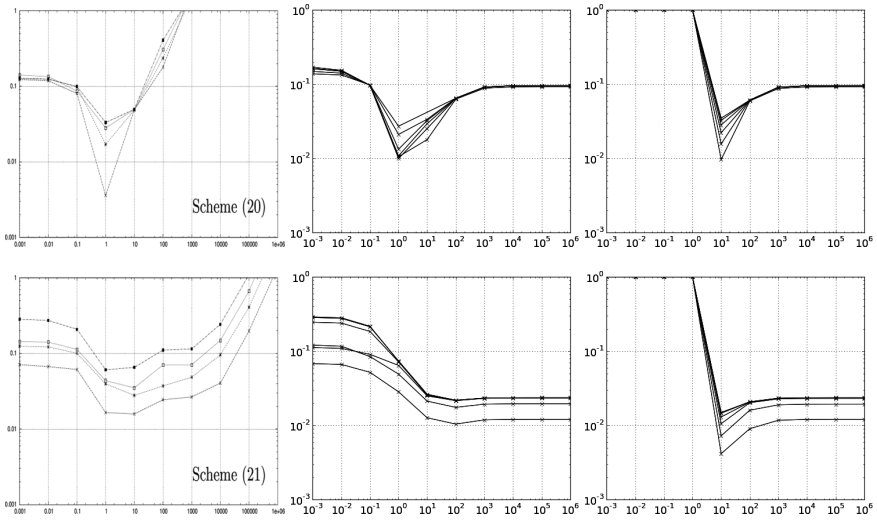

Fig. 2 convergence rates for the Poisson equation, left: NIPG method using the geometric multigrid from Johannsen [2005], center: NIPG method using our multigrid, right: SIPG method using our multigrid, top: additive smoother, bottom: multiplicative smoother

	unit sphere					unit cube				
elements	2104	8270	33418	139572	547038	2406	9386	38202	154194	635216
levels	11	11	10	10	10	11	12	13	12	11
iterations	3	3	3	3	4	3	3	3	3	3

Table 1 results for the Poisson equation using the NIPG method on unstructured tetrahedral grids with the multigrid as a preconditioner for the BiCGSTAB algorithm

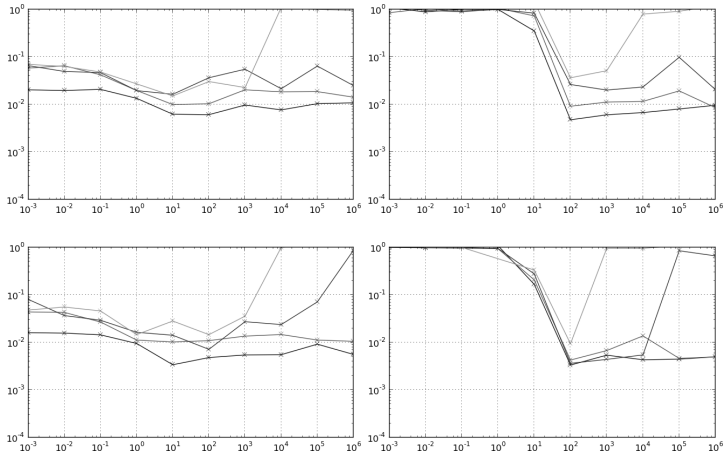

Fig. 3 convergence rates for the Stokes equation, left: NIPG method, right: SIPG method, top: additive smoother, bottom: multiplicative smoother

different penalties and grid sizes. We use the same damping parameters as before, but weight the velocity degrees of freedom differently depending on their local patch position when applying an update. The weight for the central element of a patch is set to $\frac{1}{2}$ and the weight for the outer elements is set to $\frac{1}{2m}$, where m denotes the number of outer elements in the patch. Our method is used as a preconditioner for the BiCGSTAB algorithm. The results can be seen in Fig. 3. We observe increased convergence rates when compared to the Poisson equation. In addition, we observe larger convergence rates for finer grids and larger penalties.

5 Conclusion

We proposed an algebraic multigrid method for the discontinuous Galerkin discretization of the Poisson and Stokes problem. It shows good convergence rates and is flexible enough to be applied to different types of problems, which are not covered in this paper. Currently, one drawback of the method is its large computational cost. This effort is dominated by the application of the overlapping block smoother on the finest grid level. Reducing this effort by applying different smoothing strategies has not yielded the desired convergence behavior so far. To avoid increasing convergence rates for finer grids and higher penalties, one can develop different local shift strategies. Instead of projecting into the local basis of a single element, one can investigate the possibility to project into a common basis on all aggregated elements. In or-

der to get a better understanding of the smoother, an investigation of the smoothing property might be worthwhile.

Acknowledgements This work was supported by the "German Academic Exchange Service" (DAAD) with the project 54570350 of the "German-Norwegian collaborative research support scheme".

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