Global convergence rates of some multilevel methods for variational and quasi-variational inequalities

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Outline of the talk

- some references of papers which have dealt with multilevel methods for variational inequalities
- for variational inequalities arising from the constrained minimization of a functional:
  - introduce some subspace correction algorithms in a reflexive Banach space
  - give, under a certain assumption, general convergence results (error estimations, included)
  - in the finite element spaces, the introduced algorithms are one-, two-, multilevel and multigrid methods and the constants in the error estimations are explicitly written as functions of
    - the overlapping and mesh parameters, for the one- and two-level methods
    - the number of levels, for the multigrid methods
- extensions of these results for:
  - hybrid multilevel and multigrid methods
  - one- and two-level methods for variational inequalities of the second kind
  - one- and two-level methods for quasi-variational inequalities
  - multilevel and multigrid methods for inequalities with a term given by a nonlinear operator

Some references

• first multilevel method for variational inequalities has been proposed by J. Mandel (Appl. Math. Opt., 1984) for complementarity problems

- globally convergent
- has an optimal computing complexity of iterations, i.e. it is linear with respect to the degrees of freedom of the problem

– an upper bound of the asymptotic convergence rate is given for the two-level method by J. Mandel (C. R. Acad. Sci., 1984)
– some generalizations of the method have been given by E. Gelman and J. Mandel (Math. Program., 1990)

• related methods have been previously introduced by A. Brandt and C. Cryer (SIAM J. Sci. Stat. Comput., 1983) and W. Hackbush and H. Mittelmann (Numer. Math., 1983)

• method introduced by Mandel has been studied later by R. Kornhuber (Numer. Math., 1994) in two variants

- standard monotone multigrid method
- truncated monotone multigrid method (introduced by R. Hoppe and R. Kornhuber (SIAM J. Numer. Anal., 1994) to precondition the conjugate gradient method applied to linear problems)

• these methods have been extended to variational inequalities of the second kind by R. Kornhuber (Numer. Math., 1996 and 2002)

• versions of this method have been applied to Signorini's problem in elasticity by R. Kornhuber and R. Krause (Comp. Visual. Sci., 2001) and B. Wohlmuth and R. Krause (SIAM Sci. Comput., 2003)

• other references

- L. B., X. C. Tai and J. Wang (SIAM J. Numer. Anal., 2003) - global convergence rate for a two-level multiplicative method
- X. C. Tai, (Numer. Math., 2003) - multilevel subset decomposition method
- L. B. (proceedings 17th conference on DDM, 2005) - global convergence rate for a two-level additive method
- L. B. (SIAM J. Numer. Anal., 2006) - projected multilevel relaxation method
- L. B. (IMA J. Numer. Anal., 2014) - justified theoretically the global convergence rate for the standard monotone multigrid methods

• the above list of citations is not exhaustive and, for further information, we can see the review article written by C. Gräser and R. Kornhuber (J. Comput. Math., 2009)

General background

- V - reflexive Banach space, $K \subset V$ - non empty closed convex subset
- $F : K \to \mathbf{R}$ - Gâteaux differentiable functional:

- there exist $p, q > 1$, and for any $M > 0$ there exist $\alpha_M,\ \beta_M > 0$ for which

$$\alpha_M ||v-u||^p \le < F'(v) - F'(u), v-u >, \quad ||F'(v) - F'(u)||_{V'} \le \beta_M ||v-u||^{q-1},$$

for any $u, v \in K$, $||u||, ||v|| \le M$
- F coercive, i.e. $F(v) \to \infty$ as $||v|| \to \infty$, if K is not bounded

$$\Downarrow$$

$$F \text{ convex functional, } 1 < q \le 2 \le p$$

- problem (has an unique solution)

$$u \in K : \ < F'(u), v-u > \ge 0, \text{ for any } v \in K$$

$\Leftrightarrow$

$$u \in K : F(u) \le F(v), \text{ for any } v \in K$$

One and two-level methods

Subspace correction algorithms

• $V_1, \cdots, V_m$ - closed subspaces of V

Assumption

There exists a constant $C_0 > 0$ such that for any $w, v \in K$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying

$$w + \sum_{j=1}^{i-1} w_j + v_i \in K, \quad v - w = \sum_{i=1}^{m} v_i,$$

and the stability condition (for liniar problems, in a more simle form, introduced by J. Xu (SIAM Rev., 1992))

$$\sum_{i=1}^{m} ||v_i|| \le C_0 \left(||v - w|| + \sum_{i=1}^{m} ||w_i|| \right).$$

Algorithm

We start the algorithm with an arbitrary $u^0 \in K$. *At iteration* $n+1$, *having* $u^n \in K$, $n \geq 0$, *we sequentially compute for* $i = 1, \cdots, m$,

$$w_i^{n+1} \in V_i,\ u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K:\ < F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1} > \geq 0,$$

for any $v_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$, *and then we update* $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

Theorem

On the above conditions on the spaces and the functional F, if Assumption 1 holds, we have the following error estimations:
(i) if $p = q = 2$ we have

$$||u^n - u||^2 \leq \frac{2}{\alpha_M} \left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1} \right)^n \left[F(u^0) - F(u) \right].$$

(ii) if $p > q$ we have

$$||u - u^n||^p \leq \frac{p}{\alpha_M} \frac{F(u^0) - F(u)}{\left[1 + n\tilde{C}_2 \left(F(u^0) - F(u) \right)^{\frac{p-q}{q-1}} \right]^{\frac{q-1}{p-q}}}.$$

Constants $\tilde{C}_1$ and $\tilde{C}_2$ can be explicitly written as a function of the functional F (i.e. on α_M, β_M, p and q), the number of subspaces m, the initial approximation u^0 and C_0. Constant $\tilde{C}_1$ is an increasing function and $\tilde{C}_2$ a decreasing function on C_0 in the assumption.

One-level methods

- simplicial regular mesh partition $\mathcal{T}_h$ of mesh size h over $\Omega \subset \mathbf{R}^d$
- $\Omega = \cup_{i=1}^m \Omega_i$ - domain decomposition
- $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$
- the overlapping parameter of the domain decomposition is δ
- linear finite element spaces

$$V_h = \{v \in C^0(\bar{\Omega}) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_h,\ v = 0 \text{ on } \partial\Omega\}$$
$$V_h^i = \{v \in V_h : v = 0 \text{ in } \Omega \backslash \Omega_i\}$$

- spaces V_h and V_h^i, $i = 1, \ldots, m$, are considered as subspaces of $W^{1,\sigma}$
- convex set $K_h \subset V_h$ satisfying

Property

If $v, w \in K_h$, and if $\theta \in C^0(\bar{\Omega})$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \le \theta \le 1$, then

$$L_h(\theta v + (1-\theta)w) \in K_h$$

where L_h is the P_1-Lagrangian interpolation.

Proposition

Assumption 1 holds for the linear finite element spaces, $V = V_h$ and $V_i = V_h^i$, $i = 1, \dots, m$, and for any convex set $K = K_h \subset V_h$ having Property 1. The constant C_0 in Assumption 1 can be written as

$$C_0 = C(m+1)(1 + \frac{m-1}{\delta})$$

where C is independent of the mesh parameter and the domain decomposition.

- the proof uses some functions $\theta_i \in C(\bar{\Omega})$, $\theta_i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_h$, $i = 1, \cdots, m$, associated with the domain decomposition

$$0 \le \theta_i \le 1 \text{ on } \Omega, \; \theta_i = 0 \text{ on } \cup_{j=i+1}^m \Omega_j \backslash \Omega_i \text{ and } \theta_i = 1 \text{ on } \Omega_i \backslash \cup_{j=i+1}^m \Omega_j$$

$$|\partial_{x_k} \theta_i| \le C/\delta, \; \text{ a.e. in } \Omega, \; \text{for any } k = 1, \dots, d.$$

to construct the decomposition of $v - w$ as in Assumption 1,

$$v_i = L_h \left(\theta_i (v - w - \sum_{j=1}^{i-1} v_j) + (1 - \theta_i) w_i \right),$$

for $i = 1, \cdots, m$, where L_h is the Lagrangian interpolation

Two-level methods

- two regular simplicial mesh partitions $\mathcal{T}_h$ and $\mathcal{T}_H$ on $\Omega \subset \mathbf{R}^d$, $\mathcal{T}_h$ being a refinement of $\mathcal{T}_H$
- $\mathcal{T}_h$ and the spaces: V_h, V_h^i, $i = 1, \ldots, m$, defined as for the one-level methods
- introduce the linear finite element space corresponding to the H-level,

$$V_H^0 = \left\{ v \in C^0(\bar{\Omega}_0) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_H,\ v = 0 \text{ on } \partial\Omega_0 \right\},$$

- the two-level method is obtained from the general subspace correction algorithm for $V = V_h$, $K = K_h$, and the subspaces $V_0 = V_H^0$, $V_1 = V_h^1$, $V_2 = V_h^2$, ..., $V_m = V_h^m$
- spaces $V_h, V_H^0, V_h^1, V_h^2, \ldots, V_h^m$, are considered as subspaces of $W^{1,\sigma}$ for $1 \le \sigma \le \infty$

- the following proposition shows that the constant C_0 in Assumption 1 is independent of the mesh and domain decomposition parameters if H/δ and H/h are constant

Proposition

Assumption 1 is verified for the linear finite element spaces $V = V_h$ and $V_0 = V_H^0$, $V_i = V_h^i$, $i = 1, \ldots, m$, and any convex set $K = K_h$ satisfying Property 1. The constant C_0 can be taken of the form

$$C_0 = Cm\left(1 + (m-1)\frac{H}{\delta}\right) C_{d,\sigma}(H,h),$$

in Assumption 1, where C is independent of the mesh and domain decomposition parameters, and

$$C_{d,\sigma}(H,h) = \begin{cases} 1 & \text{if } d = \sigma = 1 \text{ or } 1 \le d < \sigma \le \infty \\ \left(\ln \frac{H}{h} + 1\right)^{\frac{d-1}{d}} & \text{if } 1 < d = \sigma < \infty \\ \left(\frac{H}{h}\right)^{\frac{d-\sigma}{\sigma}} & \text{if } 1 \le \sigma < d < \infty, \end{cases}$$

- the proof uses some nonlinear interpolation operators $I_H : V_h \to V_H$ defined as follows. Let us denote by x_i a node of $\mathcal{T}_H$, by ϕ_i the linear nodal basis function associated with x_i and $\mathcal{T}_H$, and by ω_i the support of ϕ_i. Given a $v \in V_h$, we write

$$I_i^- v = \min_{x \in \omega_i} v(x)^- \text{ and } I_i^+ v = \min_{x \in \omega_i} v(x)^+,$$

where $v(x)^- = \max(0, -v(x))$ and $v(x)^+ = \max(0, v(x))$. Next, we define

$$I_H^- v := \sum_{x_i \text{node of } \mathcal{T}_H} (I_i^- v)\phi_i(x), \quad I_H^+ v := \sum_{x_i \text{node of } \mathcal{T}_H} (I_i^+ v)\phi_i(x),$$

and write

$$I_H v = I_H^+ v - I_H^- v$$

- decomposition of $v - w$ is given by

$$v_0 = w_0 + I_H(v - w - w_0),$$

and, similarly with the one-level method,

$$v_i = L_h\left(\theta_i(v - w - \sum_{j=0}^{i-1} v_j) + (1 - \theta_i)w_i\right)$$

for $i = 1, \ldots, m$

Numerical example

(one-level method versus two-level method

for the two-obstacle problem of a nonlinear elastic membrane)

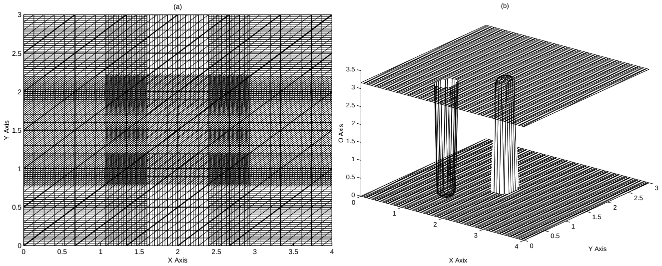
- $\Omega \subset \mathbf{R}^2$, $K = [a, b]$, $a \leq b$, $a, b \in W_0^{1,\sigma}(\Omega)$, $1 < \sigma < \infty$

$$u \in [a, b] : \int_\Omega |\nabla u|^{\sigma-2} \nabla u \nabla (v - u) \geq 0, \text{ for any } v \in [a, b]$$

(exterior forces omitted, $f = 0$)

$$\Leftrightarrow$$

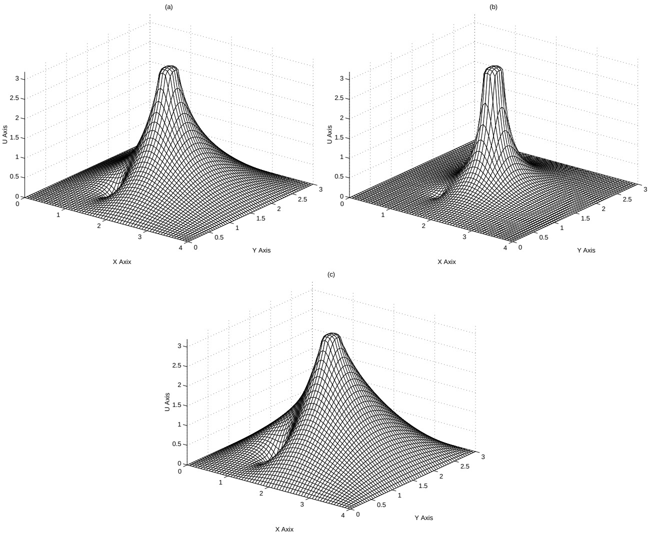
$$u \in K : F(u) = \min_{v \in K} \frac{1}{\sigma} \int_\Omega |\nabla v|^\sigma$$



(a) Meshes $\mathcal{T}_H$, $\mathcal{T}_h$, and the domain decomposition,

(b) Obstacles a and b.

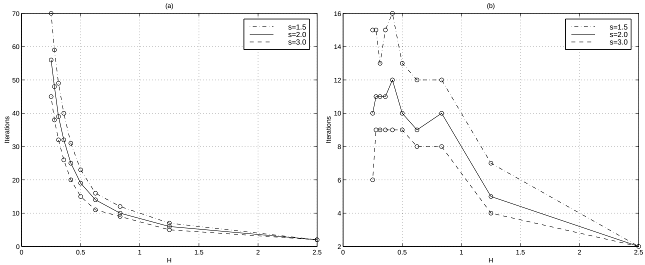
$\Omega = (0,4) \times (0,3)$; $\mathcal{T}_h$, $\mathcal{T}_H$ contain right-angled triangles; the same number of equal segments (30 for $\mathcal{T}_h$ and 6 for $\mathcal{T}_H$, in the figure) on the sides of the rectangular domain; obstacles a, b: plane + cylinder + semisphere



Solutions for: (a) σ=2, (b) σ=1.5, (c) σ=3.

We have compared by numerical experiments the convergence of the methods (by studying their dependence on H, h and δ):
- for H/h=constant, H/δ=constant

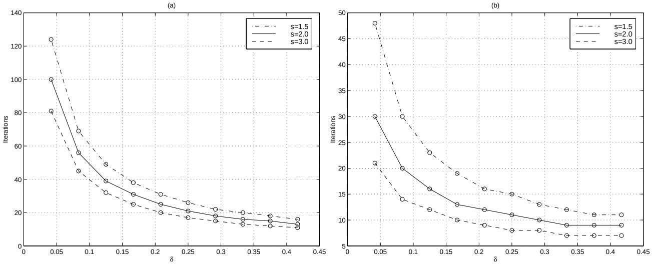
- for two of H, h or δ constant and the other one variable



Iterations for H/h and H/δ constant: (a) one level, (b) two levels.

$H/h = 6$, $H/\delta = 2$, H corresponds to 20, 18, 16, $\cdots$, 2 segments on a side.
- one-level: NI decreasing function of δ
- two-level: NI bounded

(in concordance with C_0)

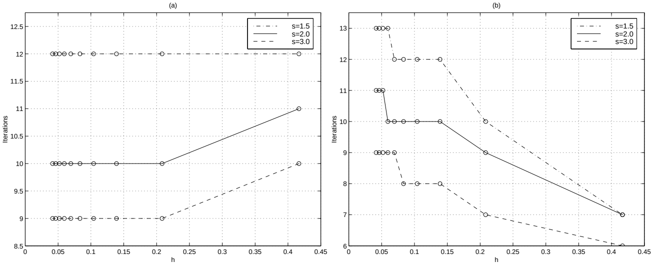


Iterations for H and h constant, and δ variable:

(a) one level, (b) two levels.

$H = 5.0/12$, $h = 5.0/120$ and $\delta = 1h, 2h, \cdots, 10h$
-in both cases, NI decreasing function of δ

(in concordance with C_0)

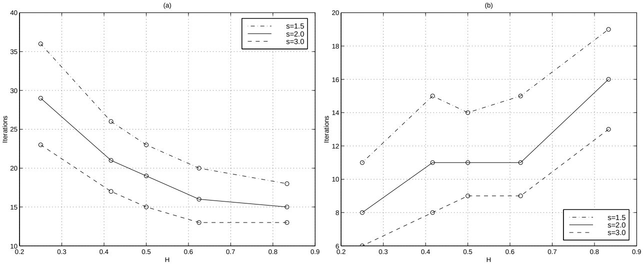


Iterations for H and δ constant, and h variable:

(a) one level, (b) two levels.

$H = 5.0/6$, $\delta = 5.0/12$, h corresponds to $2 \cdot 6, 4 \cdot 6, 6 \cdot 6, \cdots, 20 \cdot 6$ segments on a side
- one-level: NI independent of h
- two-levels: NI decreasing function of h

(in concordance with C_0)



Iterations for h and δ constant, and H variable:

(a) one level, (b) two levels.

$h = 5.0/120$, $\delta = 5.0/20$, H = 5.0/20, 5.0/12, 5.0/10, 5.0/8 and 5.0/6
- one-level: NI decreasing function of H
- two-levels: NI increasing function of H

(in concordance with C_0)

• NI two-level method $\ll$ NI one-level method, but the two-level method is more complicated than the one-level method ($K_h \subset V_h$, but we look for corrections in V_H, too)
- for $H = 5.0/10$, $h = 5.0/60$, $\delta = 5.0/20$ (nb. unknowns 3481)
one-level: NI= 23 for $\sigma = 1.5$, 19 for $\sigma = 2.0$, 15 for $\sigma = 3.0$
two-levels: NI= 13 for $\sigma = 1.5$, 10 for $\sigma = 2.0$, 9 for $\sigma = 3.0$
- computing time (PC one processor Pentium III of 600MHz):
one-level: 18min45sec for $\sigma = 1.5$, 6min16sec for $\sigma = 2.0$, 17min8sec for $\sigma = 3.0$
two-levels: 13min54sec for $\sigma = 1.5$, 4min43sec for $\sigma = 2.0$, 14min27sec for $\sigma = 3.0$

• CT for $\sigma = 2.0 \ll$ CT for $\sigma = 1.5$ or $\sigma = 3.0$ (linear equations in the relaxation method;

minimization of quadratic functionals).

Subspace correction algorithms

• $V_j, j = 1, \dots, J$ - closed subspaces of $V = V_J$ - associated with the level discretizations
$V_{ji}, i = 1, \dots, I_j$ - closed subspaces of V_j - associated with the domain decompositions of the levels
$K \subset V$ - non empty closed convex subset, denote $I = \max_{j=J,\dots,1} I_j$

• to get sharper error estimations in the case of the multigrid method

- we introduce constants $0 < \beta_{jk} \leq 1$, $\beta_{jk} = \beta_{kj}, j,\ k = J, \dots, 1$, such that

$$\langle F'(v + v_{ji}) - F'(v), v_{kl} \rangle \leq \beta_M \beta_{jk} ||v_{ji}||^{q-1} ||v_{kl}||$$

  for any $v \in V$, $v_{ji} \in V_{ji}$, $v_{kl} \in V_{kl}$ with $||v||, ||v + v_{ji}||, ||v_{kl}|| \leq M$, $i = 1, \dots, I_j$ and $l = 1, \dots, I_l$.

- we fix a constant $\frac{p}{p-q+1} \leq \sigma \leq p$ and assume that there exists a constant C_1 such that

$$||\sum_{j=1}^{J} \sum_{i=1}^{I_j} w_{ji}|| \leq C_1 (\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}||^{\sigma})^{\frac{1}{\sigma}}$$

  for any $w_{ji} \in V_{ji}$, $j = J, \dots, 1$, $i = 1, \dots, I_j$.

- evidently, in general, we can take

$$\beta_{jk} = 1,\ j,\ k = J, \dots, 1 \text{ and } \quad C_1 = (IJ)^{\frac{\sigma-1}{\sigma}}$$

Assumption

For a given $w \in K$*, we recursively introduce the level convex sets* $\mathcal{K}_j$*,* $j = J,\ J-1,\ \ldots, 1$*, (where we look for corrections) as*
- at level J*: we assume that*

$$0 \in \mathcal{K}_J,\ \mathcal{K}_J \subset \{v_J \in V_J : w + v_J \in K\}$$

and consider a $w_J \in \mathcal{K}_J$
- at a level $J - 1 \geq j \geq 1$*: we assume that*

$$0 \in \mathcal{K}_j,\ \mathcal{K}_j \subset \{v_j \in V_j : w + w_J + \ldots + w_{j+1} + v_j \in K\}$$

and consider a $w_j \in \mathcal{K}_j$

Assumption

There exists two constants $C_2,\ C_3 > 0$ such that for any $w \in K$, $w_{ji} \in V_{ji}$, $w_{j1} + \ldots + w_{ji} \in \mathcal{K}_j$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$, and $u \in K$, there exist $u_{ji} \in V_{ji}$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$, which satisfy

$$u_{j1} \in \mathcal{K}_j \text{ and } w_{j1} + \ldots + w_{ji-1} + u_{ji} \in \mathcal{K}_j,\ i = 2, \ldots, I_j,\ j = J, \ldots, 1$$

$$u - w = \sum_{j=1}^{J} \sum_{i=1}^{I_j} u_{ji}$$

$$\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||u_{ji}||^{\sigma} \leq C_2^{\sigma} ||u - w||^{\sigma} + C_3^{\sigma} \sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}||^{\sigma}$$

The convex sets $\mathcal{K}_j$, $j = J, \ldots, 1$, are constructed as in Assumption 2 with the above w and $w_j = \sum_{i=1}^{I_j} w_{ji}$, $j = J, \ldots, 1$.

Algorithm

We start with an arbitrary $u^0 \in K$. *At iteration* $n+1$ *we have* $u^n \in K$, $n \geq 0$, *and successively perform:*

- at level J: as in Assumption 2, with $w = u^n$, *we construct* $\mathcal{K}_J$.

Then, write $w_J^n = 0$, *and, for* $i = 1, \dots, I_J$, *we successively calculate*

$w_{Ji}^{n+1} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1} \in \mathcal{K}_J$,

$$\langle F'(u^n + w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}), v_{Ji} - w_{Ji}^{n+1} \rangle \geq 0$$

for any $v_{Ji} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + v_{Ji} \in \mathcal{K}_J$, *and write* $w_J^{n+\frac{i}{I_J}} = w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}$.

Algorithm (continuation)

- *at a level* $J-1 \geq j \geq 1$*: as in Assumption 2, we construct* $\mathcal{K}_j$ *with* $w = u^n$ *and* $w_J = w_J^{n+1}, \ldots, w_{j+1} = w_{j+1}^{n+1}$.
Then, write $w_j^n = 0$*, and for* $i = 1, \ldots, I_j$*, we successively calculate*

$w_{ji}^{n+1} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in \mathcal{K}_j$,

$$\langle F'(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in \mathcal{K}_j$*, and write* $w_j^{n+\frac{i}{I_j}} = w_J^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$.

- *we write* $u^{n+1} = u^n + \sum_{j=1}^{J} w_j^{n+1}$.

Theorem

Under the above conditions on the spaces and the functional F, if Assumptions 2 and 3 hold, we have the following error estimations:
(i) if $p = q = 2$ we have

$$||u^n - u||^2 \leq \frac{2}{\alpha_M}\left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1}\right)^n [F(u^0) - F(u)],$$

(ii) if $p > q$ we have

$$||u - u^n||^p \leq \frac{p}{\alpha_M} \frac{F(u^0) - F(u)}{[1 + n\tilde{C}_2(F(u^0) - F(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}},$$

Constants $\tilde{C}_1$ and $\tilde{C}_2$ depend on the functional F, the number J of levels, the maximum number I of subspaces on levels, the constants in assumptions, C_1, C_2 and C_3.

Multilevel methods

– $\mathcal{T}_{h_j}$ of mesh sizes $h_j, j = 1, \ldots, J$ - family of regular meshes over the domain $\Omega \subset \mathbf{R}^d$
– $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$
– $\{\Omega_j^i\}_{1 \le i \le I_j}$ an overlapping decomposition of Ω at each level $j = 1, \ldots, J$
– mesh partition $\mathcal{T}_{h_j}$ of Ω supplies a mesh partition for each Ω_j^i, $1 \le i \le I_j$
– introduce the linear finite element spaces,
$V_{h_j} = \{v \in C(\bar{\Omega}_j) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_{h_j},\ v = 0 \text{ on } \partial\Omega_j\}, j = 1, \ldots, J,$ - corresponding to the level meshes
$V_{h_j}^i = \{v \in V_{h_j} : v = 0 \text{ in } \Omega_j \backslash \Omega_j^i\}, i = 1, \ldots, I_j$ - associated with the level decompositions
– spaces V_{h_j} $j = 1, \ldots, J-1$, will be considered as subspaces of $W^{1,\sigma}$, $1 \le \sigma \le \infty$
– two sided obstacle problem:

$$u \in K : \ < F'(u), v - u > \ge 0, \text{ for any } v \in K,$$

where

$$K = \{v \in V_{h_J} : \varphi \le v \le \psi\},$$

with $\varphi,\ \psi \in V_{h_J}, \varphi \le \psi$

Proposition

Assumption 2 holds for the convex sets $\mathcal{K}_j$, $j = J, \ldots, 1$, *defined as,*
- for $w \in K$, *we take at the level* J

$$\varphi_J = \varphi - w,\ \psi_J = \psi - w,$$
$$\mathcal{K}_J = [\varphi_J, \psi_J],\ \text{ and consider an } w_J \in \mathcal{K}_J$$

- a level $j = J - 1, \ldots, 1$, *we define*

$$\varphi_j = I_{h_j}(\varphi_{j+1} - w_{j+1}),\ \psi_j = I_{h_j}(\psi_{j+1} - w_{j+1}),$$
$$\mathcal{K}_j = [\varphi_j, \psi_j],\ \text{ and consider an } w_j \in \mathcal{K}_j$$

$I_{h_j} : V_{h_{j+1}} \to V_{h_j}$, $j = 1, \ldots, J - 1$, *being the nonlinear interpolation operators between two consecutive levels.*

Proposition

Assumption 3 holds for the convex sets $\mathcal{K}_j$, $j = J, \ldots, 1$, defined in Proposition 3. The constants C_2 and C_3 are written as

$$C_2 = CI^{\frac{\sigma+1}{\sigma}}(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^\sigma]^{\frac{1}{\sigma}}$$

$$C_3 = CI^2(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^\sigma]^{\frac{1}{\sigma}}$$

- we make first a level decomposition of $u - w$

$$u_j = v_j - v_{j-1} = v_j - I_{h_{j-1}}(v_j - w_j) \text{ for } j = J, \ldots, 2,$$
$$u_1 = v_1 = I_{h_1}(v_2 - w_2).$$

with

$$v_J = u - w \text{ and } v_j = I_{h_j}(v_{j+1} - w_{j+1}) \text{ for } j = J-1, \ldots, 1,$$

and then, domain decompositions on each level

$$u_{ji} = L_{h_j}(\theta_j^i(u_j - \sum_{l=1}^{i-1} u_{jl}) + (1-\theta_j^i)w_{ji}), \quad i = 1, \ldots, I_j,$$

$I_{h_j} : V_{h_{j+1}} \to V_{h_j}$, $j = 1, \ldots, J$, being the nonlinear interpolation operators and θ_j^i, $j = 1, \ldots, J$, $i = 1, \ldots, I_j$, are functions constructed by means of the unity partitions

• we proved that Assumptions 2 and 3 hold, and have explicitly written constants C_2 and C_3 in function of the mesh and overlapping parameters. We can then conclude from Theorem 2 that Algorithm 2 is globally convergent.

• convergence rates depend on the functional F, the maximum number of the subdomains on each level, I, and the number of levels J. The number of subdomains on levels can be associated with the number of colors needed to mark the subdomains such that the subdomains with the same color do not intersect with each other. Since this number of colors depends in general on the dimension of the Euclidean space where the domain lies, we can conclude that the convergence rate essentially depends on the number of levels J.

• in this general framework

$$C_1 = CJ^{\frac{\sigma-1}{\sigma}} \text{ and } \max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj} = J$$

• as functions of J, we have

$$C_2 = C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J)$$

$$C_3 = C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J)$$

where

$$S_{d,\sigma}(J) = \left[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}\right]^{\frac{1}{\sigma}} = \begin{cases} (J-1)^{\frac{1}{\sigma}} & \text{if } d = \sigma = 1 \\ & \text{or } 1 \le d < \sigma < \infty \\ CJ & \text{if } 1 < d = \sigma < \infty \\ C^J & \text{if } 1 \le \sigma < d < \infty, \end{cases}$$

Multigrid methods

• in the above multilevel methods a mesh is the refinement of that one on the previous level, but the domain decompositions are almost independent from one level to another
• we obtain similar multigrid methods by decomposing the domain by the supports of the nodal basis functions of each level. Consequently, the subspaces $V_{h_j}^i$, $i = 1, \dots, I_j$, are one-dimensional spaces generated by the nodal basis functions associated with the nodes of $\mathcal{T}_{h_j}$, $j = J, \dots, 1$
• for the multigrid methods, we can take

$$C_1 = C \text{ and } \max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj} = C$$

• we write the convergence rate of the multigrid Algorithm 2 in function of the number of levels J for the typical example where

$$F(v)=\frac{1}{\sigma}|v|_{1,\sigma}^{\sigma}-L(v),\quad v\in W^{1,\sigma}(\Omega)$$

where L is a linear and continuous functional on $W^{1,\sigma}(\Omega)$, $\sigma>1$
- if $1<\sigma\leq 2$ (R. Glowinski and A. Marrocco, 1975) $\Rightarrow$ for $v,u\in W_0^{1,\sigma}(\Omega)$,

$$<F'(v)-F'(u),v-u>\geq\alpha\frac{||v-u||_{1,\sigma}^2}{(||v||_{1,\sigma}+||u||_{1,\sigma})^{2-\sigma}}$$

$$\beta||v-u||_{1,\sigma}^{\sigma-1}\geq||F'(v)-F'(u)||_{V'},$$

$\alpha,\ \beta>0$ constants $\Rightarrow \alpha_M=\frac{\alpha}{(2M)^{2-\sigma}},\ \beta_M=\beta,\ p=2,\ q=\sigma$

- if $\sigma\geq 2$ (P. G. Ciarlet, 1978) $\Rightarrow$ for $v,u\in W_0^{1,\sigma}(\Omega)$,

$$<F'(v)-F'(u),v-u>\geq\alpha||v-u||_{1,\sigma}^{\sigma}$$

$$\beta(||v||_{1,\sigma}+||u||_{1,\sigma})^{\sigma-2}||v-u||_{1,\sigma}\geq||F'(v)-F'(u)||_{V'},$$

$\alpha,\ \beta>0$ constant $\Rightarrow \alpha_M=\alpha,\ \beta_M=\beta(2M)^{\sigma-2},\ p=\sigma,\ q=2$

- for $\sigma = 2, p = q = 2$,

$$||u^n - u||_{1,2}^2 \le \tilde{C}_0 \left(1 - \frac{1}{1 + \tilde{C}_1(J)}\right)^n \text{ where}$$

$$\tilde{C}_1(J) = CJS_{d,2}(J)^2$$

- for $1 < q = \sigma < 2, p = 2$,

$$||u^n - u||_{1,\sigma}^2 \le \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{\sigma-1}{2-\sigma}}} \text{ where}$$

$$\tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\frac{1}{\sigma-1}}}, \quad \tilde{C}_3(J) = CJ^{\frac{(4-\sigma)(\sigma-1)}{\sigma}} S_{d,\sigma}(J)^2$$

- for $p = \sigma > 2, q = 2$,

$$||u^n - u||_{1,\sigma}^{\sigma} \le \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{1}{\sigma-1}}} \text{ where}$$

$$\tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\sigma-1}}, \quad \tilde{C}_3(J) = CJ^{\frac{2\sigma-3}{\sigma-1}} S_{d,\sigma}(J)^{\frac{\sigma}{\sigma-1}}$$

where $\tilde{C}_0$ is a constant independent of J.

Concluding remarks

- the results referred to problems in $W^{1,\sigma}$ with Dirichlet boundary conditions, but they also hold for Neumann or mixed boundary conditions.
- similar convergence results can be obtained for problems in $(W^{1,\sigma})^d$.
- the above convergence results give estimations of the global convergence rate, and the analysis refers to two sided obstacle problems which arise from the minimization of functionals defined on $W^{1,\sigma}$, $1 < \sigma < \infty$.
- we can compare the convergence rates we have obtained with similar ones in the literature in the case of H^1 ($p = q = 2$) and $d = 2$. In this case, we get that the global convergence rate of Algorithm 2 is $1 - \frac{1}{1+CJ^3}$. The same estimate, of $1 - \frac{1}{1+CJ^3}$, is obtained by R. Kornhuber for the asymptotic convergence rate of the standard monotone multigrid methods for the complementarity problems.

Hybrid multigrid methods

• Algorithm 2 is of multiplicative type over the levels as well as on each level, i.e. the current correction is found in function of all previous corrections on

- the previous levels
- the current level

• we can imagine hybrid algorithms: the type of the iteration over the levels is different from the type of the iteration on the levels (idea found in B. F. Smith, P. E. Bjørstad and W. Gropp, Domain Decomposition. Parallel multilevel methods for elliptic partial differential equations, Cambridge University Press, 1996)

• algorithm of multiplicative type over the levels and of additive type on each level

Algorithm

We start the algorithm with an arbitrary $u^0 \in K$. *Assuming that at iteration* $n+1$ *we have* $u^n \in K$, $n \geq 0$, *we successively perform the following steps:*

- at the level J, we construct the convex set $\mathcal{K}_J$ *as in Assumption 2, with* $w = u^n$. *Then, we simultaneously calculate* $w_{Ji}^{n+1} \in V_{Ji} \cap \mathcal{K}_J$, $i = 1, \ldots, I_J$, *the solutions of the inequalities*

$$\langle F'(u^n + w_{Ji}^{n+1}), v_{Ji} - w_{Ji}^{n+1} \rangle \geq 0,$$

for any $v_{Ji} \in V_{Ji} \cap \mathcal{K}_J$, *and write* $w_J^{n+1} = \frac{r}{I} \sum_{i=1}^{I_J} w_{Ji}^{n+1}$,

Algorithm (continuation)

- *at a level* $J-1 \geq j \geq 1$, *we construct the convex set* $\mathcal{K}_j$ *as in Assumption 2, with* $w = u^n$ *and* $w_J = w_J^{n+1}, \ldots, w_{j+1} = w_{j+1}^{n+1}$. *Then, we simultaneously calculate* $w_{ji}^{n+1} \in V_{ji} \cap \mathcal{K}_j$, $i = 1, \ldots, I_j$, *the solutions of the inequalities*

$$\langle F'(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0,$$

for any $v_{ji} \in V_{ji} \cap \mathcal{K}_j$, *and write* $w_j^{n+1} = \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}^{n+1}$,

- *we write* $u^{n+1} = u^n + \sum_{j=1}^{J} w_j^{n+1}$.

Above, r *is a constant in the interval* $(0, 1]$.

• algorithm of additive type over the levels and of multiplicative type on each level

Algorithm

We start the algorithm with an $u^0 \in K$. *Assuming that at iteration* $n+1$ *we have* $u^n \in K$, $n \geq 0$, *for* $j = 1, \dots, J$, *we simultaneously perform the following steps:*
- we construct the convex set $\mathcal{K}_j$ *as in Assumption 2 with* $w = u^n$ *and* $w_J = \dots = w_1 = 0$,
- we write $w_j^n = 0$, *and for* $i = 1, \dots, I_j$, *we successively calculate* $w_{ji}^{n+1} \in V_{ji}$,
$w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in \mathcal{K}_j$, *the solution of the inequalities*

$$\langle F'(u^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0,$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in \mathcal{K}_j$, *and write* $w_j^{n+\frac{i}{I_j}} = w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$.
Then, we write $u^{n+1} = u^n + \frac{s}{J} \sum_{j=1}^{J} w_j^{n+1}$, *with a fixed* $0 < s \leq 1$.

• algorithm of additive type over the levels as well as on each level

Algorithm

We start the algorithm with an $u^0 \in K$. *Assuming that at iteration* $n \geq 0$ *we have* $u^n \in K$, *we simultaneously perform, for* $j = 1, \ldots, J$, *the following steps:*
- we construct the convex sets $\mathcal{K}_j$ *as in Assumption 2 with* $w = u^n$ *and* $w_J = \ldots = w_1 = 0$,
- we simultaneously calculate, for $i = 1 \ldots, I_j$, $w_{ji}^{n+1} \in V_{ji} \cap \mathcal{K}_j$, *the solutions of the inequalities*

$$\langle F'(u^n + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0,$$

for any $v_{ji} \in V_{ji} \cap \mathcal{K}_j$, *and write* $w_j^{n+1} = \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}^{n+1}$, *with a fixed* $0 < r \leq 1$.

Then, we write $u^{n+1} = u^n + \frac{s}{J} \sum_{j=1}^{J} w_j^{n+1}$, *with a fixed* $0 < s \leq 1$.

- a convergence result similar with Theorem 2 can be obtained for all these algorithms
- constants C_2 and C_3 in Assumption
  - for Algorithm 4 (multiplicative – additive)

$$C_2 = CI^{\frac{1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}},$$

$$C_3 = C(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}}.$$

  - for Algorithm 6 (additive – multiplicative)

$$C_2 = CI^{\frac{\sigma+1}{\sigma}}(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}},$$

$$C_3 = CI^{\frac{\sigma+1}{\sigma}}(I+1)^{\frac{\sigma-1}{\sigma}}.$$

  - for Algorithm 7 (additive – additive)

$$C_2 = CI^{\frac{1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}} \text{ and } C_3 = 0.$$

• in function of J only

$$C_2 = C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J) \text{ for all algorithms}$$

$$C_3 = \begin{cases} C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J) & \text{for Algorithms 2 and 4} \\ C & \text{for Algorithm 6} \\ 0 & \text{for Algorithm 7.} \end{cases}$$

• convergence rate for the typical example

$$F(v) = \frac{1}{\sigma}|v|_{1,\sigma}^{\sigma} - L(v), \quad v \in W^{1,\sigma}(\Omega)$$

L is a linear and continuous functional on $W^{1,\sigma}(\Omega)$, $1 < \sigma < \infty$

- for $\sigma = p = q = 2$

$$\tilde{C}_1(J) = \begin{cases} CJS_{d,2}(J)^2 & \text{for Algorithms 2 and 4 (multiplicative over levels)} \\ CJ^2S_{d,2}(J)^2 & \text{for Algorithms 6 and 7 (additive over levels)} \end{cases}$$

$$||u^n - u||_{1,2}^2 \leq \tilde{C}_0 \left(1 - \frac{1}{1+\tilde{C}_1(J)}\right)^n,$$

where $\tilde{C}_0$ is a constant independent of J.

- for $1 < q = \sigma < 2$ and $p = 2$

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{(4-\sigma)(\sigma-1)}{\sigma}} S_{d,\sigma}(J)^2 & \text{for Algorithms 2 and 4 (multiplicative over levels)} \\ CJ^{\frac{4(\sigma-1)}{\sigma}} S_{d,\sigma}(J)^2 & \text{for Algorithms 6 and 7 (additive over levels)} \end{cases}$$

$$||u^n - u||^2_{1,\sigma} \leq \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{\sigma-1}{2-\sigma}}} \text{ with } \tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\frac{1}{\sigma-1}}}.$$

- for $p = \sigma > 2$ and $q = 2$

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{2\sigma-3}{\sigma-1}} S_{d,\sigma}(J)^{\frac{\sigma}{\sigma-1}} & \text{for Algorithms 2 and 4 (multiplicative over levels)} \\ CJ^2 S_{d,\sigma}(J)^{\frac{\sigma}{\sigma-1}} & \text{for Algorithms 6 and 7 (additive over levels)} \end{cases}$$

$$||u^n - u||^\sigma_{1,\sigma} \leq \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{1}{\sigma-1}}} \text{ where } \tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\sigma-1}}.$$

- Remarks
  - regardless of the iteration type on levels, algorithms having the same type of iterations over the levels have the same convergence rate, provided that additive iterations on levels are parallelized
  - the algorithms which are of multiplicative type over the levels converge better, by a factor of between $1/J$ and 1 (depending on σ), than their additive similar variants

One- and two-level methods for variational inequalities of the second kind

- $\varphi : K \to \mathbf{R}$ convex, lower semicontinuous, not differentiable functional and, if K is not bounded, $F + \varphi$ coercive, i.e. $F(v) + \varphi(v) \to \infty$, as $\|v\| \to \infty$, $v \in K$.
- problem (has a unique solution)

$$u \in K : \langle F'(u), v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for any } v \in K$$

$\Leftrightarrow$

$$u \in K : F(u) + \varphi(u) \leq F(v) + \varphi(v), \text{ for any } v \in K$$

– **Example** - Contact problems with Tresca friction
- $\Omega \subset \mathbf{R}^d$, $d = 2, 3$, $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N \cup \bar{\Gamma}_C$, Γ_D - Dirichlet boundary, Γ_N - Neumann boundary, Γ_C - the possible contact boundary

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \ v_n \leq 0, \text{ a.e. on } \Gamma_C \text{ and } \boldsymbol{v} = \boldsymbol{g}, \text{ a.e. on } \Gamma_D\}$$

- find $\boldsymbol{u} = \boldsymbol{u}(\tau) \in \boldsymbol{K}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j_\tau(\boldsymbol{v}) - j_\tau(\boldsymbol{u}) \geq f(\boldsymbol{v} - \boldsymbol{u}), \text{ for any } \boldsymbol{v} \in \boldsymbol{K}$$

$$j_\tau(\boldsymbol{v}) = -\int_{\Gamma_C} \mathcal{F}\tau|\boldsymbol{v}_t|, \quad \tau \in H^{-1/2}(\Gamma_C), \ \tau \leq 0$$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega E_{ijlm}\epsilon_{ij}(\boldsymbol{v})\epsilon_{ml}(\boldsymbol{u}), \quad f(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + (\boldsymbol{p}, \boldsymbol{v})_{\boldsymbol{L}^2(\Gamma_N)}$$

Algorithm

We start the algorithm with an arbitrary $u^0 \in K$*. At iteration* $n+1$*, having* $u^n \in K$*,* $n \geq 0$*, we compute sequentially for* $i = 1, \cdots, m$*, the local corrections* $w_i^{n+1} \in V_i,\ u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$ *as the solution of the variational inequality*

$$\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}} + v_i) - \varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0,$$

for any $v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K$*,and then we update* $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

– to prove the convergence, we introduce a technical assumption

$$\sum_{i=1}^{m}[\varphi(w + \sum_{j=1}^{i-1} w_j + v_i) - \varphi(w + \sum_{j=1}^{i-1} w_j + w_i)] \leq \varphi(v) - \varphi(w + \sum_{i=1}^{m} w_i)$$

for $v, w \in K$, and $v_i, w_i \in V_i$, $i = 1, \ldots, m$, in Assumption 1
– to show that this assumption holds when the finite element spaces are used, we have to take a numerical approximation of the functional φ of the form

$$\varphi(v) = \sum_{\kappa \in \mathcal{N}_h} s_\kappa(h)\phi(v(x_\kappa))$$

where $\phi : \mathbf{R} \to \mathbf{R}$ is a continuous convex function, $\mathcal{N}_h$ is the set of nodes of the mesh $\mathcal{T}_h$, and $s_\kappa(h) \geq 0$, $\kappa \in \mathcal{N}_h$, are non-negative real numbers which may depend on the mesh size h.

Theorem

Under the above assumptions on V, F and φ, let u be the solution of the problem and u^n, $n \geq 0$, be its approximations obtained from Algorithm 8. If Assumption 1 holds, then there exists $M > 0$ such that $\max(\|u\|, \|u^0\|, \max_{n\geq 0, 1\leq i\leq m} \|u^{n+\frac{i}{m}}\|) \leq M$ *and we have the following error estimations:*

(i) if $p = q = 2$ *we have*

$$\|u^n - u\|^2 \leq \frac{p}{\alpha_M} \left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1} \right)^n \left[F(u^0) + \varphi(u^0) - F(u) - \varphi(u) \right].$$

(ii) if $p > q$ *we have*

$$\|u - u^n\|^p \leq \frac{p}{\alpha_M} \frac{F(u^0) + \varphi(u^0) - F(u) - \varphi(u)}{\left[1 + n\tilde{C}_2 \left(F(u^0) + \varphi(u^0) - F(u) - \varphi(u) \right)^{\frac{p-q}{q-1}} \right]^{\frac{q-1}{p-q}}}.$$

Constants $\tilde{C}_1$ and $\tilde{C}_2$ can be explicitly written as a function of the functionals F and φ, the number of subspaces m, the initial approximation u^0 and C_0 in Assumption 1.

One- and two-level methods for quasi-variational inequalities

– consider $p = q = 2$
– $\varphi : K \times K \to \mathbf{R}$ functional such that, for any $u \in K$, $\varphi(u, \cdot) : K \to \mathbf{R}$ is convex, lower semicontinuous and, if K is not bounded, $F(\cdot) + \varphi(u, \cdot)$ is coercive, i.e. $F(v) + \varphi(u, v) \to \infty$ as $\|v\| \to \infty, v \in K$
– assume that for any $M > 0$ there exists $c_M > 0$ such that

$$|\varphi(v_1, w_2) + \varphi(v_2, w_1) - \varphi(v_1, w_1) - \varphi(v_2, w_2)| \le c_M ||v_1 - v_2|| ||w_1 - w_2||$$

for any $v_1,\ v_2,\ w_1\ w_2 \in K$, $||v_1||,\ ||v_2||,\ ||w_1||\ ||w_2|| \le M$
– problem (has a unique solution)

$$u \in K : \langle F'(u), v - u\rangle + \varphi(u, v) - \varphi(u, u) \ge 0, \text{ for any } v \in K$$

$\Leftrightarrow$

$$u \in K : F(u) + \varphi(u, u) \le F(v) + \varphi(u, v), \text{ for any } v \in K$$

– **Example** - Contact problems with non-local Coulomb friction

$$j(\boldsymbol{u}, \boldsymbol{v}) = -\int_{\Gamma_C} \mathcal{F}\, \sigma_n^*(\boldsymbol{u})\, |\boldsymbol{v}_t|\, .$$

where $\sigma_n^* = \omega * \sigma_n$, the convolution, $\omega \in \mathcal{D}(-\eta, \eta)$, $\int_{-\eta}^{\eta} \omega = 1$, $\eta \in \mathbf{R}$, $\eta > 0$

– three algorithms depending on the first argument of φ:

Algorithm

We start the algorithm with an arbitrary $u^0 \in K$. *At iteration* $n+1$, *having* $u^n \in K$, $n \geq 0$, *we compute sequentially for* $i = 1, \cdots, m$, *the local corrections* $w_i^{n+1} \in V_i,\ u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, *satisfying*

$$\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}, u^{n+\frac{i-1}{m}} + v_i) - \varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0,$$

for any $v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K$, *and then we update* $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

Algorithm

We start the algorithm with an arbitrary $u^0 \in K$. *At iteration* $n+1$, *having* $u^n \in K$, $n \geq 0$, *we compute sequentially for* $i = 1, \cdots, m$, *the local corrections* $w_i^{n+1} \in V_i,\ u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, *satisfying*

$$\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}}, u^{n+\frac{i-1}{m}} + v_i) - \varphi(u^{n+\frac{i-1}{m}}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0,$$

for any $v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K$, *and then we update* $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

Algorithm

We start the algorithm with an arbitrary $u^0 \in K$. *At iteration* $n+1$, *having* $u^n \in K$, $n \geq 0$, *we compute sequentially for* $i = 1, \cdots, m$, *the local corrections* $w_i^{n+1} \in V_i,\ u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, *satisfying*

$$\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^n, u^{n+\frac{i-1}{m}} + v_i) - \varphi(u^n, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0,$$

for any $v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K$, *and then we update* $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

- similarly with the case of the inequalities of the second kind, we introduce the technical assumption

$$\sum_{i=1}^{m}[\varphi(u,w+\sum_{j=1}^{i-1}w_j+v_i)-\varphi(u,w+\sum_{j=1}^{i-1}w_j+w_i)]\le$$
$$\varphi(u,v)-\varphi(u,w+\sum_{i=1}^{m}w_i)$$

for any $u\in K$ and for $v,w\in K$ and $v_i,w_i\in V_i,\ u^{n+\frac{i-1}{m}}+v_i\in K, i=1,\dots,m$, in Assumption 1
- in the finite element spaces, φ is approximated by

$$\varphi(u,v)=\sum_{\kappa\in\mathcal{N}_h}I_\kappa(\phi(u,v(x_\kappa)))$$

where, $I_\kappa : L^2(\Omega)\to\mathbf{R}$ and $\phi : K_h\times\mathbf{R}\to L^2(\Omega)$ are assumed to be continuous, and, for any $u\in K_h$, $I_\kappa(\phi(u,\cdot)) : \mathbf{R}\to\mathbf{R}$, $\kappa\in\mathcal{N}_h$, are convex functions

Theorem

Under the above assumptions on V, F and φ, let u be the solution of the problem and u^n, $n \geq 0$, be its approximations obtained from one of the multiplicative Algorithms 9–11. If Assumption 1 holds, and if

$$\frac{\alpha_M}{2} \geq mC_M + \sqrt{2m(25C_0 + 8)\beta_M C_M},$$

for any $M > 0$, then there exists an $M > 0$ such that $\max(\|u\|, \|u^0\|, \max_{n\geq 0, 1\leq i\leq m} \|u^{n+\frac{i}{m}}\|) \leq M$ and we have the following error estimation

$$\|u^n - u\|^2 \leq \frac{2}{\alpha_M}\left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1}\right)^n \left[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u, u)\right].$$

Constant $\tilde{C}_1$ depends on the functionals F and φ, the number of subspaces m, the initial approximation u^0, and is an increasing function on C_0 in Assumption 1.

Remark

1. extension of the previous methods (given for variational inequalities of the second kind and quasi-variational inequalities) to methods with more than two levels, having an optimal rate of convergence, is not very evident because of the technical conditions we have introduced which are not satisfied when domain decompositions on the coarse levels are considered

2. by using Newton linearizations of φ, R. Kornhuber introduced multigrid methods for complementarity problems and estimated asymptotic convergence rates

3. we can estimate the global convergence rate of a multigrid method for the particular case of quasi-variational inequalities when the inequality contains a term given by a contraction operator

Multigrid methods for

inequalities with a term given by a contraction operator

- consider $p = q = 2$, $\alpha_M = \alpha$, $\beta_M = \beta$
- $T : V \to V'$ a Lipschitz continuous operator

$$||T(v) - T(u)||_{V'} \leq \gamma ||v - u||$$

for any $v, u \in V$
- problem

$$u \in K : \langle F'(u), v - u \rangle + \langle T(u), v - u \rangle \geq 0 \text{ for any } v \in K$$

– in the following algorithm, each iteration contains κ intermediate iterations in which the argument of T is kept unchanged
– these intermediate iterations are in fact iterations of the multigrid algorithm for usual variational inequalities (of the first kind)

Algorithm

We start the algorithm with an arbitrary $u^0 \in K$. *Assuming that at iteration* $n+1$ *we have* $u^n \in K$, $n \geq 0$, *we write* $\tilde{u}^n = u^n$ *and carry out the following two steps:*

1. *We perform* $\kappa \geq 1$ *iterations, keeping the argument of* T *equal with* u^n. *We start with* $\tilde{u}^n$ *and having* $\tilde{u}^{n+k-1}$ *at iteration* $1 \leq k \leq \kappa$, *we successively calculate level corrections and compute* $\tilde{u}^{n+k}$:

- at the level J, we construct the convex set $\mathcal{K}_J$ *as in Assumption 2, with* $w = \tilde{u}^{n+k-1}$. *Then, we first write* $w_J^k = 0$, *and, for* $i = 1, \ldots, I_J$, *we successively calculate* $w_{Ji}^{k+1} \in V_{Ji}$, $w_J^{k+\frac{i-1}{I_J}} + w_{Ji}^{k+1} \in \mathcal{K}_J$, *the solution of the inequality*

$$\langle F'(\tilde{u}^{n+k-1} + w_J^{k+\frac{i-1}{I_J}} + w_{Ji}^{k+1}), v_{Ji} - w_{Ji}^{k+1}\rangle + \langle T(u^n), v_{Ji} - w_{Ji}^{k+1}\rangle \geq 0,$$

for any $v_{Ji} \in V_{Ji}$, $w_J^{k+\frac{i-1}{I_J}} + v_{Ji} \in \mathcal{K}_J$, *and write* $w_J^{k+\frac{i}{I_J}} = w_J^{k+\frac{i-1}{I_J}} + w_{Ji}^{k+1}$,

Algorithm (continuation)

- at a level $J-1 \geq j \geq 1$, we construct the convex set $\mathcal{K}_j$ as in Assumption 2 with $w = \tilde{u}^{n+k-1}$ and $w_J = w_J^{k+1}, \ldots, w_{j+1} = w_{j+1}^{k+1}$. Then, we write $w_j^{k+1} = 0$, and for $i = 1, \ldots, I_j$, we successively calculate $w_{ji}^{k+1} \in V_{ji}$, $w_j^{j+\frac{i-1}{I_j}} + w_{ji}^{k+1} \in \mathcal{K}_j$, the solution of the inequality

$$\langle F'(\tilde{u}^{n+k-1} + \sum_{l=j+1}^{J} w_l^{k+1} + w_j^{j+\frac{i-1}{I_j}} + w_{ji}^{k+1}), v_{ji} - w_{ji}^{k+1}\rangle + \langle T(u^n), v_{ji} - w_{ji}^{k+1}\rangle \geq 0,$$

for any $v_{ji} \in V_{ji}$, $w_j^{k+\frac{i-1}{I_j}} + v_{ji} \in \mathcal{K}_j$, and write $w_j^{k+\frac{i}{I_j}} = w_J^{k+\frac{i-1}{I_j}} + w_{ji}^{k+1}$,

- we write $\tilde{u}^{n+k} = \tilde{u}^{n+k-1} + \sum_{j=1}^{J} w_j^{k+1}$.

2. We write $u^{n+1} = \tilde{u}^{n+\kappa}$.

Theorem

We assume that V, F and T satisfy the above conditions and that Assumptions 2-3 hold. Then, if

$$\gamma/\alpha < 1/2$$

and κ satisfies

$$\left(\frac{\tilde{C}}{\tilde{C}+1}\right)^{\kappa} < \frac{1-2\frac{\gamma}{\alpha}}{1+3\frac{\gamma}{\alpha}+4\frac{\gamma^2}{\alpha^2}+\frac{\gamma^3}{\alpha^3}},$$

Algorithm 12 is convergent and we have the following error estimation

$$\|u^n - u\|^2 \le \frac{2}{\alpha}\left[2\frac{\gamma}{\alpha} + \left(\frac{\tilde{C}}{\tilde{C}+1}\right)^{\kappa}\left(1+3\frac{\gamma}{\alpha}+4\frac{\gamma^2}{\alpha^2}+\frac{\gamma^3}{\alpha^3}\right)\right]^n \cdot [F(u^0) + \langle T(u), u^0\rangle - F(u) - \langle T(u), u\rangle].$$

Constant $\tilde{C}$ can be written as

$$\tilde{C} = \frac{1}{C_2\varepsilon}\left[1 + C_2 + C_1C_2 + \frac{C_2}{\varepsilon}\right],\ \varepsilon = \frac{\alpha}{2\beta I(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})C_2}.$$