Recent Advances in Robust Coarse Space Construction

An ASM type theory for P.L. Lions algorithm – Optimized Schwarz Methods

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DD23 2015, Jeju Island, Korea

GOAL: Endow P.L. Lions algorithm (1988) with an "ASM-like" theory

$$
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-\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1, \\
u_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\
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(n_1 and n_2 are the outward normal on the boundary of the subdomains)

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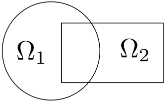
with $\alpha > 0$. Overlap is not necessary for convergence.
Parameter α can be optimized for.
Extended to the Helmholtz equation (B. Desprès, 1991)
a.k.a FETI 2 LM (Two-Lagrange Multiplier) Method, 1998.

Outline

The original Schwarz Method (H.A. Schwarz, 1870)



$$-\Delta(u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Schwarz Method : $(u_1^n, u_2^n) \to (u_1^{n+1}, u_2^{n+1})$ with

$$-\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1$$
$$u_1^{n+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega$$
$$u_1^{n+1} = u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}.$$

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Parallel algorithm.

An introduction to Additive Schwarz

Consider the discretized Poisson problem: $Au = f \in \mathbb{R}^n$.

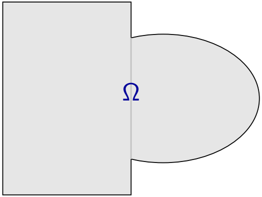
Given a decomposition of $[\![1; n]\!]$, $(\mathcal{N}_1, \mathcal{N}_2)$, define:

- the restriction operator R_i from $\mathbb{R}^{[\![1;n]\!]}$ into $\mathbb{R}^{\mathcal{N}_i}$,
- R_i^T as the extension by 0 from $\mathbb{R}^{\mathcal{N}_i}$ into $\mathbb{R}^{[\![1;n]\!]}$.

$u^m \longrightarrow u^{m+1}$ by solving concurrently:

$$u_1^{m+1} = u_1^m + A_1^{-1} R_1 (f - Au^m) \qquad u_2^{m+1} = u_2^m + A_2^{-1} R_2 (f - Au^m)$$

where $u_i^m = R_i u^m$ and $A_i := R_i A R_i^T$.



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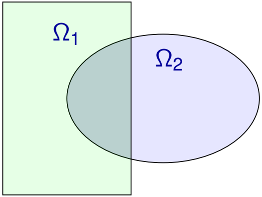

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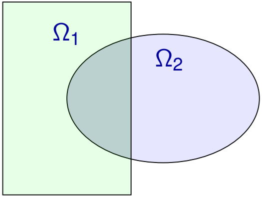
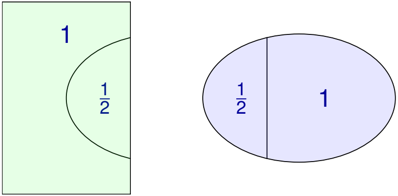


We have effectively divided, but we have yet to conquer.

Duplicated unknowns coupled via a *partition of unity*:

$$I = \sum_{i=1}^{N} R_i^T D_i R_i.$$



Then, $u^{m+1} = \sum_{i=1}^{N} R_i^T D_i u_i^{m+1}$.

$$M_{RAS}^{-1} = \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i.$$

RAS algorithm (Cai & Sarkis, 1999)

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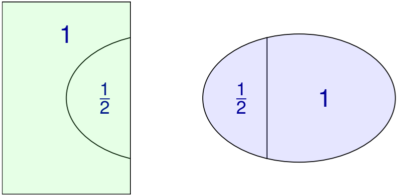

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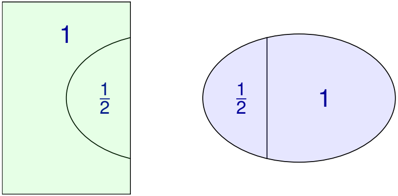

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RAS algorithm (Cai & Sarkis, 1999)

Schwarz algorithm iterates on a pair of local functions (u_m^1, u_m^2)
RAS algorithm iterates on the global function u^m

Schwarz and RAS

Discretization of the classical Schwarz algorithm and the iterative RAS algorithm:

$$U^{n+1} = U^n + M_{RAS}^{-1} r^n \, , r^n := F - A\, U^n .$$

are equivalent

$$U^n = R_1^T D_1 U_1^n + R_2^T D_2 U_2^n \, .$$

(Efstathiou and Gander, 2002).

Operator M_{RAS}^{-1} is used as a preconditioner in Krylov methods for non symmetric problems.

Algebraic formulation - RAS and ASM

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ASM: a symmetrized version of RAS

$$M_{RAS}^{-1} := \sum_{i=1}^{N} R_i^T \, D_i \, A_i^{-1} \, R_i \, .$$

A symmetrized version: Additive Schwarz Method (ASM),

$$M_{ASM}^{-1} := \sum_{i=1}^{N} R_i^T \, A_i^{-1} \, R_i \qquad (1)$$

is used as a preconditioner for the conjugate gradient (CG) method.

Although RAS is more efficient, ASM is amenable to to condition number estimates.

Chronological curiosity: First paper on Additive Schwarz dates back to 1989 whereas RAS paper was published in 1998

Adding a coarse space

One level methods are not scalable.
We add a coarse space correction (*aka* second level)
Let V_H be the coarse space and Z be a basis, $V_H = \text{span}\, Z$, writing $R_0 = Z^T$ we define the two level preconditioner as:

$$M^{-1}_{ASM,2} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^N R_i^T A_i^{-1} R_i .$$

The Nicolaides approach (1987) is to use the kernel of the operator as a coarse space, this is the constant vectors, in local form this writes:

$$Z := (R_i^T\, D_i R_i \mathbf{1})_{1\le i\le N}$$

where D_i are chosen so that we have a partition of unity:

$$\sum_{i=1}^N R_i^T D_i R_i = Id .$$

Key notion: Stable splitting (J. Xu, 1989)

Theoretical convergence result

Theorem (Widlund, Dryija)

Let $M^{-1}_{ASM,2}$ be the two-level additive Schwarz method:

$$\kappa(M^{-1}_{ASM,2}\, A) \le C \left(1 + \frac{H}{\delta}\right)$$

where δ is the size of the overlap between the subdomains and H the subdomain size.

This does indeed work very well

Number of subdomains	8	16	32	64
ASM	18	35	66	128
ASM + Nicolaides	20	27	28	27

Fails for highly heterogeneous problems
You need a larger and adaptive coarse space

GenEO

Strategy

Define an appropriate coarse space $V_{H2} = \mathrm{span}(Z_2)$ and use the framework previously introduced, writing $R_0 = Z_2^T$ the two level preconditioner is:

$$P_{ASM2}^{-1} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i.$$

The coarse space must be

- Local (calculated on each subdomain) $\to$ parallel
- Adaptive (calculated automatically)
- Easy and cheap to compute
- Robust (must lead to an algorithm whose convergence is proven not to depend on the partition nor the jumps in coefficients)

Adaptive Coarse space for highly heterogeneous Darcy and (compressible) elasticity problems:
Geneo .EVP per subdomain:

Find $V_{j,k} \in \mathbb{R}^{\mathcal{N}_j}$ and $\lambda_{j,k} \geq 0$:

$$D_j R_j A R_j^T D_j V_{j,k} = \lambda_{j,k} A_j^{Neu} V_{j,k}$$

In the two-level ASM, let τ be a user chosen parameter:
Choose eigenvectors $\lambda_{j,k} \geq \tau$ per subdomain:

$$Z := \left(R_j^T D_j V_{j,k} \right)_{\lambda_{j,k} \geq \tau}^{j=1,\dots,N}$$

This automatically includes Nicolaides CS made of Zero Energy Modes.

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Theory of GenEO

Two technical assumptions.

Theorem (Spillane, Dolean, Hauret, N., Pechstein, Scheichl (Num. Math. 2013))

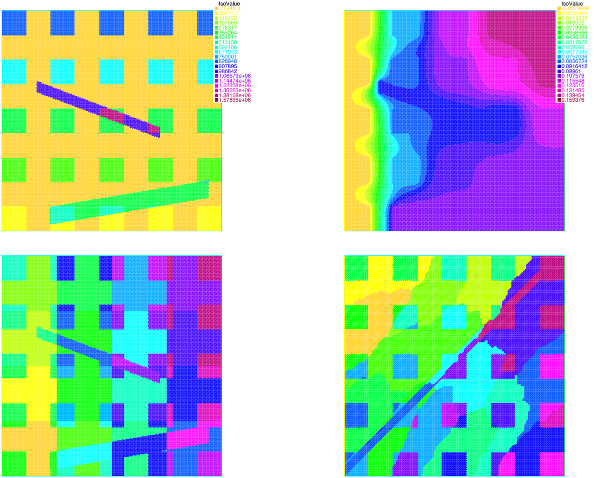
If for all j*:* $0 < \lambda_{j,m_{j+1}} < \infty$*:*

$$\kappa(M_{ASM,2}^{-1}A) \;\leq\; (1+k_0)\Big[2 + k_0\,(2k_0+1)\,\Big(1+\tau\Big)\Big]$$

Possible criterion for picking τ: (used in our Numerics)

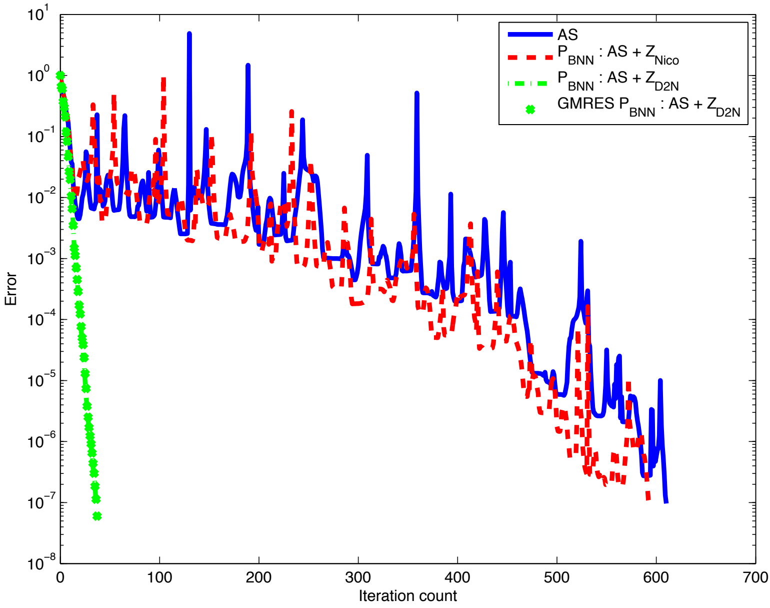
$$\tau := \min_{j=1,\dots,N} \frac{H_j}{\delta_j}$$

$H_j \dots$ subdomain diameter, $\delta_j \dots$ overlap



Channels and inclusions: $1 \leq \alpha \leq 1.5 \times 10^6$, the solution and partitionings (Metis or not)

Convergence



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Parameter α can be optimized for.
Extended to the Helmholtz equation (B. Desprès, 1991)
a.k.a FETI 2 LM (Two-Lagrange Multiplier) Method, 1998.

GOAL of this work

(Recap) $A_i := R_i A R_i^T$, $1 \leq i \leq N$

1. Schwarz algorithm at the continuous level (partial differential equation)
2. Algebraic reformulation $\Rightarrow M_{RAS}^{-1} := \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i$
3. Symmetric variant $\Rightarrow M_{AS}^{-1} := \sum_{i=1}^{N} R_i^T A_i^{-1} R_i$
4. Adaptive Coarse space with prescribed targeted convergence rate
   $\Rightarrow$ Find $V_{j,k} \in \mathbb{R}^{\mathcal{N}_j}$ and $\lambda_{j,k} \geq 0$:

$$D_j A_j D_j V_{j,k} = \lambda_{j,k} A_j^{Neu} V_{j,k}$$

GOAL: Develop a theory and computational framework for P.L. Lions algorithm similar to what was done for Schwarz algorithm for a S.P.D. matrix A.

ORAS: Optimized RAS

1. P.L. Lions algorithm at the continuous level (partial differential equation)
2. Algebraic formulation for overlapping subdomains ⇒ Let B_i be the matrix of the Robin subproblem in each subdomain $1 \le i \le N$, define $M_{ORAS}^{-1} := \sum_{i=1}^{N} R_i^T D_i B_i^{-1} R_i$, *Optimized multiplicative, additive, and restricted additive Schwarz preconditioning, St Cyr et al, 2007*
3. Symmetric variant ⇒
   1. $M_{OAS}^{-1} := \sum_{i=1}^{N} R_i^T B_i^{-1} R_i$ (Natural but K.O.)
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P.L. Lions and ORAS

Provided subdomains overlap, discretization of the classical P.L. Lions algorithm and the iterative ORAS algorithm:

$$U^{n+1} = U^n + M^{-1}_{ORAS} r^n \,, r^n := F - A\, U^n.$$

are equivalent

$$U^n = R_1^T D_1 U_1^n + R_2^T D_2 U_2^n \,,$$

(St Cyr, Gander and Thomas, 2007).

- **Huge** simplification in the implementation: no boundary right hand side discretization
- Operator M^{-1}_{ORAS} is used as a preconditioner in Krylov methods for non symmetric problems.
- *First step in a global theory*

Outline

Fictitious Space Lemma

Lemma (Fictitious Space Lemma, Nepomnyaschikh 1991)

Let H and H_D be two Hilbert spaces. Let a be a symmetric positive bilinear form on H and b on H_D. Suppose that there exists a linear operator $\mathcal{R} : H_D \to H$, such that

- *$\mathcal{R}$ is surjective.*
- *there exists a positive constant c_R such that*

$$a(\mathcal{R}u_D, \mathcal{R}u_D) \leq c_R \cdot b(u_D, u_D) \;\; \forall u_D \in H_D\,. \tag{2}$$

- *Stable decomposition: there exists a positive constant c_T such that for all $u \in H$ there exists $u_D \in H_D$ with $\mathcal{R}u_D = u$ and*

$$c_T \cdot b(u_D, u_D) \leq a(\mathcal{R}u_D, \mathcal{R}u_D) = a(u, u)\,. \tag{3}$$

Fictitious Space Lemma (continued)

Lemma (FSL continued)

We introduce the adjoint operator $\mathcal{R}^* : H \to H_D$ *by* $(\mathcal{R}u_D, u) = (u_D, \mathcal{R}^*u)_D$ *for all* $u_D \in H_D$ *and* $u \in H$. *Then we have the following spectral estimate*

$$c_T \cdot a(u,u) \leq a\left(\mathcal{R}B^{-1}\mathcal{R}^*Au,\, u\right) \leq c_R \cdot a(u,u)\,, \quad \forall u \in H \qquad (4)$$

which proves that the eigenvalues of operator $\mathcal{R}B^{-1}\mathcal{R}^*A$ *are bounded from below by* c_T *and from above by* c_R.

FSL and DDM

FSL lemma is the Lax-Milgram lemma of domain decomposition methods.

In combination with GenEO techniques it yields adaptive coarse spaces with a targeted condition number.

- Additive Schwarz method
- Hybrid Schwarz method
- Balancing Neumann Neumann and FETI
- Optimized Schwarz method

For a comprehensive presentation:

"An Introduction to Domain Decomposition Methods: algorithms, theory and parallel implementation", V. Dolean, P. Jolivet and F Nataf, https://hal.archives-ouvertes.fr/cel-01100932 , Lecture Notes to appear in SIAM collection, 2015.

FSL and one level SORAS

- $H := \mathbb{R}^{\#\mathcal{N}}$ and the a-bilinear form:

$$a(\mathbf{U},\mathbf{V}) := \mathbf{V}^T A \mathbf{U}. \tag{5}$$

  where A is the matrix of the problem we want to solve.

- H_D is a product space and b a bilinear form defined by

$$H_D := \prod_{i=1}^{N} \mathbb{R}^{\#\mathcal{N}_i} \text{ and } b(\mathcal{U},\mathcal{V}) := \sum_{i=1}^{N} \mathbf{V}_i^T B_i \mathbf{U}_i, \,. \tag{6}$$

- The linear operator $\mathcal{R}_{SORAS}$ is defined as

$$\mathcal{R}_{SORAS} : H_D \longrightarrow H,\, \mathcal{R}_{SORAS}(\mathcal{U}) := \sum_{i=1}^{N} R_i^T D_i \mathbf{U}_i. \tag{7}$$

We have: $M_{SORAS}^{-1} = \mathcal{R}_{SORAS}\, B^{-1}\, \mathcal{R}_{SORAS}^{*}$.

Estimate for the **one level** SORAS

Let k_0 be the maximum number of neighbors of a subdomain and γ_1 be defined as:

$$\gamma_1 := \max_{1 \leq i \leq N} \max_{\mathbf{U}_i \in \mathbb{R}^{\#\mathcal{N}_i} \setminus \{0\}} \frac{(D_i \mathbf{U}_i)^T A_i (D_i \mathbf{U}_i)}{\mathbf{U}_i^T B_i \mathbf{U}_i}$$

We can take $c_R := k_0 \, \gamma_1$.
Let k_1 be the maximum multiplicity of the intersection between subdomains and τ_1 be defined as:

$$\tau_1 := \min_{1 \leq i \leq N} \min_{U_i \in \mathbb{R}^{\#\mathcal{N}_i} \setminus \{0\}} \frac{\mathbf{U}_i{}^T A_i^{Neu} \mathbf{U}_i}{\mathbf{U}_i{}^T B_i \mathbf{U}_i} \, .$$

We can take $c_T := \frac{\tau_1}{k_1}$.
We have:

$$\frac{\tau_1}{k_1} \leq \lambda(M_{SORAS}^{-1} A) \leq k_0 \, \gamma_1 \, .$$

Definition (Generalized Eigenvalue Problem for the upper bound)

$$\text{Find } (\mathbf{U}_{ik}, \mu_{ik}) \in \mathbb{R}^{\#\mathcal{N}_i} \setminus \{0\} \times \mathbb{R} \text{ such that}$$
$$D_i A_i D_i \mathbf{U}_{ik} = \mu_{ik} B_i \, \mathbf{U}_{ik} \quad . \tag{8}$$

Let $\gamma > 0$ be a user-defined threshold, we define $Z_{geneo}^{\gamma} \subset \mathbb{R}^{\#\mathcal{N}}$ as the vector space spanned by the family of vectors $(R_i^T D_i \mathbf{U}_{ik})_{\mu_{ik} > \gamma, 1 \le i \le N}$ corresponding to eigenvalues larger than γ.

Control of the lower bound

Definition (Generalized Eigenvalue Problem for the lower bound)

For each subdomain $1 \leq j \leq N$, we introduce the generalized eigenvalue problem

$$\begin{array}{c}\text{Find } (\mathbf{V}_{jk}, \lambda_{jk}) \in \mathbb{R}^{\#\mathcal{N}_j} \setminus \{0\} \times \mathbb{R} \text{ such that} \\ A_j^{Neu} \mathbf{V}_{jk} = \lambda_{jk} B_j \mathbf{V}_{jk} \ .\end{array} \tag{9}$$

Let $\tau > 0$ be a user-defined threshold, we define $Z_{geneo}^{\tau} \subset \mathbb{R}^{\#\mathcal{N}}$ as the vector space spanned by the family of vectors $(R_j^T D_j \mathbf{V}_{jk})_{\lambda_{jk} < \tau, 1 \leq j \leq N}$ corresponding to eigenvalues smaller than τ.

Two level SORAS-GENEO-2 preconditioner

Definition (Two level SORAS-GENEO-2 preconditioner)

Let P_0 denote the a-orthogonal projection on the SORAS-GENEO-2 coarse space

$$Z_{\text{GenEO-2}} := Z^{\tau}_{geneo} \bigoplus Z^{\gamma}_{geneo} \,,$$

the two-level SORAS-GENEO-2 preconditioner is defined:

$$\boxed{M^{-1}_{SORAS,2} := P_0 A^{-1} + (I_d - P_0)\, M^{-1}_{SORAS}\, (I_d - P_0^T)}$$

where $P_0 A^{-1} = R_0^T (R_0\, A\, R_0^T)^{-1} R_0$, see J. Mandel, 1992.

Two level SORAS-GENEO-2 preconditioner

Theorem (Haferssas, Jolivet and N., 2015)

Let γ and τ be user-defined targets. Then, the eigenvalues of the two-level SORAS-GenEO-2 preconditioned system satisfy the following estimate

$$\frac{1}{1+\frac{k_1}{\tau}} \leq \lambda(M^{-1}_{SORAS,2}\, A) \leq \max(1, k_0\, \gamma)$$

What if one level method is M^{-1}_{OAS}:

$$\text{Find } (\mathbf{V}_{jk}, \lambda_{jk}) \in \mathbb{R}^{\#\mathcal{N}_j} \setminus \{0\} \times \mathbb{R} \text{ such that}$$
$$A_j^{Neu}\mathbf{V}_{jk} = \lambda_{jk} D_j B_j D_j \mathbf{V}_{jk} \ .$$

Outline

Nearly incompressible elasticity

Material properties: Young modulus E and Poisson ratio ν or alternatively by its Lamé coefficients λ and μ:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \text{ and } \mu = \frac{E}{2(1+\nu)}\,.$$

For ν close to $1/2$, the variational problem consists in finding $(\boldsymbol{u}_h, p_h) \in \mathcal{V}_h := \mathbb{P}_2^d \cap H_0^1(\Omega) \times \mathbb{P}_1$ such that for all $(\boldsymbol{v}_h, q_h) \in \mathcal{V}_h$

$$\begin{cases} \int_\Omega 2\mu \underline{\underline{\varepsilon}}(\boldsymbol{u}_h) : \underline{\underline{\varepsilon}}(\boldsymbol{v}_h) dx & - \int_\Omega p_h \operatorname{div}(\boldsymbol{v}_h) dx = \int_\Omega \boldsymbol{f}\boldsymbol{v}_h dx \\ - \int_\Omega \operatorname{div}(\boldsymbol{u}_h) q_h dx & - \int_\Omega \frac{1}{\lambda} p_h q_h = 0 \end{cases}$$

$$\Longrightarrow A\mathbf{U} = \begin{bmatrix} H & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathrm{f} \\ 0 \end{bmatrix} = \mathbf{F}.$$

A is symmetric but no longer positive.

"Robin" interface condition for nearly incompressible elasticity

(Lube, 1998)

$$\sigma(\boldsymbol{u}).n + \mathcal{L}(\alpha)\,\boldsymbol{u} = 0. \text{ on } \partial\Omega_i \backslash \partial\Omega$$

Where $\mathcal{L}$ is constructed from the Lamé coefficient of the material and it is defined as follows

$$\mathcal{L}(\alpha, \lambda, \mu) := \frac{2\alpha\mu(2\mu + \lambda)}{\lambda + 3\mu}.$$

Parameter α in the range $(1., 10.)$.

Figure: 2D Elasticity: Sandwich of steel $(E_1, \nu_1) = (210 \cdot 10^9, 0.3)$ and rubber $(E_2, \nu_2) = (0.1 \cdot 10^9, 0.4999)$.

Metis partitioning

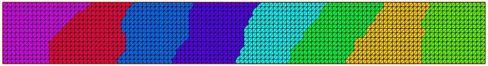


Table: 2D Elasticity. GMRES iteration counts

		AS	SORAS	AS+CS(ZEM)		SORAS +CS(ZEM)		AS-GenEO		SORAS -GenEO-2	
Nb DOFs	Nb subdom	iteration	iteration	iteration	*dim*	iteration	*dim*	iteration	*dim*	iteration	*dim*
35841	8	150	184	117	*24*	79	*24*	110	*184*	13	*145*
70590	16	276	337	170	*48*	144	*48*	153	*400*	17	*303*
141375	32	497	++1000	261	*96*	200	*96*	171	*800*	22	*561*
279561	64	++1000	++1000	333	*192*	335	*192*	496	*1600*	24	*855*
561531	128	++1000	++1000	329	*384*	400	*384*	++1000	*2304*	29	*1220*
1077141	256	++1000	++1000	369	*768*	++1000	*768*	++1000	*3840*	36	*1971*

Numerical results via a Domain Specific Language

`FreeFem++` (`http://www.freefem.org/ff++`), `F.Hecht` interfaced with

- Metis Karypis and Kumar 1998
- SCOTCH Chevalier and Pellegrini 2008
- UMFPACK Davis 2004
- ARPACK Lehoucq et al. 1998
- MPI Snir et al.
- Intel MKL
- PARDISO Schenk et al. 2004
- MUMPS Amestoy et al. 1998
- PETSc solvers Balay et al.
- Slepc via PETSc

Runs on PC (Linux, OSX, Windows, Smartphones) and HPC (Babel@CNRS, HPC1@LJLL, Titane@CEA via GENCI PRACE)

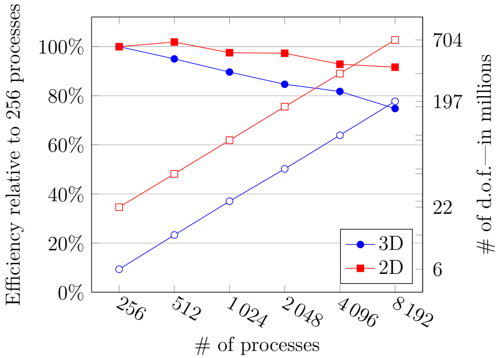
Why use a DS(E)L instead of `C/C++/Fortran/..` ?

- performances close to low-level language implementation,
- hard to beat something as simple as:

$$\textbf{varf}\ \mathrm{a(u,\ v)} = \textbf{int3d}(\mathrm{mesh})([\textbf{dx}(\mathrm{u}),\ \textbf{dy}(\mathrm{u}),\ \textbf{dz}(\mathrm{u})]'\ *\ [\textbf{dx}(\mathrm{v}),\ \textbf{dy}(\mathrm{v}),\ \textbf{dz}(\mathrm{v})]) \\ + \textbf{int3d}(\mathrm{mesh})(\mathrm{f}\ *\ \mathrm{v}) + \textbf{on}(\mathrm{boundary\_mesh})(\mathrm{u} = 0)$$

Weak scalability for heterogeneous elasticity (with FreeFem++ and HPDDM)

Rubber Steel sandwich with automatic mesh partition



(a) Timings of various simulations

200 millions unknowns in 3D wall-clock time: 200. sec.

IBM/Blue Gene Q machine with 1.6 GHz Power A2 processors.

Hours provided by an IDRIS-GENCI project.

Strong scalability in two and three dimensions (with FreeFem++ and HPDDM)

Stokes problem with automatic mesh partition. Driven cavity problem

	N	Factorization	Deflation	Solution	# of it.	Total	# of d.o.f.
3D	1 024	79.2 s	229.0 s	76.3 s	45	387.5 s	$50.63 \cdot 10^6$
	2 048	29.5 s	76.5 s	34.8 s	42	143.9 s	
	4 096	11.1 s	45.8 s	19.8 s	42	80.9 s	
	8 192	4.7 s	26.1 s	14.9 s	41	56.8 s	

Peak performance: 50 millions d.o.f's in 3D in 57 sec.

IBM/Blue Gene Q machine with 1.6 GHz Power A2 processors.

Hours provided by an IDRIS-GENCI project.

HPDDM `https://github.com/hpddm/hpddm` is a framework in C++/MPI for high-performance domain decomposition methods with a Plain Old Data (POD) interface

Outline

Conclusion

Summary

- SORAS preconditioner

$$M_{SORAS}^{-1} := \sum_{i=1}^{N} R_i^T D_i B_i^{-1} D_i R_i$$

  is amenable to a fruitful theory for OSM
- Using two generalized eigenvalue problems, we are able to achieve a targeted convergence rate for OSM
- Freely available via HPDDM library or FreeFem++

Future work

- Another look at parameter α optimization
- Nonlinear time dependent problem (Coarse space reuse)
- Multigrid like three (or more) level methods

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THANK YOU FOR YOUR ATTENTION!

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