Optimized Schwarz method with two-sided transmission conditions in an unsymmetric domain decomposition

Martin J. Gander[1] and Yingxiang Xu[2]

1 Introduction

Domain decomposition (DD) methods are important techniques for designing parallel algorithms for solving partial differential equations. Since the decomposition is often performed using automatic mesh partitioning tools, one can in general not make any assumptions on the shape or physical size of the subdomains, especially if local mesh refinement is used. In many of the popular domain decomposition methods, neighboring subdomains are not using the same type of boundary conditions, e.g. the Dirichlet-Neumann methods invented by [2], or the two-sided optimized Schwarz methods proposed in [3], and one has to decide which subdomain uses which boundary condition. A similar question also arises in mortar methods, see [1], where one has to decide on the master and slave side at the interfaces. In [4], it was found that for optimized Schwarz methods, the subdomain geometry and problem boundary conditions influence the optimized Robin parameters for symmetrical finite domain decompositions, and in [5], it was observed numerically that swapping the optimized two-sided Robin parameters can accelerate the convergence for a circular domain decomposition.

We study in this paper two-sided optimized Schwarz methods for a model decomposition into a larger and a smaller subomain, to investigate which Robin parameter should be used on which subdomain in order to get fast convergence. We consider the model problem

$$\Delta u - \eta u = f \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1}$$

1. Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, CP 64, 1211 Genève 4, Suisse. Martin.Gander@unige.ch. · 2. School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China. yxxu@nenu.edu.cn, partly supported by NSFC-11201061, 11471047, 11271065, CPSF-2012M520657 and the Science and Technology Development Planning of Jilin Province 20140520058JH.

where $\Omega = \{(x, y) \in \mathbb{R}^2 | -a \le x \le b\}$ is decomposed into two subdomains $\Omega = \Omega_1 \cup \Omega_2$, with $\Omega_1 = \{(x, y) \in \mathbb{R}^2 | -a \le x \le L\}$, $\Omega_2 = \{(x, y) \in \mathbb{R}^2 | 0 \le x \le b\}$, and $L \ge 0$ is the overlap between subdomains, $a + L < b$. Note here in the y-direction, the domain Ω is still infinite, but this will not affect our theoretical findings, since in numerical computations the Fourier frequency lies in between $k_{\min}$ and $k_{\max}$, the lowest and the highest frequencies involved in the computation, and we will use this in our analysis.

We focus in this short paper on the parallel Schwarz method

$$\begin{array}{ll} \Delta u_1^n - \eta u_1^n = f, \quad \text{in } \Omega_1, & \Delta u_2^n - \eta u_2^n = f, \quad \text{in } \Omega_2, \\ u_1^n(-a, y) = 0, & u_2^n(b, y) = 0, \end{array} \tag{2}$$

with two-sided Robin transmission conditions

$$\begin{array}{c} (\partial_x + p_1) u_1^n(L, \cdot) = (\partial_x + p_1) u_2^{n-1}(L, \cdot), \\ (\partial_x - p_2) u_2^n(0, \cdot) = (\partial_x - p_2) u_1^{n-1}(0, \cdot), \end{array} \tag{3}$$

where p_1, p_2 are positive constants.

2 Optimized two-sided Robin transmission conditions

Inserting a Fourier expansion of the iterates, $u_i^n(x, y) = \sum_{k=-\infty}^{\infty} \hat{u}_i^n(x, k) e^{iky}$, into (2) and iterating the solutions between subdomains through the transmission condition (3), see for example [3], we obtain for each Fourier mode k the contraction factor

$$\begin{aligned} \rho(k, \eta, L, p_1, p_2, a, b) = \; & \frac{\sqrt{\eta+k^2}(1+e^{-2\sqrt{\eta+k^2}(b-L)}) - p_1(1-e^{-2\sqrt{\eta+k^2}(b-L)})}{\sqrt{\eta+k^2}(1+e^{-2\sqrt{\eta+k^2}(a+L)}) + p_1(1-e^{-2\sqrt{\eta+k^2}(a+L)})} \cdot \\ & \frac{\sqrt{\eta+k^2}(1+e^{-2\sqrt{\eta+k^2}a}) - p_2(1-e^{-2\sqrt{\eta+k^2}a})}{\sqrt{\eta+k^2}(1+e^{-2\sqrt{\eta+k^2}b}) + p_2(1-e^{-2\sqrt{\eta+k^2}b})} \cdot e^{-2\sqrt{\eta+k^2}L}. \end{aligned} \tag{4}$$

To obtain the fastest method for all relevant Fourier modes k, we have to solve the optimization problem

$$\min_{p_1, p_2 > 0} \rho_{\max}(L, p_1, p_2), \tag{5}$$

where $\rho_{\max}(L, p_1, p_2) := \max_{k_{\min} \le k \le k_{\max}} |\rho(k, \eta, L, p_1, p_2, a, b)|$ and $k_{\min}$, $k_{\max}$ are estimates of the lowest and the highest frequencies involved in the computation. If h is the mesh size along the interface, and the interface length is c, one can estimate $k_{\min} = \pi/c$ and $k_{\max} = \pi/h$, see [3].

Since the frequency k is involved in the contraction factor in a complicated fashion, (5) can not be solved analytically. We show here a new idea, namely to approximate ρ for large k asymptotically accurately in order to solve the optimimization problem (5). To this end, we introduce

$$\rho_{app}(k,\eta,L,p_1,p_2) = \frac{\sqrt{\eta+k^2}-p_1}{\sqrt{\eta+k^2}+p_1} \cdot \frac{\sqrt{\eta+k^2}-p_2}{\sqrt{\eta+k^2}+p_2} \cdot e^{-2\sqrt{\eta+k^2}L}, \qquad (6)$$

which is the contraction factor obtained by [3] in the infinite, symmetric domain decomposition analysis.

Theorem 1 (Approximation to the contraction factor). *The difference between the exact and approximate contraction factor satisfies the estimate*

$$|\rho(k,\eta,L,p_1,p_2,a,b) - \rho_{app}(k,\eta,L,p_1,p_2)| \le 4e^{-2\sqrt{\eta+k^2}(a+L)}. \qquad (7)$$

Proof. The contraction factor ρ can be rewritten in the form

$$\rho = \rho_{app} + (1-\rho_{app})\left(\frac{\sqrt{\eta+k^2}-p_2}{\sqrt{\eta+k^2}+p_2}e^{-2\sqrt{\eta+k^2}b} + \frac{\sqrt{\eta+k^2}-p_1}{\sqrt{\eta+k^2}+p_1}e^{-2\sqrt{\eta+k^2}(a+L)}\right), \qquad (8)$$

and the result then follows by the triangle inequality and using that $-1 \le \rho_{app} \le 1$. □

Theorem 1 shows that ρ_{app} is a good approximation for k large, but not for k small. We thus propose to only use the approximation for k large, and the exact ρ for k small, in order to solve the min-max problem (5) asymptotically. We obtain the following theorems, whose proofs are beyond the scope of this short paper, see our forthcoming paper [6].

Theorem 2 (Optimized parameters, overlapping case). *With the overlap $L > 0$, the parameters $p_1^* = G^{\frac{4}{5}}L^{-\frac{1}{5}}$, $p_2^* = G^{\frac{2}{5}}L^{-\frac{3}{5}}$ solve asymptotically the equioscillation equations*

$$\rho(k_{\min},\eta,L,p_1^*,p_2^*,a,b) = -\rho_{app}(\bar{k}_1,\eta,L,p_1^*,p_2^*) = \rho_{app}(\bar{k}_2,\eta,L,p_1^*,p_2^*), \quad (9)$$

where $G = G(k_{\min},\eta,a,b) := \frac{\sqrt{\eta+k_{\min}^2}}{2}\frac{1-e^{-2\sqrt{\eta+k_{\min}^2}(a+b)}}{(1-e^{-2\sqrt{\eta+k_{\min}^2}a})(1-e^{-2\sqrt{\eta+k_{\min}^2}b})}$, and $\bar{k}_1 = G^{\frac{3}{5}}L^{-\frac{2}{5}}$ and $\bar{k}_2 = G^{\frac{1}{5}}L^{-\frac{4}{5}}$ are the locations of the interior maxima of ρ_{app}. Furthermore, p_1^*, p_2^* approximately solve the min-max problem (5) as $L \to 0$ with the error estimate

$$|\rho_{\max}(L,p_1^*,p_2^*) - \min_{p_1,p_2>0}\rho_{\max}(L,p_1,p_2)| \le 4e^{-2\sqrt{\eta+\underline{k}_1^2}(a+L)}, \qquad (10)$$

where $\underline{k}_1 = cL^{-\frac{1}{5}}$, and c is some constant. The associated contraction factor is

$$\rho_{\max}(L,p_1^*,p_2^*) = 1 - 4G^{\frac{1}{5}}L^{\frac{1}{5}} + O(L^{\frac{2}{5}}). \qquad (11)$$

Theorem 3 (Optimized parameters, nonoverlapping case). *When $L = 0$, the parameters $\bar{p}_1 = 2^{\frac{1}{4}}G^{\frac{3}{4}}k_{\max}^{\frac{1}{4}}$, $\bar{p}_2 = 2^{\frac{3}{4}}G^{\frac{1}{4}}k_{\max}^{\frac{3}{4}}$ solve asymptotically the equioscillation equations*

$$\rho(k_{\min},\eta,0,\bar{p}_1,\bar{p}_2,a,b) = -\rho_{app}(\bar{k},\eta,0,\bar{p}_1,\bar{p}_2) = \rho_{app}(k_{\max},\eta,0,\bar{p}_1,\bar{p}_2), \tag{12}$$

where $\bar{k} = (2G)^{\frac{1}{2}} k_{\max}^{\frac{1}{2}}$ *is the location of the interior maximum of* ρ_{app}*. Furthermore,* $\bar{p}_1$*,* $\bar{p}_2$ *solve approximately the min-max problem (5) as* $k_{\max} \to \infty$ *with the error estimate*

$$|\rho_{\max}(0,\bar{p}_1,\bar{p}_2) - \min_{p_1,p_2>0} \rho_{\max}(0,p_1,p_2)| \le 4e^{-2\sqrt{\eta+\underline{k}_0^2}a}, \tag{13}$$

where $\underline{k}_0 = ck_{\max}^{\frac{1}{4}}$*, and c is some constant. The associated contraction factor is*

$$\rho_{\max}(0,\bar{p}_1,\bar{p}_2) = 1 - 2^{\frac{7}{4}} G^{\frac{1}{4}} k_{\max}^{-\frac{1}{4}} + O(k_{\max}^{-\frac{1}{2}}). \tag{14}$$

3 Swapping the Robin parameters

Theorems 2 and 3 do not allow us to see which Robin parameter of the two should be used on which subdomain, swapping them leads to the same asymptotic results. To see the influence of the domain size, we have to push the asymptotic analysis further.

The overlapping case: To get further insight, we compute one more term in the asymptotic expansions of the equioscillation equations (9) both for the parameter ordering given, and swapped. We obtain at the interior maximum points $\bar{k}_1$ and $\bar{k}_2$ the same result $\rho_{\max} = 1 - 4G^{\frac{1}{5}}L^{\frac{1}{5}} + 8G^{\frac{2}{5}}L^{\frac{2}{5}} + O(L^{\frac{3}{5}})$, while at $k_{\min}$

$$\rho(k_{\min},\eta,L,p_1^*,p_2^*,a,b) = 1 - 4G^{\frac{1}{5}}L^{\frac{1}{5}} + 8G^{\frac{2}{5}}L^{\frac{2}{5}}(1+d) + O(L^{\frac{3}{5}}), \tag{15}$$

$$\rho(k_{\min},\eta,L,p_2^*,p_1^*,a,b) = 1 - 4G^{\frac{1}{5}}L^{\frac{1}{5}} + 8G^{\frac{2}{5}}L^{\frac{2}{5}}(1-d) + O(L^{\frac{3}{5}}), \tag{16}$$

where the additional term

$$d := \frac{e^{-2\sqrt{\eta+k_{\min}^2}a} - e^{-2\sqrt{\eta+k_{\min}^2}b}}{1 - e^{-2\sqrt{\eta+k_{\min}^2}(a+b)}} \tag{17}$$

appears. Hence, if $d > 0$, i.e. $a < b$, one should swap the parameters to get a uniform contraction factor bounded by $\rho_{\max}$, since $G > 0$, and we get

Theorem 4. *If* $a < b$ *and* L *is small, swapping the transmission parameters* p_1^* *and* p_2^* *improves the performance of the optimized Schwarz method (2), and the bigger the value of* d *in (17) is, the larger the improvement becomes.*

The natural next question is: from which overlap on should one swap the transmission parameters to get better performance? Notice that $|\rho|$ has the same asymptotic expansions at $\bar{k}_1$ and $\bar{k}_2$ up to $O(L^{\frac{3}{5}})$. We should thus look for an $L^* > 0$ such that when $L < L^*$

$$|\rho(k_{\min},\eta,L,p_2^*,p_1^*,a,b)| < |\rho(k_{\min},\eta,L,p_1^*,p_2^*,a,b)|. \tag{18}$$

Though it is hard to obtain an explicit expression of such an L^*, the inequality (18) can be used numerically as a necessary condition for judging when we should swap the optimized transmission parameters. A sufficient condition can be obtained as follows: if

$$p_1^* > \sqrt{\eta+k_{\min}^2}\,\frac{1+e^{-2\sqrt{\eta+k_{min}^2}a}}{1-e^{-2\sqrt{\eta+k_{\min}^2}a}}, \tag{19}$$

then (4) implies $\rho(k_{\min},\eta,L,p_2^*,p_1^*,a,b) > 0$. Now using (8), we obtain with a direct comparison after a short calculation $\rho(k_{\min},\eta,L,p_2^*,p_1^*,a,b) < \rho(k_{\min},\eta,L,p_1^*,p_2^*,a,b)$, which together with the positivity implies (18). Solving (19) asymptotically yields

$$L < \frac{1}{16\sqrt{\eta+k_{\min}^2}}\,\frac{(1-e^{-2\sqrt{\eta+k_{\min}^2}(a+b)})^4(1-e^{-2\sqrt{\eta+k_{\min}^2}a})}{(1+e^{-2\sqrt{\eta+k_{\min}^2}a})^5(1-e^{-2\sqrt{\eta+k_{\min}^2}b})^4} =: L^*. \tag{20}$$

Noting that $GL^* < 1$, all the above mentioned asymptotic expansions converge, and we arrive at

Theorem 5. *For $a < b$, with an overlap $L < L^*$, where L^* is defined in (20), swapping the transmission parameters p_1^* and p_2^* in the optimized Schwarz method (2) improves the performance.*

The nonoverlapping case: we again compute one more term in the expansions of the equioscillation equations (12), both for the parameter ordering given, and swapped. We obtain as in the overlapping case at $\bar{k}$ and $k_{\max}$ the same result, $1 - 2^{\frac{7}{4}}G^{\frac{1}{4}}k_{\max}^{-\frac{1}{4}} + 2^{\frac{5}{2}}G^{\frac{1}{2}}k_{\max}^{-\frac{1}{2}} + O(k_{\max}^{-\frac{3}{4}})$, while at $k_{\min}$

$$\rho(k_{\min},\eta,0,\bar{p}_1,\bar{p}_2,a,b) = 1-2^{\frac{7}{4}}G^{\frac{1}{4}}k_{\max}^{-\frac{1}{4}}+2^{\frac{5}{2}}G^{\frac{1}{2}}k_{\max}^{-\frac{1}{2}}(1+d)+O(k_{\max}^{-\frac{3}{4}}), \tag{21}$$

$$\rho(k_{\min},\eta,0,\bar{p}_2,\bar{p}_1,a,b) = 1-2^{\frac{7}{4}}G^{\frac{1}{4}}k_{\max}^{-\frac{1}{4}}+2^{\frac{5}{2}}G^{\frac{1}{2}}k_{\max}^{-\frac{1}{2}}(1-d)+O(k_{\max}^{-\frac{3}{4}}), \tag{22}$$

where the same term d from (17) appears. Hence, as in the overlapping case, if

$$|\rho(k_{\min},\eta,0,\bar{p}_2,\bar{p}_1,a,b)| < |\rho(k_{\min},\eta,0,\bar{p}_1,\bar{p}_2,a,b)|, \tag{23}$$

swapping the transmission parameters in the optimized Schwarz method (2) improves the performance. Solving $\bar{p}_1 > \sqrt{\eta+k_{\min}^2}\frac{1+e^{-2\sqrt{\eta+k_{\min}^2}a}}{1-e^{-2\sqrt{\eta+k_{\min}^2}a}}$ with $k_{\max} = \pi/h$ gives an $\bar{h} = 2G^3\pi(\frac{1}{\sqrt{\eta+k_{\min}^2}}\frac{1-e^{-2\sqrt{\eta+k_{\min}^2}a}}{1+e^{-2\sqrt{\eta+k_{\min}^2}a}})^4$ such that for any $h < \bar{h}$ inequality (23) holds, and we get

Theorem 6. *If $a < b$ and there is no overlap, and if $h < \bar{h}$, swapping the transmission parameters $\bar{p}_1$ and $\bar{p}_2$ of the optimized Schwarz method (2) improves the performance.*

L	Transmission parameters h=	1/50	1/100	1/200	1/400	1/800
h	$p_1 = p_1^*(p_2^*), p_2 = p_2^*(p_1^*)$	6(9)	7(9)	8(9)	10(10)	12(11)
h	$p_1 = p_{1,\mathtt{inf}}^*(p_{2,\mathtt{inf}}^*), p_2 = p_{2,\mathtt{inf}}^*(p_{1,\mathtt{inf}}^*)$	7(12)	8(15)	10(18)	11(18)	13(15)
0	$p_1 = \bar{p}_1(\bar{p}_2), p_2 = \bar{p}_2(\bar{p}_1)$	11(10)	14(12)	17(14)	19(16)	24(20)
0	$p_1 = \bar{p}_{1,\mathtt{inf}}(\bar{p}_{2,\mathtt{inf}}), p_2 = \bar{p}_{2,\mathtt{inf}}(\bar{p}_{1,\mathtt{inf}})$	13(14)	16(14)	18(17)	22(20)	27(24)

Table 1 Number of iterations required by the various optimized Schwarz methods.

4 Numerical experiments

We consider the model problem (1), where $\eta = 2$, and the domain $\Omega = (-a, b) \times (0, 1)$ is decomposed into $\Omega_1 = (-a, L) \times (0, 1)$, $\Omega_2 = (0, b) \times (0, 1)$, with $a = 0.1$, and $b = 0.5$. We discretize (2) with the classical five-point finite difference scheme on a uniform mesh with mesh parameter h, and simulate directly the error equations, i.e. $f = 0$. The initial guesses on the interfaces are chosen randomly so that all frequencies are present. We count the number of iterations required to reach an error reduction of $1e-6$, and compare the results obtained with our parameters to those obtained with parameters from the infinite domain decomposition analysis in [3], denoted by the subscript "inf". Table 1 shows the corresponding results, both for the overlapping case, $L = h$, and the nonoverlapping case, $L = 0$, with the results after parameter swapping in parentheses. In both cases, our parameters require less iterations than those from the infinite domain decomposition analysis. For the new parameters in the nonoverlapping case, the swapped transmission parameters perform better, which is in agreement with Theorem 6, since all the mesh sizes involved in this computation are less than $\bar{h} \approx 0.0234$. For the overlapping case, we see that swapping for the larger mesh sizes is not advantageous, but as soon as the mesh size becomes small, the swapped parameters catch up to give lower iteration numbers. The situation is similar for the parameters from the infinite domain decomposition analysis, but a more refined mesh would be required. We also plot all the results in Figure 1, on the left for the overlapping case and on the right for the nonoverlapping case. We observe that each method performs as predicted by the asymptotic analysis, except in the case of the infinite domain decomposition analysis with overlap where a more refined mesh would be needed to reach the asymptotic regime.

We next illustrate numerically that there is indeed a critical value L^* so that when the ovelap $L < L^*$, swapping the parameters can improve the performance, as predicted by Theorem 5. Table 2 shows the error after 10 iterations of the optimized Schwarz method with varying overlap and $h = 1/800$. We see that in this case L^* lies in between $3h$ and $4h$.

To finally test how well our analysis predicts the optimal parameters to be used in a numerical setting, we vary the parameters p_1 and p_2 with 51 equidistant samples of each for a fixed problem with $h = 1/200$ and count for each parameter pair (p_1, p_2) the number of iterations to reach a residual of $1e-6$. In the left column of Figure 2 we show a contour plot before transmission

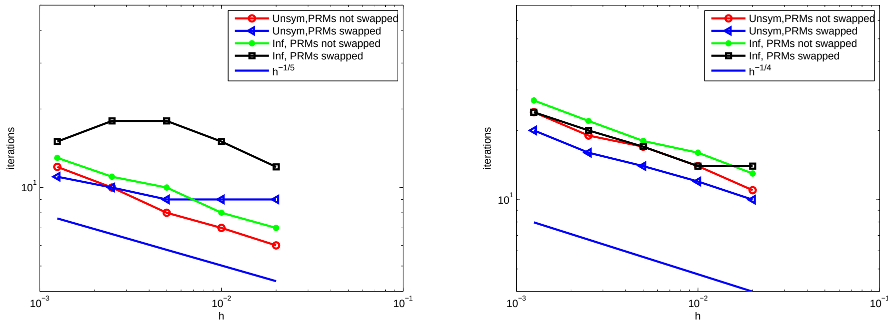

Fig. 1 Number of iterations required by the optimized Schwarz methods: overlapping case on the left, nonoverlapping case on the right.

L	$2h$	$3h$	$4h$	$5h$
$p_1 = p_1^*, p_2 = p_2^*$	$1.5467e-06$	$1.6942e-07$	$1.6390e-08$	$1.0579e-08$
$p_1 = p_2^*, p_2 = p_1^*$	$1.4941e-07$	$3.7717e-08$	$3.4429e-08$	$2.9584e-08$

Table 2 Error reduction of each optimized Schwarz method.

parameter swapping, the overlapping case (on the top) and the nonoverlapping case (at the bottom), and in the right column the corresponding contour plots with transmission parameters swapped. We see that the transmission parameters obtained by our analysis ($*$) are always closer to the numerical optimum than those from the infinite domain decomposition analysis ($\circ$). The left and right columns of Figure 2 also show numerically that there exist at least 2 local minimizers and the swapped parameters are close to the one resulting in the smaller contraction factor.

5 Conclusion

We have shown that when there are two different transmission conditions to be imposed between subdomains, the geometry, in our case the size of the subdomain, can indicate which subdomain should use which transmission condition. Using asymptotic analysis for a two subdomain model problem, we developed a necessary and a sufficient condition on the overlap or mesh size for when transmission conditions should be swapped between neighboring subdomains of different size to get better performance. Numerical experiments confirm well our theoretical findings. We also observed numerically that the min-max problem (5) has at least two local minimizers, but a more refined pre-asymptotic study of this problem is needed for a complete understanding of (5).

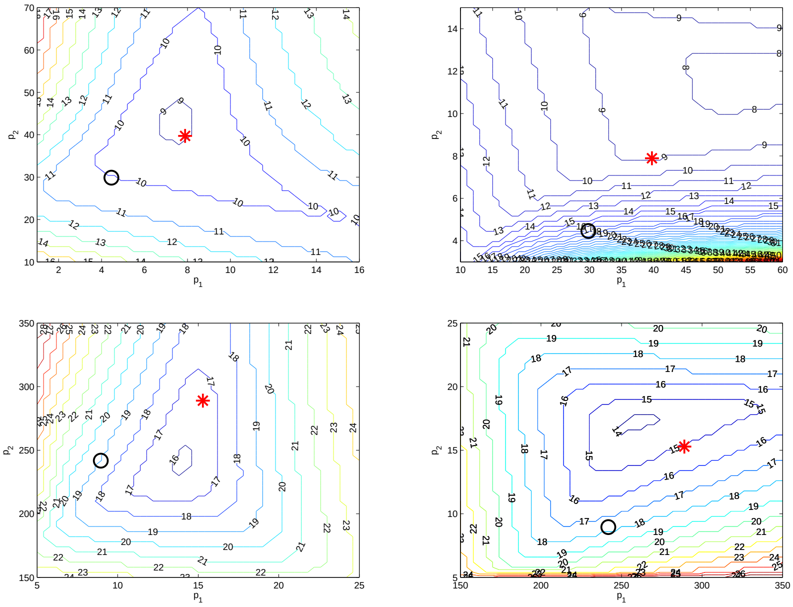

Fig. 2 Optimized parameters found by our analysis (∗), as well as by the infinite domain decomposition analysis (∘), compared to the performance of other values of the parameters: first row for the overlapping case, second row for the nonoverlapping case, with the parameters before parameter swapping in the left column and after swapping in the right column.

References

[1] Christine Bernardi, Yvon Maday, and Anthony T. Patera. Domain decomposition by the mortar element method. In *Asymptotic and numerical methods for partial differential equations with critical parameters*, pages 269–286. Springer, 1993.

[2] Petter E. Bjørstad and Olof B. Widlund. Iterative methods for the solution of elliptic problems on regions partitioned into substructures. *SIAM J. Numer. Anal.*, 23(6):1097–1120, 1986.

[3] Martin J. Gander. Optimized Schwarz methods. *SIAM J. Numer. Anal.*, 44(2):699–731, 2006.

[4] Martin J. Gander. On the influence of geometry on optimized Schwarz methods. *SeMA Journal*, 53(1):71–78, 2011.

[5] Martin J. Gander and Yingxiang Xu. Optimized Schwarz methods for circular domain decompositions with overlap. *SIAM J. Numer. Anal.*, 52(4):1981–2004, 2014.

[6] Yingxiang Xu. Optimized Schwarz methods with unsymmetric domain decomposition. *Submitted*, 2015.