BDDC deluxe Domain Decomposition

Olof B. Widlund[1] and Clark R. Dohrmann[2]

1 Introduction

We will consider BDDC domain decomposition algorithms for finite element approximations of a variety of elliptic problems. The BDDC (Balancing Domain Decomposition by Constraints) algorithms were introduced by Dohrmann [2003], following the introduction of FETI–DP by Farhat et al. [2001]. These two families are closely related algorithmically and have a common theory. The design of a BDDC algorithm involves the choice of a set of *primal degrees of freedom* and the choice of an *averaging operator*, which restores the continuity of the approximate solution across the interface between the subdomains into which the domain of the given problem has been partitioned. We will also refer to these operators as *scalings*.

This paper principally address the latter choice. All our efforts aim at developing effective *preconditioners* for the stiffness matrices. These approximate inverses are then combined with the conjugate gradient method. We are primarily interested in hard problems with very many subdomains and to obtain convergence rates independent of that number and with rates that deteriorate slowly with the size of the subdomain problems. Our bounds can often be made independent of jumps in the coefficients between subdomains and our numerical results indicate that our new BDDC deluxe algorithm is quite promising and robust.

Among our applications are problems formulated in $H(\mathbf{curl})$, $H(\mathrm{div})$, and for Reissner-Mindlin plates. We have worked mostly with the lowest order finite element methods for self-adjoint elliptic problems but we have also helped develop solvers for isogeometric analysis. After introducing the general ideas, we will focus on our recent work on three-dimensional problems in $H(\mathbf{curl})$, see Dohrmann and Widlund [2014], since other applications are

Courant Institute, 251 Mercer Street, New York, NY 10012 widlund@cims.nyu.edu · Sandia National Laboratories, Albuquerque, New Mexico crdohrm@sandia.gov

discussed in other papers of this volume or elsewhere; cf. Beirão da Veiga et al. [2014a], Beirão da Veiga et al. [2014b], Calvo [2014], Chung and Kim [2014], Dryja et al. [2014], Lee [2014], Oh et al. [2013], and Oh [2014].

2 BDDC, finite element meshes, and equivalence classes

The BDDC algorithms work with decompositions of the domain Ω of the elliptic problem into non-overlapping subdomains Ω_i, each often with tens of thousands of degrees of freedom. In-between the subdomains is the interface Γ, which does not cut through any elements. The local interface of Ω_i is defined by $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$.

For nodal finite element methods, most nodes are typically interior to individual subdomains while the others belong to several subdomain interfaces or to the boundary of the given region. We partition the nodes on Γ into equivalence classes determined by the set of indices of the local interfaces Γ_j to which they belong. In three dimensions, we have equivalence classes of face nodes, associated with two local interfaces, and classes of edge nodes and subdomain vertex nodes typically associated with more than two. For $H(\mathbf{curl})$ and Nédélec (edge) elements, there are only equivalence classes of element edges for subdomain faces and for subdomain edges. For $H(\mathrm{div})$ and Raviart-Thomas elements, we only have degrees of freedom for element faces and the only equivalence classes are associated with the subdomain faces. These equivalence classes play a central role in the design, analysis, and programming of domain decomposition methods.

These preconditioners are constructed using partially subassembled stiffness matrices built from the subdomain stiffness matrices $A^{(i)}$ of the subdomains $\Omega_i,\ i = 1, \ldots, N$. We will first consider nodal finite element problems. The nodes of $\Omega_i \cup \Gamma_i$ are divided into those in the interior (I) and those on the interface (Γ). The interface set is further divided into a primal set (Π) and a dual set (Δ).

We can then represent the subdomain stiffness matrix, of Ω_i, as

$$A^{(i)} \;=\; \begin{pmatrix} A^{(i)}_{II} & A^{(i)}_{I\Delta} & A^{(i)}_{I\Pi} \\ A^{(i)}_{\Delta I} & A^{(i)}_{\Delta\Delta} & A^{(i)}_{\Delta\Pi} \\ A^{(i)}_{\Pi I} & A^{(i)}_{\Pi\Delta} & A^{(i)}_{\Pi\Pi} \end{pmatrix}.$$

This matrix represents the stiffness contributed by Ω_i. Throughout the iteration, we enforce continuity of the primal variables, as in the given finite element model, but allow multiple values of the dual variables when working with a *partially subassembled* model as in Figure 1. Partially subassembling the subdomain matrices and noting that the matrix $\tilde{A}_{\Pi\Pi}$ is assembled from the submatrices $A^{(i)}_{\Pi\Pi}$, we obtain

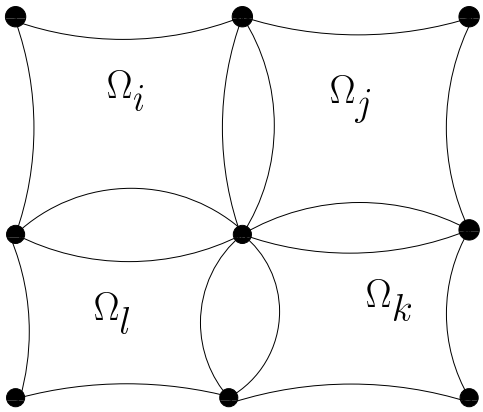

Fig. 1 Torn 2D scalar elliptic problem; primal variables at subdomain vertices only.

$$\widetilde{A} = \begin{pmatrix} A_{II}^{(1)} & A_{I\Delta}^{(1)} & & & A_{I\Pi}^{(1)} \\ A_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & & & A_{\Delta\Pi}^{(1)} \\ & & \ddots & & \vdots \\ & & & A_{II}^{(N)} \; A_{I\Delta}^{(N)} & A_{I\Pi}^{(N)} \\ & & & A_{\Delta I}^{(N)} \; A_{\Delta\Delta}^{(N)} & A_{\Delta\Pi}^{(N)} \\ A_{\Pi I}^{(1)} & A_{\Pi\Delta}^{(1)} & \cdots & A_{\Pi I}^{(N)} \; A_{\Pi\Delta}^{(N)} & \tilde{A}_{\Pi\Pi} \end{pmatrix}.$$

This partially subassembled stiffness matrix of this alternative finite element model is an important component of the BDDC preconditioners. The primal variables provide a necessary, global component of the preconditioners and they make the partially assembled matrix invertible.

Solving a linear system with the matrix $\widetilde{A}$ is much cheaper than when using the fully assembled model but results in multiple values of the dual interface variables. When using BDDC, we therefore restore the continuity of the original finite element problem by averaging across the interface. When using FETI–DP, we instead employ Lagrange multipliers.

For scalar second order elliptic equations in the plane, as in Figure 1, the approach outlined yields a condition number bound of $C(1 + \log(H/h))^2$, where $H/h := \max_i(H_i/h_i)$ with H_i the diameter of Ω_i and h_i that of the smallest of the elements of Ω_i. These results can be made independent of jumps in the coefficients, if the interface averages are chosen carefully, but for three dimensions the primal set of variables should include averages (and possibly first moments) of the displacements over subdomain edges (and possibly subdomain faces) to obtain competitive algorithms. After introducing primal variables of this type, we can change the variables to allow us to represent the partially subassembled system matrix as above.

We note that parallel, public domain BDDC software, developed by Zampini [2013], is available. We also note Farhat, Pierson, et al. and Klawonn and Rheinbach have been pioneers in developing FETI–DP software for elasticity problems.

The BDDC and FETI–DP algorithms can be described in terms of three product spaces of finite element functions/vectors defined by their interface nodal values:

$$\widehat{W}_\Gamma \subset \widetilde{W}_\Gamma \subset W_\Gamma.$$

W_Γ: no constraints; $\widehat{W}_\Gamma$: continuity at every point on Γ; $\widetilde{W}_\Gamma$: common values of the primal variables. After eliminating the interior variables, we can write the resulting subdomain Schur complements as

$$S^{(i)} := \begin{pmatrix} S^{(i)}_{\Delta\Delta} & S^{(i)}_{\Delta\Pi} \\ S^{(i)}_{\Pi\Delta} & S^{(i)}_{\Pi\Pi} \end{pmatrix} := \begin{pmatrix} A^{(i)}_{\Delta\Delta} & A^{(i)}_{\Delta\Pi} \\ A^{(i)}_{\Pi\Delta} & A^{(i)}_{\Pi\Pi} \end{pmatrix} - \begin{pmatrix} A^{(i)}_{\Delta I} \\ A^{(i)}_{\Pi I} \end{pmatrix} \left(A^{(i)}_{II}\right)^{-1} \begin{pmatrix} A^{(i)}_{I\Delta} & A^{(i)}_{I\Pi} \end{pmatrix}.$$

By partially subassembling the $S^{(i)}$, we obtain $\tilde{S}$.

Let us denote the BDDC averaging operator, which maps $\widetilde{W}_\Gamma$ into $\widehat{W}_\Gamma$, by E_D. In each BDDC iteration, we first compute the residual of the fully assembled Schur complement equation. We then apply E_D^T to obtain a right-hand side for the partially subassembled linear system. We solve this system and then apply E_D. In the conventional BDDC algorithms the averaging across the interface is done point-wise and that leads to non-zero residuals at the nodes next to Γ. In each iteration, subdomain Dirichlet solves are then used to eliminate them, but in the deluxe variant this step is not needed. The iteration is accelerated by using a preconditioned conjugate gradient method.

The core of any theory for BDDC algorithms is the norm of the average operator E_D. By an algebraic argument known, for FETI–DP, since the publication of [Klawonn et al., 2002, Proof of Theorem 1], we have

$$\kappa(M^{-1}A) \leq \|E_D\|_{\tilde{S}},$$

which then provides an upper bound for the number of iterations required of the preconditioned conjugate gradient method; for details on the BDDC case, see, e.g., Beirão da Veiga et al. [2014a]. Here M^{-1} represents the action of the preconditioner.

3 The new algorithmic idea

When designing a BDDC algorithm, we have to choose an effective set of primal constraints and also a good recipe for the averaging across the interface. Traditional averaging recipes were found not to work uniformly well for three dimensional problems in $H(\mathbf{curl})$; see Dohrmann and Widlund [2013]. This is directly related to the fact that there are two material parameters.

An alternative was found and will be outlined in this section. It has also proven to be very robust for $H(\mathrm{div})$ problems, see Oh et al. [2013], and for Reissner-Mindlin plates, see Lee [2014].

We note that experimentally, the condition numbers are often quite small. For Reissner-Mindlin plates, in Lee's experiments, they are ≤ 4 while without preconditioning the condition numbers can exceed 10^{11} for very thin plates with the parameter $t = 10^{-5}$. The results in the $H(\mathrm{div})$-study are quite similar and experiments with the deluxe version of BDDC for isogeometric analysis show considerable improvement over older variants.

Across a subdomain face $F \subset \Gamma$, common to two subdomains Ω_i and Ω_j, the deluxe E_D is defined in terms of two Schur complements, which are principal minors of $S^{(i)}$ and $S^{(j)}$, respectively:

$$S_F^{(k)} := A_{FF}^{(k)} - A_{FI}^{(k)} {A_{II}^{(k)}}^{-1} A_{IF}^{(k)}, \ k = i, j.$$

The face contribution of the deluxe averaging operator is then defined by

$$\bar{w}_F := (E_D w)_F := (S_F^{(i)} + S_F^{(j)})^{-1}(S_F^{(i)} w_F^{(i)} + S_F^{(j)} w_F^{(j)}).$$

This action of this component of E_D can be implemented by solving a Dirichlet problem on $\Omega_i \cup F \cup \Omega_j$. This local problem is larger than those of the conventional BDDC algorithms, and we are currently exploring the effects of using cheaper, inexact solvers for these subproblems.

Similar formulas can also be written down for subdomain edges and other equivalence classes of interface variables. The operator E_D is assembled from these components.

We will now show that the analysis of BDDC deluxe can be reduced to bounds for individual subdomains. Arbitrary jumps in two coefficients can then often be well accommodated. We also note that the analysis of traditional BDDC algorithms requires an extension theorem; the deluxe version does not.

Instead of estimating $(R_F^T \bar{w}_F)^T S^{(i)} R_F^T \bar{w}_F$, where the restriction operator R_F maps the values on Γ onto those on F, we will work with the norm of $R_F^T(w_F^{(i)} - \bar{w}_F)$. Thus, instead of estimating the norm of E_D, we will estimate the norm of $I - E_D$; a bound on the norm of E_D will, as we previously have noted, give a bound on the condition number.

We easily find that

$$w_F^{(i)} - \bar{w}_F = (S_F^{(i)} + S_F^{(j)})^{-1} S_F^{(j)}(w_F^{(i)} - w_F^{(j)}).$$

By some more algebra and noting that $R_F S^{(i)} R_F^T = S_F^{(i)}$, we find that

$$(R_F^T(w_F^{(i)} - \bar{w}_F))^T S^{(i)}(R_F^T(w_F^{(i)} - \bar{w}_F)) =$$

$$(w_F^{(i)} - w_F^{(j)})^T S_F^{(j)}(S_F^{(i)} + S_F^{(j)})^{-1} S_F^{(i)}(S_F^{(i)} + S_F^{(j)})^{-1} S_F^{(j)}(w_F^{(i)} - w_F^{(j)}).$$

We now add the corresponding expression for the subdomain Ω_j and, after a simplification, find that this sum can be written as

$$(w_F^{(i)} - w_F^{(j)})^T (S_F^{(i)-1} + S_F^{(j)-1})^{-1} (w_F^{(i)} - w_F^{(j)}).$$

We then find that, for any element w_Π in the primal space,

$$(R_F^T(w_F^{(i)} - \bar{w}_F))^T S^{(i)} R_F^T(w_F^{(i)} - \bar{w}_F) + (R_F^T(w_F^{(j)} - \bar{w}_F))^T S^{(j)} R_F^T(w_F^{(j)} - \bar{w}_F)$$

$$\leq 2(w_F^{(i)} - R_F w_\Pi)^T S_F^{(i)} (w_F^{(i)} - R_F w_\Pi) + 2(w_F^{(j)} - R_F w_\Pi)^T S_F^{(j)} (w_F^{(j)} - R_F w_\Pi).$$

Each of the two terms on the right hand side are local to only one subdomain.

For the subdomain faces, what now remains is to estimate, after a suitable shift w_Π, $(w_F^{(i)} - R_F w_\Pi)^T S_F^{(i)} (w_F^{(i)} - R_F w_\Pi)$ by ${w^{(i)}}^T S^{(i)} w^{(i)}$. This is routine for $H^1(\Omega_i)$ using standard estimates in the domain decomposition literature such as a *face lemma* [Toselli and Widlund, 2005, Lemma 4.24] in which we estimate the energy of the extension of the face values by zero to the rest of Γ_i with that of the minimal energy extension. A factor of $C(1 + \log(H/h))^2$ results. For $H^1(\Omega_i)$, all these estimates have been available for twenty years. But for $H(\mathrm{div})$ and $H(\mathbf{curl})$, new tools have been required.

Similar estimates are required for subdomain edges. Let R_E be the restriction matrix which maps the values on Γ onto those on a subdomain edge E. If this edge is common to three subdomains Ω_i, Ω_j, and Ω_k, the edge average $\bar{w}_E$ is defined by

$$\bar{w}_E := (S_E^{(i)} + S_E^{(j)} + S_E^{(k)})^{-1} (S_E^{(i)} w_E^{(i)} + S_E^{(j)} w_E^{(j)} + S_E^{(k)} w_E^{(k)}).$$

Here $S_E^{(i)} := R_E S^{(i)} R_E^T$, etc. We can show that,

$$(R_E^T(w_E^{(i)} - \bar{w}_E))^T \ S^{(i)} \ R_E^T(w_E^{(i)} - \bar{w}_E) \leq$$
$$3(w_E^{(i)})^T S_E^{(j)} w_E^{(i)} + 3/4(w_E^{(j)})^T \ S_E^{(j)} \ w_E^{(j)} + 3/4(w_E^{(k)})^T S_E^{(k)} w_E^{(k)}.$$

Other bounds, e.g., with a shift with an element of the primal space, can also be developed, but the one given here has proven of use in our work on problems in $H(\mathbf{curl})$. We can also develop similar bounds for any edge, common to more than three subdomains, using the same kinds of arguments.

4 $H(\mathbf{curl})$ problems in three dimensions

Consider the variational problem: Find $\mathbf{u} \in H_0(\mathbf{curl}; \Omega)$ such that

$$a(\mathbf{u}, \mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in H_0(\mathbf{curl}; \Omega),$$

where $\mathbf{u} \times \mathbf{n} = 0$ on $\partial\Omega$ and where

$$a(\mathbf{u},\mathbf{v})_\Omega := \int_\Omega [\alpha \nabla \times \mathbf{u} \cdot \nabla \times \mathbf{v} + \beta \mathbf{u} \cdot \mathbf{v}]\, dx, \quad (\mathbf{f},\mathbf{v})_\Omega := \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx.$$

Here, $\alpha(x) \geq 0$ and $\beta(x)$ strictly positive. For coefficients constant in each subdomain, we have

$$a(\mathbf{u},\mathbf{v})_\Omega = \sum_{i=1}^{N} (\alpha_i (\nabla \times \mathbf{u}, \nabla \times \mathbf{v})_{\Omega_i} + \beta_i (\mathbf{u},\mathbf{v})_{\Omega_i}).$$

In our work, there are two relevant finite element spaces, namely $W^{h_i}_{\text{curl}}$ of the lowest order triangular Nédeléc elements and $W^{h_i}_{\text{grad}}$ of the standard piecewise linear, continuous elements, on the same triangulation.

The space of Nédélec finite element functions, $W^{h_i}_{\text{curl}}$, can be represented as the range of an interpolation operator Π^h which is well defined, for sufficiently smooth elements of $\mathbf{w} \in H(\mathbf{curl}, \Omega)$, by

$$\Pi^h(\mathbf{w}) := \sum_e \lambda_e(\mathbf{w}) \mathbf{N}_e \quad where \quad \lambda_e(\mathbf{w}) := \frac{1}{|e|} \int_e \mathbf{w} \cdot \mathbf{t}_e ds.$$

Here $\mathbf{t}_e$ is a unit vector in the direction of the element edge e and $\mathbf{N}_e$ the standard Nédélec basis function.

We have been able to build on the work by Toselli [2006]. Thus, for Nédélec elements, the use of the basis based on $\{\mathbf{N}_e\}$ results in a poor result since the coupling between the subdomain faces and edges is far too strong. Following Toselli, we change the variables associated with the subdomain edges using a constant along each subdomain edge and the gradient of the standard $W^{h_i}_{\text{grad}}$ basis functions for all the interior nodes of the subdomain edges. After this change of variables, a quite stable decomposition can be found.

Domain decomposition theory always involves establishing the stability of a decomposition; in our context, a new auxiliary result is then needed:

Lemma 1. *Let F be a face of a polyhedral subdomain Ω_i. Further, let $f_{\partial F} \in W^{h_i}_{grad}(\Omega_i)$ have vanishing nodal values everywhere in Ω_i except on ∂F. There then exists a $\mathbf{g}_{iF} \in W^{h_i}_{curl}(\Omega_i)$ such that $\lambda_e(\mathbf{g}_{iF}) = \lambda_e(\nabla f_{\partial F})$ for all element edges in the interior of the face F, $\lambda_e(\mathbf{g}_{iF}) = 0$ for all other element edges on $\partial\Omega_i$, and*

$$\|\mathbf{g}_{iF}\|^2_{L^2(\Omega_i)} \leq C((1+\log(H_i/h_i))\|f_{\partial F}\|^2_{L^2(\partial F)} + H_i^2 \|\nabla f_{\partial F} \cdot \mathbf{t}_{\partial F}\|^2_{L^2(\partial F)}),$$
$$\|\nabla \times \mathbf{g}_{iF}\|^2_{L^2(\Omega_i)} \leq C(1+\log(H_i/h_i))\|\nabla f_{\partial F} \cdot \mathbf{t}_{\partial F}\|^2_{L^2(\partial F)}.$$

For a proof of this result, see [Dohrmann and Widlund, 2014, Lemma 3.5]. We also use several standard auxiliary results for $W^{h_i}_{\text{grad}}$ as collected in [Toselli and Widlund, 2005, Subsection 4.6].

A key to our work is also a result by [Hiptmair and Xu, 2007, Lemma 5.1]:

Lemma 2. *For any polyhedral subdomain Ω_i and any $\mathbf{u}_h \in W^{h_i}_{curl}(\Omega_i)$, there exist $\mathbf{\Psi}_h \in (W^{h_i}_{grad}(\Omega_i))^3$, $p_h \in W^{h_i}_{grad}(\Omega_i)$, and $\mathbf{q}_h \in W^{h_i}_{curl}(\Omega_i)$, such that*

$$\begin{aligned}
\mathbf{w}_i &= \mathbf{q}_i + \Pi^{h_i}(\mathbf{\Psi}_i) + \nabla p_i, \\
\|\nabla p_i\|^2_{L^2(\Omega_i)} &\leq C(\|\mathbf{w}_i\|^2_{L^2(\Omega_i)} + H_i^2\|\nabla \times \mathbf{w}_i\|^2_{L^2(\Omega_i)}), \\
\|h_i^{-1}\mathbf{q}_i\|^2_{L^2(\Omega_i)} + \|\mathbf{\Psi}_i\|^2_{H^1(\Omega_i)} &\leq C\|\nabla \times \mathbf{w}_i\|^2_{L^2(\Omega_i)}.
\end{aligned}$$

We note that these bounds are local and that the result has been established for polyhedra which are not necessarily convex.

This result is essential to Hiptmair and Xu's work on algebraic multigrid algorithms for $H(\mathbf{curl})$ in which AMG Poisson solvers are used.

In contrast to earlier results on domain decomposition algorithms for $H(\mathbf{curl})$, we do not have to rely on any trace theorem in our proof.

Toselli primarily advocates the use of two primal variables for each subdomain edge: the average and first moment and so do we. We have improved Toselli's condition number bound from

$$C \max_i (1 + \frac{\beta_i H_i^2}{\alpha_i})(1 + \log(H_i/h_i))^4$$

to an estimate, with the best possible power of $(1 + \log(H_i/h_i)$:

$$C \max_i \min((H_i/h_i)^2, (1 + \frac{\beta_i H_i^2}{\alpha_i}))(1 + \log(H_i/h_i))^2.$$

We have fewer restrictions on the coefficients than Toselli; our constant C is independent of the α_i and β_i.

So far, we have not mastered the case where $\frac{\beta_i H_i^2}{\alpha_i}$ is large. We note that for $H(\text{div})$, one simple primal space works well in all cases; not so for $H(\mathbf{curl})$.

5 Numerical Results

Numerical results are presented in this section, which confirm the theory and demonstrate that in certain cases, the deluxe BDDC algorithm is much more robust than older BDDC variants. In our tables *iter* and *cond* denote the number of iterations and the estimated condition numbers obtained using a relative tolerance of 10^{-8} of the ℓ_2-norm of the residual as a stopping criterion.

The subdomain problems are discretized using the lowest order hexahedral edge elements, for which our theory is equally valid.

In the first example, a unit cube is subdivided into N_d^3 smaller cubes, which are each subdivided into $64 = 4^3$ elements. Table 1 illustrates that the rate of convergence is independent of the number of subdomains.

Table 1 Results for unit cube decomposed into smaller cubical subdomains with $H/h = 4$. The material properties are constant with $\alpha_i = \alpha$ and $\beta_i = 1$.

	$\alpha = 10^{-4}$		$\alpha = 10^{-2}$		$\alpha = 1$		$\alpha = 10^{2}$		$\alpha = 10^{4}$	
N_d	iter	cond	iter	cond	iter	cond	iter	cond	iter	cond
2	9	2.49	8	1.59	10	1.99	10	2.03	10	2.03
4	12	2.36	10	1.79	14	2.63	15	2.70	16	2.70
6	11	2.12	12	2.07	15	2.81	16	2.88	17	2.88
8	11	2.02	13	2.25	15	2.87	16	2.95	17	2.95
10	11	1.97	13	2.35	16	2.91	17	2.98	18	2.98
12	11	1.92	14	2.44	16	2.93	17	2.99	18	2.99

In the next set of experiments, we study the behavior of our algorithm for increasing values of H/h, the number of elements across each subdomain. We note a much more rapid growth of the condition number for the mass-dominated cases, i.e., with $\beta_i H_i^2 >> \alpha_i$, represented by the first column of Table 2.

Table 2 Results for unit cube decomposed into 27 smaller cubical subdomains. The material properties are constant with $\alpha_i = \alpha$ and $\beta_i = 1$.

	$\alpha = 10^{-7}$		$\alpha = 10^{-2}$		$\alpha = 1$		$\alpha = 10^{2}$		$\alpha = 10^{4}$	
H/h	iter	cond	iter	cond	iter	cond	iter	cond	iter	cond
4	12	2.74	9	1.63	13	2.41	13	2.47	14	2.47
6	15	4.51	12	2.15	14	2.93	15	3.01	16	3.01
8	19	6.89	14	2.70	16	3.34	17	3.44	18	3.44
10	22	9.98	15	3.22	17	3.69	18	3.79	19	3.79
12	24	13.8	16	3.69	17	3.98	19	4.09	20	4.10
14	28	18.3	17	4.13	18	4.24	19	4.36	21	4.36
16	30	23.5	18	4.55	19	4.47	20	4.60	22	4.60

In our final table, Table 3, we consider a case of a three-dimensional checkerboard arrangement of the material parameters with $\alpha_i = 10^4$, $\beta_i = 10^{-2}$ for the red subdomains and $\alpha_i = 10^2$, $\beta_i = 1$ for the black. We indeed find a considerable improvement for the deluxe variant over two standard scalings. In the final columns, marked e-deluxe, results of replacing the solver over pairs of subdomains with a common face, by a solver over only a thin neighborhood of the face, which just includes the element edges next to the face, are given.

Table 3 Results for unit cube decomposed into 27 smaller smaller cubical subdomains with a checkerboard arrangement of material properties.

	stiffness scaling		cardinality scaling		deluxe scaling		e-deluxe scaling	
H/h	iter	cond	iter	cond	iter	cond	iter	cond
4	50	272	80	156	6	1.06	6	1.06
6	67	342	100	207	7	1.20	7	1.20
8	78	398	117	247	8	1.33	8	1.33
10	87	445	128	281	9	1.45	9	1.45
12	95	486	140	310	10	1.55	10	1.55
14	102	522	151	336	10	1.63	10	1.63
16	109	554	160	360	11	1.71	11	1.71

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