A FETI-DP algorithm for saddle point problems in three dimensions

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1 Introduction

In [2, 7, 6], a new class of FETI-DP type domain decomposition algorithms was introduced and analyzed by the authors for solving incompressible Stokes equations in two dimensions. Both discontinuous and continuous pressures can be used in the mixed finite element discretization. In both cases, the indefinite system of linear equations can be reduced to a symmetric positive semi-definite system. Therefore, the preconditioned conjugate gradient method can be applied.

Both lumped and Dirichlet preconditioners have been studied in [2, 7, 6]. For the lumped preconditioner, it has been proved in [2] that the coarse level space can be chosen as simple as for solving scalar elliptic problems corresponding to each velocity component to achieve a scalable convergence rate. However, for the Dirichlet preconditioner, most existing FETI-DP and BDDC type algorithms [4, 1, 3] for Stokes problems use subdomain Stokes extensions in the preconditioners and the coarse level velocity space has to contain sufficient components to enforce divergence free subdomain boundary velocity conditions. Due to this divergence free requirement, the coarse space becomes very complicated, especially for three-dimensional problems as discussed in [3]. For the Dirichlet preconditioner introduced in [7, 6], an application of subdomain harmonic extension instead of Stoke extension in the preconditioner makes it possible to remove the divergence free constraints for the coarse level velocity space. Unfortunately, the analysis provided for the algorithms in [7, 6] still requires the divergence free constraints.

In this paper, we provide a new analysis for the algorithms in [7, 6], which can not only analyze both lumped and Dirichlet preconditioners in a same

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framework, but also remove the divergence free constraints for the Dirichlet preconditioner. We then extended this class of algorithms [2, 7, 6] to three dimensional problems; see [8] for more details.

2 Discretization, domain decomposition, and a reduced interface system

Let Ω be a bounded, three-dimensional polyhedral domain. We consider solving the following saddle point problem: find $\mathbf{u}^* \in \left(H_0^1(\Omega)\right)^3 = \{\mathbf{v} \in (H^1(\Omega))^3 \mid \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\}$ and $p^* \in L^2(\Omega)$, such that

$$\begin{cases} a(\mathbf{u}^*, \mathbf{v}) + b(\mathbf{v}, p^*) = (\mathbf{f}, \mathbf{v}), \forall \mathbf{v} \in \left(H_0^1(\Omega)\right)^3, \\ b(\mathbf{u}^*, q) \qquad\qquad\quad = 0, \qquad \forall q \in L^2(\Omega)\ , \end{cases} \tag{1}$$

where

$$a(\mathbf{u}^*, \mathbf{v}) = \int_\Omega \nabla \mathbf{u}^* \cdot \nabla \mathbf{v}, \quad b(\mathbf{u}^*, q) = -\int_\Omega (\nabla \cdot \mathbf{u}^*) q, \quad (\mathbf{f}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}.$$

The solution of (1) is not unique and the pressure p^* is determined up to an additive constant.

The domain Ω is partitioned into shape-regular rectangular elements of characteristic size h, and the Q_2-Q_1 Taylor-Hood mixed finite element is used to solve (1). The pressure finite element space, $Q \subset L^2(\Omega)$, is taken as the space of continuous piecewise trilinear functions while the velocity finite element space, $\mathbf{W} \in \left(H_0^1(\Omega)\right)^3$, is formed by the continuous piecewise triquadratic functions.

The finite element solution $(\mathbf{u}, p) \in \mathbf{W} \bigoplus Q$ of (1) satisfies

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}, \tag{2}$$

where A, B, and $\mathbf{f}$ represent, respectively, the restrictions of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $(\mathbf{f}, \cdot)$ to the finite-dimensional spaces $\mathbf{W}$ and Q. The solution of (2) always exists and is uniquely determined when the pressure is required to have a zero average.

The Q_2-Q_1 Taylor-Hood mixed finite element space $\mathbf{W} \times Q$ is inf-sup stable in the sense that there exists a positive constant β, independent of h, such that, in matrix/vector form,

$$\sup_{\mathbf{w} \in \mathbf{W}} \frac{\langle q, B\mathbf{w} \rangle^2}{\langle \mathbf{w}, A\mathbf{w} \rangle} \geq \beta^2 \langle q, Zq \rangle, \qquad \forall q \in Q/Ker(B^T). \tag{3}$$

Here the matrix Z represents the mass matrix defined on the pressure finite element space Q, i.e., for any $q \in Q$, $\|q\|_{L^2}^2 = \langle q, Zq \rangle$. It is easy to see, cf. [5, Lemma B.31], that Z is spectrally equivalent to $h^3 I$ for three-dimensional problems, where I represents the identity matrix of the same dimension.

The domain Ω is decomposed into N non-overlapping polyhedral subdomains Ω_i, $i = 1, 2, ..., N$. Each subdomain is the union of a bounded number of elements, with the diameter of the subdomain in the order of H. The nodes on the interface Γ of neighboring subdomains match across the subdomain boundaries and Γ is composed of subdomain faces, which are regarded as open subsets of Γ shared by two subdomains, subdomain edges, which are regarded as open subsets of Γ shared by more than two subdomains, and of the subdomain vertices, which are end points of edges.

The velocity and pressure finite element spaces $\mathbf{W}$ and Q are decomposed into

$$\mathbf{W} = \mathbf{W}_I \bigoplus \mathbf{W}_\Gamma, \quad Q = Q_I \bigoplus Q_\Gamma,$$

where $\mathbf{W}_I$ and Q_I are direct sums of independent subdomain interior velocity spaces $\mathbf{W}_I^{(i)}$, and interior pressure spaces $Q_I^{(i)}$, respectively. $\mathbf{W}_\Gamma$ and Q_Γ are subdomain interface velocity and pressure spaces, respectively. All functions in $\mathbf{W}_\Gamma$ and Q_Γ are continuous across Γ; their degrees of freedom are shared by neighboring subdomains. A partially sub-assembled subdomain interface velocity space $\widetilde{\mathbf{W}}_\Gamma$ is defined as

$$\widetilde{\mathbf{W}}_\Gamma = \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi = \left(\bigoplus_{i=1}^{N} \mathbf{W}_\Delta^{(i)} \right) \bigoplus \mathbf{W}_\Pi.$$

$\mathbf{W}_\Pi$ is the continuous, coarse level, primal velocity space which is typically spanned by subdomain vertex nodal basis functions, and/or by interface edge/face-cutoff functions with constant nodal values on each edge/face, or with values of positive weights on these edges/faces. The primal, coarse level velocity degrees of freedom are shared by neighboring subdomains. The complimentary space $\mathbf{W}_\Delta$ is the direct sum of independent subdomain dual interface velocity spaces $\mathbf{W}_\Delta^{(i)}$, which correspond to the remaining subdomain interface velocity degrees of freedom and are spanned by basis functions which vanish at the primal degrees of freedom. Thus, an element in $\widetilde{\mathbf{W}}_\Gamma$ typically has a continuous primal velocity component and a discontinuous dual velocity component.

We construct a matrix B_Δ from $\{0, 1, -1\}$ to enforce the continuity for dual velocity components. For any $\mathbf{w}_\Delta$ in $\mathbf{W}_\Delta$, each row of $B_\Delta \mathbf{w}_\Delta = 0$ implies that the two independent degrees of freedom from the neighboring subdomains be the same. The range of B_Δ applied on $\mathbf{W}_\Delta$ is a vector space of the Lagrange multipliers, denoted by Λ. For each node x on the subdomain boundary Γ, we define a positive scaling factor $\delta^\dagger(x) = 1/\mathcal{N}_x$, where $\mathcal{N}_x$ represents the number of subdomains sharing x. Multiplying the entries on each row of B_Δ by the corresponding scaling factor $\delta^\dagger(x)$ gives us $B_{\Delta,D}$.

The original system (2) is equivalent to: find $(\mathbf{u}_I,\ p_I,\ \mathbf{u}_\Delta,\ \mathbf{u}_\Pi,\ p_\Gamma,\ \lambda) \in \mathbf{W}_I \bigoplus Q_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi \bigoplus Q_\Gamma \bigoplus \Lambda$, such that

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} & B_{\Gamma I}^T & 0 \\ B_{II} & 0 & B_{I\Delta} & B_{I\Pi} & 0 & 0 \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{\Delta\Pi} & B_{\Gamma\Delta}^T & B_\Delta^T \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} & A_{\Pi\Pi} & B_{\Gamma\Pi}^T & 0 \\ B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} & 0 & 0 \\ 0 & 0 & B_\Delta & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_\Pi \\ p_\Gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \\ 0 \\ 0 \end{bmatrix}, \tag{4}$$

where the sub-blocks in the coefficient matrix represent the restrictions of A and B of (2) to appropriate subspaces. The leading three-by-three block can be ordered to become block diagonal with each diagonal block representing one independent subdomain problem.

Lemma 1. *[8, Lemma 4] The basis vector in the null space of (4), corresponding to the one-dimensional null space of the original incompressible Stokes system (2) , is*

$$\left(\mathbf{0},\, 1_{p_I},\, \mathbf{0},\, \mathbf{0},\, 1_{p_\Gamma},\, -B_{\Delta,D}[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix}\right). \tag{5}$$

Here $1_{p_I} \in Q_I$ and $1_{p_\Gamma} \in Q_\Gamma$ represent vectors with each entry equal to 1.

System (4) can be reduced to a Schur complement problem for the variables $(p_\Gamma,\ \lambda)$

$$G \begin{bmatrix} p_\Gamma \\ \lambda \end{bmatrix} = g, \tag{6}$$

where

$$G = B_C \widetilde{A}^{-1} B_C^T, \qquad g = B_C \widetilde{A}^{-1} \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \end{bmatrix}, \tag{7}$$

with

$$\widetilde{A} = \begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} \\ B_{II} & 0 & B_{I\Delta} & B_{I\Pi} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{\Delta\Pi} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix} \quad \text{and} \quad B_C = \begin{bmatrix} B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} \\ 0 & 0 & B_\Delta & 0 \end{bmatrix}. \tag{8}$$

G is symmetric positive semi-definite. The null space of G can be derived from Lemma 1, and its basis has the form

$$\left(1_{p_\Gamma},\ -B_{\Delta,D}[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix} \right).$$

Let $X = Q_\Gamma \bigoplus \Lambda$. The range of G, denoted by R_G, is the subspace of X, which is orthogonal to the null space of G and has the form

$$R_G = \left\{ \begin{bmatrix} g_{p_\Gamma} \\ g_\lambda \end{bmatrix} \in X \;\middle|\; g_{p_\Gamma}^T 1_{p_\Gamma} - g_\lambda^T \left(B_{\Delta,D}[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix} \right) = 0 \right\}. \quad (9)$$

The restriction of G to its range R_G is positive definite. The conjugate gradient method will be used to solve (6), with preconditioners given in the next section.

We denote

$$A_{rr} = \begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} \\ B_{II} & 0 & B_{I\Delta} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} \end{bmatrix},\ A_{\Pi r} = A_{r\Pi}^T = \begin{bmatrix} A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} \end{bmatrix},\ f_r = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \end{bmatrix},$$

and define the Schur complement $S_\Pi = A_{\Pi\Pi} - A_{\Pi r} A_{rr}^{-1} A_{r\Pi}$, which is symmetric positive definite and defines the coarse level problem of this algorithm.

The main operation in the implementation of multiplying G by a vector is the product of $\widetilde{A}^{-1}$ with a vector consisting of f_r and $\mathbf{f}_\Pi$. This product can be represented by

$$\begin{bmatrix} A_{rr}^{-1} f_r \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} -A_{rr}^{-1} A_{r\Pi} \\ I_\Pi \end{bmatrix} S_\Pi^{-1} \left(\mathbf{f}_\Pi - A_{\Pi r} A_{rr}^{-1} f_r \right),$$

which requires solving the coarse level problem once and independent subdomain Stokes problems with Neumann type boundary conditions twice.

3 Preconditioners and condition number bounds

We define $\widetilde{V} = \mathbf{W}_I \bigoplus Q_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi$, and its subspace

$$\widetilde{V}_0 = \left\{ w = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{V} : B_{II}\mathbf{w}_I + B_{I\Delta}\mathbf{w}_\Delta + B_{I\Pi}\mathbf{w}_\Pi = 0 \right\}.$$

For any $v \in \widetilde{V}_0$, the value $\langle v, v \rangle_{\widetilde{A}} = v^T \widetilde{A} v$ is independent of its pressure component p_I. $\langle \cdot, \cdot \rangle_{\widetilde{A}}$ defines a semi-inner product on $\widetilde{V}_0$; $\langle v, v \rangle_{\widetilde{A}} = 0$ if and only if the velocity component of v is zero while its pressure component can be arbitrary. We denote the restriction operator from $\widetilde{V}$ onto $\mathbf{W}_\Delta$ by $\widetilde{R}_\Delta$ such that for any $v = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{V}$, $\widetilde{R}_\Delta v = \mathbf{w}_\Delta$.

Let H_Δ represent the direct sum of discrete subdomain harmonic extension operators. Let $M_{L,\lambda}^{-1} = B_{\Delta,D}\widetilde{R}_\Delta \widetilde{A}\widetilde{R}_\Delta^T B_{\Delta,D}^T$ and $M_{D,\lambda}^{-1} = B_{\Delta,D} H_\Delta B_{\Delta,D}^T$. The lumped and Dirichlet preconditioners M_L^{-1} and M_D^{-1} for solving (6) are given by

$$M_L^{-1} = \begin{bmatrix} \frac{\alpha}{h^3} I_{p_\Gamma} & \\ & M_{L,\lambda}^{-1} \end{bmatrix} \quad \text{and} \quad M_D^{-1} = \begin{bmatrix} \frac{\alpha}{h^3} I_{p_\Gamma} & \\ & M_{D,\lambda}^{-1} \end{bmatrix}.$$

Here I_{p_Γ} is the identity matrix of the same length as p_Γ. α is a given constant, whose value is typically taken as 1. We introduce α in the preconditioner just for the convenience in the numerical experiments to demonstrate the convergence rates of the proposed algorithm.

For both lumped and Dirichlet preconditioners, the coarse space includes only subdomain corner and edge-average variables for each velocity component, just as for solving scalar elliptic problems. Such coarse space is sufficient for this algorithm to achieve scalable convergence rates as given in the following theorem for both type preconditioners, denoted here by M^{-1}.

Lemma 2. *[8, Lemma 10]. There exists a constant C, such that for all $v \in \widetilde{V}_0$,*

$$\left\langle M^{-1} B_C v, B_C v \right\rangle \leq C \left(\alpha + \Phi(H/h)\right) \left\langle \widetilde{A} v, v \right\rangle.$$

Here, for the lumped preconditioner, $\Phi(H,h) = C(H/h)(1+\log(H/h))$, and for the Dirichlet preconditioner, $\Phi(H,h) = C(1+\log(H/h))^2$.

Lemma 3. *[8, Lemma 11] There exists a constant C, such that for any nonzero $y = (g_{p_\Gamma}, g_\lambda) \in R_G$, there exits $v \in \widetilde{V}_0$, which satisfies $B_C v = y$, $\langle v, v\rangle_{\widetilde{A}} \neq 0$, and*

$$\left\langle \widetilde{A} v, v \right\rangle \leq C \max\left\{1, \tfrac{1}{\alpha}\right\} \left(1 + \tfrac{1}{\beta^2}\right) \left\langle M^{-1} y, y \right\rangle.$$

Theorem 1. *[8, Theorem 1] There exist positive constants c and C, such that for all x in the range of $M^{-1}G$,*

$$\min\{1, \alpha\} \frac{c\beta^2}{(1+\beta^2)} \langle Mx, x\rangle \leq \langle Gx, x\rangle \leq C\left(\alpha + \Phi(H/h)\right) \langle Mx, x\rangle.$$

4 Numerical experiments

We solve the saddle point problem (1) on the cube $\Omega = [0,1]^3$ with a zero Dirichlet boundary condition. The right-hand side $\mathbf{f}$ is chosen such that the exact solution is

$$\mathbf{u} = \begin{bmatrix} \sin^2(\pi x)\left(\sin(2\pi y)\sin(\pi z) - \sin(\pi y)\sin(2\pi z)\right) \\ \sin^2(\pi y)\left(\sin(2\pi z)\sin(\pi x) - \sin(\pi z)\sin(2\pi x)\right) \\ \sin^2(\pi z)\left(\sin(2\pi x)\sin(\pi y) - \sin(\pi x)\sin(2\pi y)\right) \end{bmatrix}, \quad p = xyz - \frac{1}{8}.$$

The Q_2-Q_1 Taylor-Hood mixed finite element is used and the preconditioned system is solved by a conjugate gradient (CG) method. The CG iteration is stopped when the L^2-norm of the residual is reduced by a factor of 10^{-6}. We use the tridiagonal Lanczos matrix generated in the iteration to estimate the extreme eigenvalues of $M^{-1}G$.

For both preconditioners, the coarse level velocity space is the same as for solving scalar elliptic problems in [5, Algorithm 6.25] corresponding to each velocity component, which is spanned by the subdomain vertex nodal basis functions and subdomain edge-cutoff functions.

We take $\alpha = 1$ in Table 1 and $\alpha = 1/2$ in Table 2, to demonstrate more clearly the upper eigenvalue bound in Theorem 1. Using the Dirichlet preconditioner can reduce $\Phi(H/h)$ compared with the lumped preconditioner. However, for a small value of H/h, $\alpha = 1$ will be dominant in the upper bound and the effect of $\Phi(H/h)$ on the convergence rate is not visible in Table 1. When α is reduced to 1/2, $\Phi(H/h)$ becomes visible and the upper eigenvalue bounds in Table 2 exhibit the pattern of $\Phi(H/h)$ for both preconditioners. They are independent of the number of subdomains for fixed H/h; for fixed number of subdomains, they depend on H/h in the order of $(H/h)(1+\log(H/h))$ for the lumped preconditioner, and $(1+\log(H/h))^2$ for the Dirichlet preconditioner. The lower eigenvalue bounds in Table 2 are half of those in Table 1 since α is reduced by half, and they are also independent of the mesh size, consistent with Theorem 1.

We also comment that the inf-sup stability constant β of the mixed finite element space determines the lower eigenvalue bound in Theorem 1, which is quite small as shown in Tables 1 and 2 for this example. Some mixed finite element spaces with discontinuous pressures have better inf-sup stability and as a result give better lower eigenvalue bounds in Theorem 1.

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Table 1 Performance of solving three-dimensional problem on $[0,1]^3$, $\alpha = 1$.

		lumped			Dirichlet		
H/h	#sub	λ_{min}	λ_{max}	iteration	λ_{min}	λ_{max}	iteration
4	$3 \times 3 \times 3$	0.0776	9.13	56	0.0776	8.97	56
	$4 \times 4 \times 4$	0.0775	9.35	54	0.0774	9.19	55
	$6 \times 6 \times 6$	0.0773	9.41	58	0.0773	9.23	59
	$8 \times 8 \times 8$	0.0773	9.51	57	0.0772	9.34	61
#sub	H/h	λ_{min}	λ_{max}	iteration	λ_{min}	λ_{max}	iteration
$3 \times 3 \times 3$	3	0.0760	8.06	54	0.0760	7.96	54
	4	0.0776	9.13	56	0.0776	8.97	56
	6	0.0780	11.88	53	0.0780	9.35	55
	8	0.0780	16.64	57	0.0780	9.44	55

Table 2 Performance of solving three-dimensional problem on $[0,1]^3$, $\alpha = 1/2$.

		lumped			Dirichlet		
H/h	#sub	λ_{min}	λ_{max}	iteration	λ_{min}	λ_{max}	iteration
4	$3 \times 3 \times 3$	0.0395	7.20	59	0.0395	4.89	54
	$4 \times 4 \times 4$	0.0394	8.15	66	0.0394	5.01	53
	$6 \times 6 \times 6$	0.0393	8.85	70	0.0393	5.03	55
	$8 \times 8 \times 8$	0.0393	9.09	72	0.0393	5.09	56
#sub	H/h	λ_{min}	λ_{max}	iteration	λ_{min}	λ_{max}	iteration
$3 \times 3 \times 3$	3	0.0387	5.15	55	0.0387	4.35	53
	4	0.0395	7.20	57	0.0395	4.89	54
	6	0.0397	11.70	63	0.0397	5.11	52
	8	0.0397	16.52	73	0.0397	5.17	52

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