Schwarz Preconditioner for the Stochastic Finite Element Method

Waad Subber[1] and Sébastien Loisel[2]

1 Abstract

The intrusive polynomial chaos approach for uncertainty quantification in numerous engineering problems constitutes a computationally challenging task. Indeed, Galerkin projection in the spectral stochastic finite element method (SSFEM) leads to a large-scale linear system for the polynomial chaos coefficients of the solution process. The development of robust and efficient solution strategies for the resulting linear system therefore is of paramount importance for the applicability of the SSFEM to practical engineering problems. The solution algorithms should be parallel and scalable in order to exploit the available multiprocessor supercomputers. Therefore, we formulate a two-level Schwarz preconditioner for the polynomial chaos based uncertainty quantification of large-scale computational models.

2 Introduction

For large-scale problems, domain decomposition techniques are a natural way to split the problem into smaller subproblems that can be solved in parallel on multiprocessors computers. To this end, stochastic versions of FETI-DP and BDDC domain decomposition techniques for uncertainty quantification of large-scale problems have been recently proposed in [2, 6, 7]. In this paper, we formulate two-level Schwarz domain decomposition technique for the solution of the large-scale linear system arising from the SSFEM discretization. In the stochastic Schwarz preconditioner, we partition the spatial domain and preserve all the couplings along the stochastic directions. Consequently, stochastic Dirichlet problems are defined and solved on each subdomain concurrently. The solution of these local problems are used to define the first level of the preconditioner. A coarse grid correction is added to the one-level preconditioner to provide a global mechanism to propagate information over the subdomains. This global exchange of information across the spacial and stochastic directions leads to a scalable preconditioner. It turns out that the one-level stochastic Schwarz preconditioner based on the mean properties can be viewed as a parallel generalization of the block-diagonal mean based preconditioner [3], whereby the associated deterministic problems are solved in parallel using the deterministic Schwarz preconditioner. For the numerical illustrations, a two dimensional stochastic elliptic PDE with spatially varying

School of Mathematical and Computer Sciences, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom w.subber@hw.ac.uk · s.loisel@hw.ac.uk

random coefficients is considered. The numerical scalability of the algorithm is investigated with respect to the geometric parameters and the strength of the input uncertainty, dimension and order of the stochastic expansion.

3 Mathematical Formulations

We consider the case of finite dimensional noise in a suitable probability space $(\Theta, \Sigma, \mathcal{P})$ [1]. That is we assume that there exist a finite set of independent and identically distributed random variables $\boldsymbol{\xi}(\theta) = \{\xi_1(\theta), \xi_2(\theta), \cdots, \xi_M(\theta)\}$ with joint probability density function $p(\boldsymbol{\xi}) = p_1(\xi_1)p_2(\xi_2)\cdots p_M(\xi_M)$ which can be used to paramatrize the input uncertainty. Consider the following stochastic boundary value problem: Find a random function $u(\mathbf{x}, \xi(\theta)) : \Omega \times \Gamma \to \mathbb{R}$ such that:

$$\begin{aligned} -\nabla \cdot (\kappa(\mathbf{x}, \xi(\theta))\nabla u(\mathbf{x}, \xi(\theta))) &= f(\mathbf{x}), \text{ in } \quad \Omega \times \Gamma, \\ u(\mathbf{x}, \xi(\theta)) &= 0, \qquad \text{on } \partial\Omega \times \Gamma, \end{aligned} \tag{1}$$

where $(\Omega \subset \mathbb{R}^d, d = 1, 2, 3)$ denotes a bounded domain with Lipschitz boundary $\partial\Omega$ and $\Gamma = \Gamma_1 \times \Gamma_2 \cdots \times \Gamma_M \subset \mathbb{R}^M$ is the support of the joint probability density function $p(\boldsymbol{\xi})$ of the random vector $\boldsymbol{\xi}(\theta)$. Here we assume that the input uncertainty $\kappa(\mathbf{x}, \boldsymbol{\xi}(\theta)) : \Omega \times \Gamma \to \mathbb{R}$ is a $\mathcal{P}$-almost surely bounded and strictly positive random field, that is

$$0 < \kappa_{min} \leq \kappa(\mathbf{x}, \boldsymbol{\xi}(\theta)) \leq \kappa_{max} < +\infty, \quad \text{a.e. in} \quad \Omega \times \Gamma. \tag{2}$$

The weak form of the stochastic boundary value problem (1), can be stated as: Find $u(\mathbf{x}, \boldsymbol{\xi}) \in V$ such that for all $v \in V$

$$\int_\Gamma \left(\int_\Omega \kappa(\mathbf{x}, \boldsymbol{\xi}) \nabla u(\mathbf{x}, \boldsymbol{\xi}) \nabla v(\mathbf{x}, \boldsymbol{\xi}) \mathrm{d}\mathbf{x} \right) p(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi} = \int_\Gamma \left(\int_\Omega f(\mathbf{x}) v(\mathbf{x}, \boldsymbol{\xi}) \mathrm{d}\mathbf{x} \right) p(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi}$$

where the tensor product function space $V = H_0^1(\Omega) \otimes L^2(\Gamma)$ is defined as

$$V = \{v(\mathbf{x}, \boldsymbol{\xi}(\theta)) : \Omega \times \Gamma \to \mathbb{R} \mid \|v\|_V^2 < \infty\} \ \subset H_0^1(\Omega) \otimes L^2(\Gamma), \tag{3}$$

here $H_0^1(\Omega)$ and $L^2(\Gamma)$ represent the deterministic Hilbert space and the space of second-order random variables, respectively. The energy norm $\| \cdot \|_V^2$ is given by

$$\|v(\mathbf{x}, \boldsymbol{\xi}(\theta))\|_V^2 = \int_\Gamma \left(\int_\Omega \kappa(\mathbf{x}, \boldsymbol{\xi}) |\nabla v(\mathbf{x}, \boldsymbol{\xi})|^2 \mathrm{d}\mathbf{x} \right) p(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi}. \tag{4}$$

4 Stochastic Process Representation

Let $\kappa_0(\mathbf{x})$ and $C_{\kappa\kappa}(\mathbf{x}_1, \mathbf{x}_2)$ denote the mean and covariance function of the input uncertainty, then the Karhunen-Loéve expansion (KLE) can be used to represent $\kappa(\mathbf{x}, \boldsymbol{\xi})$ as

$$\kappa(\mathbf{x},\theta) = \sum_{i=0}^{M} \kappa_i(\mathbf{x})\xi_i(\theta), \tag{5}$$

where $\xi_0(\theta) = 1$ and $\kappa_i(\mathbf{x}) = \sigma\sqrt{\lambda_i}\phi_i(\mathbf{x});\ i \geq 1$, here σ denotes the standard deviation of the input process and λ_i and $\phi_i(\mathbf{x})$ are the eigenpairs of the covariance kernel and can be obtained from the solution of the following integral equation

$$\int_{\Omega} C_{\kappa\kappa}(\mathbf{x}_1,\mathbf{x}_2)\phi_i(\mathbf{x}_1)\mathrm{d}\mathbf{x}_1 = \lambda_i\phi_i(\mathbf{x}_2), \tag{6}$$

The solution process (with *a priori* unknown mean and covariance function) can be approximated using the PC expansion as

$$u(\mathbf{x},\theta) = \sum_{j=0}^{N} u_j(\mathbf{x})\Psi_j(\boldsymbol{\xi}(\theta)), \tag{7}$$

where $N+1$ denote the total number of terms in PCE and $u_j(\mathbf{x})$ are the deterministic PC coefficients to be determined and $\Psi_j(\boldsymbol{\xi})$ are a set of multivariate orthogonal random polynomials with the following properties

$$\langle\Psi_0\rangle = \int_{\Gamma} \Psi_0(\boldsymbol{\xi})p(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = 1, \quad \langle\Psi_j\rangle = 0, j > 0, \quad \text{and} \quad \langle\Psi_j\Psi_k\rangle = \delta_{jk}\langle\Psi_j^2\rangle.$$

5 The Stochastic Finite Element Discretization

Let $\mathcal{T}_h$ denote the triangulation of the physical domain Ω with a maximum element size h, and let the associated finite element space $\mathcal{X}_h \subset H_0^1(\Omega)$ be spanned by the traditional nodal basis functions $\{\phi_l(\mathbf{x})\}_{l=1}^{L}$. Further, for the stochastic discretization, let $\mathcal{Y}_p \subset L_2(\Gamma)$ be a finite dimensional space spanned by the PC basis functions $\{\Psi_j(\boldsymbol{\xi})\}_{j=0}^{N}$ in the random variables $\boldsymbol{\xi}$. Thus, the approximate SSFEM solution u_{hp} in the discrete tensor product space $\mathcal{X}_h \otimes \mathcal{Y}_p \subset H_0^1(\Omega) \otimes L_2(\Gamma)$ can be expressed as

$$u_{hp}(\mathbf{x},\boldsymbol{\xi}) = \sum_{j=0}^{N}\sum_{l=1}^{L} u_{jl}\phi_l(\mathbf{x})\Psi_j(\boldsymbol{\xi}). \tag{8}$$

Using (5) and (8), we can translate the stochastic weak form defined in (3) into the following coupled set of deterministic linear system

$$\sum_{j=0}^{N}\sum_{i=0}^{M}\sum_{l=1}^{L} u_{jl}\left(\int_{\Gamma} \boldsymbol{\xi}_i\Psi_j(\boldsymbol{\xi})\Psi_k(\boldsymbol{\xi})p(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}\right)\left(\int_{\Omega} \kappa_i(\mathbf{x})\nabla\phi_l(\mathbf{x})\cdot\nabla\phi_m(\mathbf{x})\mathrm{d}\mathbf{x}\right) =$$
$$\int_{\Gamma}\left(\int_{\Omega} f(\mathbf{x})\phi_m(\mathbf{x})\mathrm{d}\mathbf{x}\right)\Psi_k(\boldsymbol{\xi})p(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \quad m = 1,\cdots,L, \quad k = 0,\cdots,N \tag{9}$$

The linear system arising from (9) can be expressed as follows

$$\sum_{i=0}^{M} \mathbf{A}^{(i)}\mathbf{U}\mathbf{C}^{(i)} = \mathbf{F}, \tag{10}$$

where we define

$$\mathbf{A}^{(i)}_{lm} = \int_{\Omega} \kappa_i \nabla\phi_l \cdot \nabla\phi_m \mathrm{d}\mathbf{x}, \quad \mathbf{C}^{(i)}_{jk} = \int_{\Gamma} \boldsymbol{\xi}_i \Psi_j(\boldsymbol{\xi})\Psi_k(\boldsymbol{\xi})p(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}. \tag{11}$$

$$F_{mk} = \int_{\Gamma} \left(\int_{\Omega} f(\mathbf{x})\phi_m(\mathbf{x})\mathrm{d}\mathbf{x} \right) \Psi_k(\boldsymbol{\xi})p(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}. \tag{12}$$

Equation(10) can be vectorized by taking the vec$(\cdot)$ operator for the both sides leading to the following concise form

$$\mathcal{A}\mathcal{U} = \mathcal{F}, \tag{13}$$

where

$$\mathcal{A} = \sum_{i=0}^{M} \mathbf{C}^{(i)} \otimes \mathbf{A}^{(i)}, \quad \mathcal{U} = \mathrm{vec}(\mathbf{U}) \quad \text{and} \quad \mathcal{F} = \mathrm{vec}(\mathbf{F}). \tag{14}$$

6 Schwarz Preconditioner for Stochastic PDEs

In the Schwarz preconditioner for the stochastic problem, the physical domain Ω is partitioned into a number of overlapping subdomain $\{\Omega_s, 1 \leq s \leq S\}$ by splitting the vertices of the computational mesh. For each subdomain $\Omega_s \subset \Omega$, let $\mathbf{R}_s$ be a restriction matrix of size $n_s \times n$ (where n_s and n are the size of the subdomain and global unknowns) to extract the local nodal values from the global unknowns vector as

$$\mathbf{U}_s = \mathbf{R}_s\mathbf{U}, \tag{15}$$

applying the vec$(\cdot)$ operator to (15), leads to

$$\mathrm{vec}(\mathbf{U}_\mathrm{s}) = (\mathbf{I} \otimes \mathbf{R}_\mathrm{s})\mathrm{vec}(\mathbf{U}), \tag{16}$$

here $\mathbf{I}$ is $(N+1)\times(N+1)$ identity matrix. Let $\mathcal{U}_s = \mathrm{vec}(\mathbf{U}_\mathrm{s})$ and $\mathcal{R}_s = (\mathbf{I}\otimes\mathbf{R}_s)$ denote the stochastic subdomain nodal values and the stochastic restriction matrix, then (16) becomes

$$\mathcal{U}_s = \mathcal{R}_s\mathcal{U}, \tag{17}$$

Consequently, the stochastic stiffness matrix for subdomain Ω_s can be defined as a block extracted from the global stiffness matrix $\mathcal{A}$ as

$$\mathcal{A}_s = \mathcal{R}_s \mathcal{A} \mathcal{R}_s^T, \tag{18}$$

$$= (\mathbf{I} \otimes \mathbf{R}_s) \left(\sum_{i=0}^{M} \mathbf{C}^{(i)} \otimes \mathbf{A}^{(i)} \right) (\mathbf{I} \otimes \mathbf{R}_s^T), \tag{19}$$

$$= \sum_{i=0}^{M} \mathbf{C}^{(i)} \otimes \mathbf{A}_s^{(i)}, \tag{20}$$

Next, we define the one-level stochastic Schwarz preconditioner as a direct sum of the solution of the local stochastic Dirichlet problems as:

$$\mathcal{M}^{-1} = \sum_{s=1}^{S} \mathcal{R}_s^T \mathcal{A}_s^{-1} \mathcal{R}_s, \tag{21}$$

which can be expressed as follows

$$\mathcal{M}^{-1} = \sum_{s=1}^{S} (\mathbf{I} \otimes \mathbf{R}_s^T) \left(\sum_{i=0}^{M} \mathbf{C}^{(i)} \otimes \mathbf{A}_s^{(i)} \right)^{-1} (\mathbf{I} \otimes \mathbf{R}_s). \tag{22}$$

Remark 1. The stochastic Schwarz preconditioner has the same structure as the stochastic Neumann-Neumann preconditioner in [5].

Remark 2. The stochastic Schwarz preconditioner based on the mean properties can be obtained from (22) by setting $i = 0$ which gives

$$\mathcal{M}_0^{-1} = [\mathbf{C}^{(0)}]^{-1} \otimes \sum_{s=1}^{S} \mathbf{R}_s^T [\mathbf{A}_s^{(0)}]^{-1} \mathbf{R}_s, \tag{23}$$

where $\mathbf{A}_s^{(0)} = \mathbf{R}_s \mathbf{A}^{(0)} \mathbf{R}_s^T$ and $\mathbf{C}^{(0)} = \delta_{ij} \langle \Psi_i^2 \rangle$.

Remark 3. For one subdomain $S = 1$ and normalized PC basis functions, $\mathbf{C}^{(0)} = \mathbf{I}$, the mean-based Schwarz preconditioner defined in (23) becomes

$$\mathcal{M}_0^{-1} = \mathbf{I} \otimes [\mathbf{A}^{(0)}]^{-1}. \tag{24}$$

Remark 4. The one-level stochastic Schwarz preconditioner based on the mean properties is a generalization of the block-diagonal mean based preconditioner [3] whereby the associated deterministic problem is solved in parallel using the deterministic Schwarz preconditioner.

7 Coarse Grid Correction

Domain decomposition preconditioners can achieve a scalable performance provided that they are equipped with a coarse grid correction for global communication. To define a coarse problem for the stochastic Schwarz preconditioner, let $\mathbf{R}_0^T \in \mathbb{R}^{n_i \times n_0}$ be an interpolation matrix defined as

$$\mathbf{R}_0^T = \begin{bmatrix} \psi_1(\mathbf{x}_1) & \psi_2(\mathbf{x}_1) & \cdots & \psi_{n_0}(\mathbf{x}_1) \\ \psi_1(\mathbf{x}_2) & \psi_2(\mathbf{x}_2) & \cdots & \psi_{n_0}(\mathbf{x}_2) \\ \vdots & \vdots & \cdots & \vdots \\ \psi_1(\mathbf{x}_{n_i}) & \psi_2(\mathbf{x}_{n_i}) & \cdots & \psi_{n_0}(\mathbf{x}_{n_i}) \end{bmatrix} \tag{25}$$

where $\{\psi_i(\mathbf{x})\}_{i=1}^{n_0}$ is a set of linear basis functions, here n_0 denotes the dimension of the coarse space and $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{n_i})$ are the coordinates of the nodal points of the fine mesh. The corresponding *stochastic coarse space interpolation operator* can be defined as

$$\mathcal{R}_0 = \mathbf{I} \otimes \mathbf{R}_0, \tag{26}$$

and thus the coarse grid correction for the stochastic problem can be obtained as

$$\mathcal{A}_0 = \mathcal{R}_0^T \mathcal{A} \mathcal{R}_0. \tag{27}$$

According, the two-level stochastic Schwarz preconditioner can be defined by adding the coarse grid correction to the one-level preconditioner in (21) leading to

$$\mathcal{M}^{-1} = \mathcal{R}_0^T \mathcal{A}_0^{-1} \mathcal{R}_0 + \sum_{s=1}^{S} \mathcal{R}_s^T \mathcal{A}_s^{-1} \mathcal{R}_s. \tag{28}$$

Theorem 1. *There exists positive constants C and d that are independent of the geometric parameters (i.e. mesh size h, subdomain size H and the overlap distance δ) and the stochastic parameters (i.e. strength of randomness σ, dimension M and order p of the stochastic expansion), such that*

$$cond(\mathcal{M}^{-1}\mathcal{A}) \leq C(d+1)^2 \left(\frac{\kappa_{max}}{\kappa_{min}}\right)^2 \frac{H}{\delta}. \tag{29}$$

Proof. See [4]

8 Numerical Results

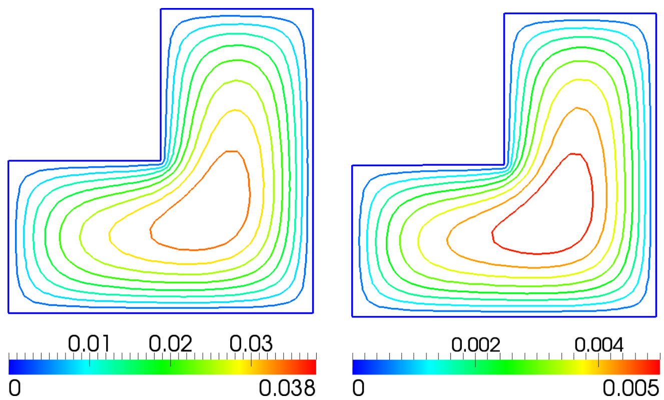
In this section, we illustrate the performance of the two-level stochastic Schwarz preconditioner defined in (28). In particular, we consider the following elliptic SPDE

$$\begin{aligned} -\nabla \cdot (\kappa(\mathbf{x},\theta)\nabla u(\mathbf{x},\theta)) &= f(\mathbf{x}), \text{ in } \quad \Omega \times \Theta, \\ u(\mathbf{x},\theta) &= 0, \qquad \text{on } \partial\Omega \times \Theta, \end{aligned} \tag{30}$$

where $f(\mathbf{x})$ denotes the source term taken as unity. The diffusivity coefficient $\kappa(\mathbf{x},\theta)$ is modelled as a uniform random field with an invariant mean and the following exponential covariance function

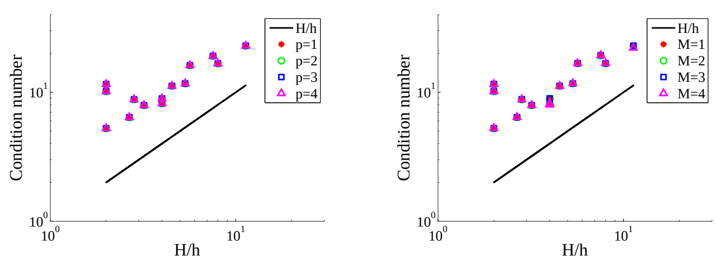
$$C_{\kappa\kappa}(\mathbf{x},\mathbf{y}) = \sigma^2 \exp\left(\frac{-|x_1 - y_1|}{b_1} + \frac{-|x_2 - y_2|}{b_2}\right), \tag{31}$$

Fig.(1(a)) and Fig.(1(b)) show the mean and standard deviation of the solution process. In Fig.(2(a)) and Fig.(2(b)), we show the condition number growth of the Schwarz preconditioner for fixed number of random variables $M = 2$ and fixed order $p = 2$, respectively, while increasing the global problem size by adding more subdomains with fixed problem size per subdomain. Table(1) and Table(2) show the condition number and iterations count of the preconditioned conjugate gradient solver equipped with Schwarz preconditioner with respect to dimension and order and coefficient of variation (CoV), respectively.



(a) mean (b) st. deviation

Fig. 1 The mean and standard deviation of the solution process



(a) $M = 2$ (b) $p = 2$

Fig. 2 Condition number growth with respect to fixed problem size per subdomain

9 Conclusion

A two-level Schwarz domain decomposition preconditioner is proposed for the iterative solution of the large-scale linear system arising from the stochastic

Table 1 Condition number and iterations count with respect to M and p

M	p	*cond*	*iter*
1	1	10.1642	17
	2	10.1706	19
	3	10.1725	19
	4	10.1733	19
2	1	10.1781	19
	2	10.1834	19
	3	10.1861	19
	4	10.1876	19
3	1	10.1785	19
	2	10.1842	19
	3	10.1873	19
	4	10.1892	19
4	1	10.1816	19
	2	10.1887	20
	3	10.1926	20
	4	10.1951	20

Table 2 Condition number and iterations count with respect to the CoV

$\frac{\sigma}{\mu}$	p	*cond*	*iter*
0.2	1	10.1760	19
	2	10.1812	19
	3	10.1841	19
	4	10.1860	19
0.3	1	10.1816	19
	2	10.1887	20
	3	10.1926	20
	4	10.1951	20
0.4	1	10.1871	19
	2	10.1959	20
	3	10.2006	20
	4	10.2035	20
0.5	1	10.1925	20
	2	10.2030	20
	3	10.2085	20
	4	10.2122	20

finite element discretization. The proposed preconditioner demonstrates a scalable performance with respect to the mesh parameters, strength of randomness, dimension and order of the stochastic expansion

References

[1] I. Babuska and P. Chatzipantelidis. On solving elliptic stochastic partial differential equations. *CMAME*, 191:4093–4122, 2002.

[2] D. Ghosh, Ph. Avery, and Ch. Farhat. A FETI-preconditioned conjugate gradient method for large-scale stochastic finite element problems. *IJNME*, 80:914–931, 2009.

[3] C. Powell and H. Elman. Block-diagonal preconditioning for spectral stochastic finite-element systems. *IMA-JNA*, 29:350–375, 2009.

[4] W. Subber and S. Loisel. Schwarz preconditioners for stochastic elliptic PDEs. *CMAME*, 272(0):34 – 57, 2014.

[5] W. Subber and A. Sarkar. Domain decomposition of stochastic PDEs: A novel preconditioner and its parallel performance. In *HPCS*, volume 5976, pages 251–268. Springer, 2010.

[6] W. Subber and A. Sarkar. Dual-primal domain decomposition method for uncertainty quantification. *CMAME*, 266(0):112 – 124, 2013.

[7] W. Subber and A. Sarkar. A domain decomposition method of stochastic PDEs: An iterative solution techniques using a two-level scalable preconditioner. *JCP*, 257, Part A(0):298 – 317, 2014.