Robust Solution Strategies for Fluid-Structure Interaction Problems with Applications

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Applications

Wind Energy

Ocean/Marine Engineering

Biomedical Devices

FSI: The Augmented Lagrangian Framework

Define an Augmented Lagrangian for the FSI problem:

$$\begin{aligned}\mathbf{N}(\{\mathbf{u}_1, p\}, \mathbf{u}_2, \boldsymbol{\lambda}) = {} & \mathbf{N}_1(\{\mathbf{u}_1, p\}) + \mathbf{N}_2(\mathbf{u}_2) \\ & + \int_{(\Gamma_{\mathrm{I}})_t} \boldsymbol{\lambda} \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, \mathrm{d}\Gamma \\ & + \frac{1}{2} \int_{(\Gamma_{\mathrm{I}})_t} \beta (\mathbf{u}_1 - \mathbf{u}_2) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, \mathrm{d}\Gamma\end{aligned}$$

FSI: The Augmented Lagrangian Framework

Take the variation with respect to the fluid, structure and Lagrange multiplier unknowns:

$$B_1(\{\mathbf{w}_1, q\}, \{\mathbf{u}_1, p\}) - F_1(\{\mathbf{w}_1, q\}) + \int_{(\Gamma_\mathrm{I})_t} \mathbf{w}_1 \cdot \boldsymbol{\lambda} \, \mathrm{d}\Gamma + \int_{(\Gamma_\mathrm{I})_t} \mathbf{w}_1 \cdot \beta(\mathbf{u}_1 - \mathbf{u}_2) \, \mathrm{d}\Gamma = 0,$$

$$B_2(\mathbf{w}_2, \mathbf{u}_2) - F_2(\mathbf{w}_2) - \int_{(\Gamma_\mathrm{I})_t} \mathbf{w}_2 \cdot \boldsymbol{\lambda} \, \mathrm{d}\Gamma - \int_{(\Gamma_\mathrm{I})_t} \mathbf{w}_2 \cdot \beta(\mathbf{u}_1 - \mathbf{u}_2) \, \mathrm{d}\Gamma = 0,$$

$$\int_{(\Gamma_\mathrm{I})_t} \delta\boldsymbol{\lambda} \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, \mathrm{d}\Gamma = 0$$

Interpretation of the Lagrange multiplier and compatibility of tractions:

$$\begin{aligned} \boldsymbol{\lambda} &= -\boldsymbol{\sigma}_1 \mathbf{n}_1 - \beta(\mathbf{u}_1 - \mathbf{u}_2), \\ \boldsymbol{\lambda} &= \boldsymbol{\sigma}_2 \mathbf{n}_2 - \beta(\mathbf{u}_1 - \mathbf{u}_2) \end{aligned} \implies \boldsymbol{\sigma}_1 \mathbf{n}_1 + \boldsymbol{\sigma}_2 \mathbf{n}_2 = \mathbf{0}$$

$$\mathbf{u}_1 = \mathbf{u}_2 \implies \boldsymbol{\lambda} = -\boldsymbol{\sigma}_1 \mathbf{n}_1 = \boldsymbol{\sigma}_2 \mathbf{n}_2$$

FSI: The Augmented Lagrangian Framework

Take a convex combination of the fluid and structure traction vectors:

$$\boldsymbol{\lambda} = -\alpha \boldsymbol{\sigma}_1 \mathbf{n}_1 + (1-\alpha)\boldsymbol{\sigma}_2 \mathbf{n}_2$$

Take its variation with respect to the fluid and structural mechanics unknowns:

$$\delta\boldsymbol{\lambda} = -\alpha \delta_{\{\mathbf{u}_1, p\}} \boldsymbol{\sigma}_1 \mathbf{n}_1(\{\mathbf{w}_1, q\}) + (1-\alpha)\delta_{\mathbf{u}_2} \boldsymbol{\sigma}_2 \mathbf{n}_2(\mathbf{w}_2)$$

Formally eliminate the Lagrange multiplier:

$$\begin{aligned}
& B_1(\{\mathbf{w}_1, q\}, \{\mathbf{u}_1, p\}) - F_1(\{\mathbf{w}_1, q\}) + B_2(\mathbf{w}_2, \mathbf{u}_2) - F_2(\mathbf{w}_2) \\
& + \int_{(\Gamma_\mathrm{I})_t} (\mathbf{w}_1 - \mathbf{w}_2) \cdot (-\alpha \boldsymbol{\sigma}_1 \mathbf{n}_1 + (1-\alpha)\boldsymbol{\sigma}_2 \mathbf{n}_2) \; \mathrm{d}\Gamma \\
& + \gamma \int_{(\Gamma_\mathrm{I})_t} \big(-\alpha \delta_{\{\mathbf{u}_1, p\}} \boldsymbol{\sigma}_1 \mathbf{n}_1(\{\mathbf{w}_1, q\}) + (1-\alpha)\delta_{\mathbf{u}_2} \boldsymbol{\sigma}_2 \mathbf{n}_2(\mathbf{w}_2)\big) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \; \mathrm{d}\Gamma \\
& + \int_{(\Gamma_\mathrm{I})_t} (\mathbf{w}_1 - \mathbf{w}_2) \cdot \beta(\mathbf{u}_1 - \mathbf{u}_2) \; \mathrm{d}\Gamma = 0
\end{aligned}$$

FSI: The Augmented Lagrangian Framework

Take a convex combination of the fluid and structure traction vectors:

$$\boldsymbol{\lambda} = -\alpha \boldsymbol{\sigma}_1 \mathbf{n}_1 + (1-\alpha)\boldsymbol{\sigma}_2 \mathbf{n}_2$$

Take its variation with respect to the fluid and structural mechanics unknowns:

$$\delta \boldsymbol{\lambda} = -\alpha \delta_{\{\mathbf{u}_1, p\}} \boldsymbol{\sigma}_1 \mathbf{n}_1(\{\mathbf{w}_1, q\}) + (1-\alpha)\delta_{\mathbf{u}_2} \boldsymbol{\sigma}_2 \mathbf{n}_2(\mathbf{w}_2)$$

Formally eliminate the Lagrange multiplier:

$$B_1(\{\mathbf{w}_1, q\}, \{\mathbf{u}_1, p\}) - F_1(\{\mathbf{w}_1, q\}) + B_2(\mathbf{w}_2, \mathbf{u}_2) - F_2(\mathbf{w}_2)$$

$$+ \int_{(\Gamma_{\mathrm{I}})_t} (\mathbf{w}_1 - \mathbf{w}_2) \cdot (-\alpha \boldsymbol{\sigma}_1 \mathbf{n}_1 + (1-\alpha)\boldsymbol{\sigma}_2 \mathbf{n}_2) \ \mathrm{d}\Gamma$$

$$+ \gamma \int_{(\Gamma_{\mathrm{I}})_t} (-\alpha \delta_{\{\mathbf{u}_1, p\}} \boldsymbol{\sigma}_1 \mathbf{n}_1(\{\mathbf{w}_1, q\}) + (1-\alpha)\delta_{\mathbf{u}_2} \boldsymbol{\sigma}_2 \mathbf{n}_2(\mathbf{w}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \ \mathrm{d}\Gamma$$

$$+ \int_{(\Gamma_{\mathrm{I}})_t} (\mathbf{w}_1 - \mathbf{w}_2) \cdot \beta(\mathbf{u}_1 - \mathbf{u}_2) \ \mathrm{d}\Gamma = 0$$

Terms disappear
in the case of matching
interface discretization!

FSI: The Augmented Lagrangian Framework

Setting: $\alpha = 1 \text{ and } \gamma = 1$

Weak Dirichlet BC formulation of fluid mechanics:

$$B_1(\{\mathbf{w}_1, q\}, \{\mathbf{u}_1, p\}) - F_1(\{\mathbf{w}_1, q\})$$
$$- \int_{(\Gamma_\mathrm{I})_t} \mathbf{w}_1 \cdot \boldsymbol{\sigma}_1 \mathbf{n}_1 \ \mathrm{d}\Gamma$$
$$- \int_{(\Gamma_\mathrm{I})_t} \left(\delta_{\{\mathbf{u}_1, p\}} \boldsymbol{\sigma}_1 \mathbf{n}_1(\{\mathbf{w}_1, q\})\right) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \ \mathrm{d}\Gamma$$
$$+ \int_{(\Gamma_\mathrm{I})_t} \mathbf{w}_1 \cdot \beta(\mathbf{u}_1 - \mathbf{u}_2) \ \mathrm{d}\Gamma = 0$$

Neumann BC formulation of structural mechanics:

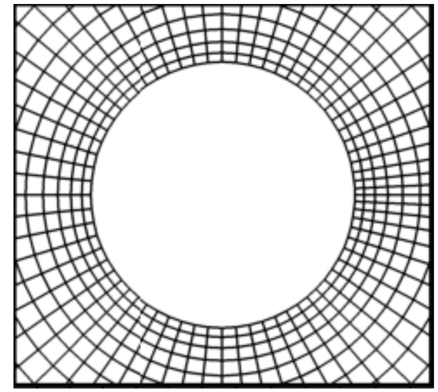
$$B_2(\mathbf{w}_2, \mathbf{u}_2) - F_2(\mathbf{w}_2) + \int_{(\Gamma_\mathrm{I})_t} \mathbf{w}_2 \cdot (\boldsymbol{\sigma}_1 \mathbf{n}_1 + \beta\, (\mathbf{u}_2 - \mathbf{u}_1)) \ \mathrm{d}\Gamma = 0$$

□

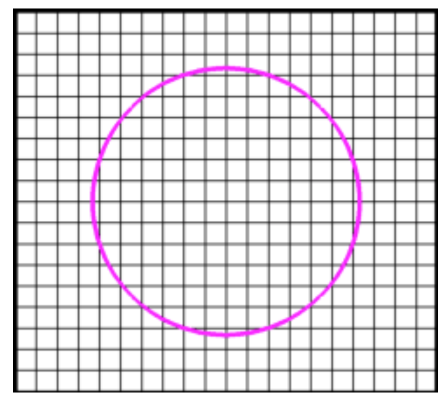
FSI: The Augmented Lagrangian Approach

Fluid mechanics:

$$B_1(\{\mathbf{w}_1, q\}, \{\mathbf{u}_1, p\}) - F_1(\{\mathbf{w}_1, q\})$$
$$- \int_{(\Gamma_I)_t} \mathbf{w}_1 \cdot \boldsymbol{\sigma}_1 \mathbf{n}_1 \ \mathrm{d}\Gamma$$
$$- \int_{(\Gamma_I)_t} \left(\delta_{\{\mathbf{u}_1, p\}} \boldsymbol{\sigma}_1 \mathbf{n}_1(\{\mathbf{w}_1, q\})\right) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \ \mathrm{d}\Gamma$$
$$+ \int_{(\Gamma_I)_t} \mathbf{w}_1 \cdot \beta(\mathbf{u}_1 - \mathbf{u}_2) \ \mathrm{d}\Gamma = 0$$



Typical case



Why not ???

Structural mechanics:

Definition of fluid traction

$$B_2(\mathbf{w}_2, \mathbf{u}_2) - F_2(\mathbf{w}_2) + \int_{(\Gamma_I)_t} \mathbf{w}_2 \cdot \overbrace{\left(\boldsymbol{\sigma}_1 \mathbf{n}_1 + \beta \left(\mathbf{u}_2 - \mathbf{u}_1\right)\right)} \ \mathrm{d}\Gamma = 0$$

Fluid Mechanics and Turbulence: ALE-VMS

Find $\mathbf{u}^h \in \mathcal{S}_u^h$ and $p^h \in \mathcal{S}_p^h$, such that $\forall \mathbf{w}^h \in \mathcal{V}_u^h$ and $q^h \in \mathcal{V}_p^h$:

Galerkin

$$\int_{\Omega_t} \mathbf{w}^h \cdot \rho \left(\left. \frac{\partial \mathbf{u}^h}{\partial t} \right|_{\hat{x}} + \left(\mathbf{u}^h - \hat{\mathbf{u}}^h \right) \cdot \boldsymbol{\nabla} \mathbf{u}^h - \mathbf{f}^h \right) \mathrm{d}\Omega + \int_{\Omega_t} \boldsymbol{\varepsilon}\left(\mathbf{w}^h\right) : \boldsymbol{\sigma}\left(\mathbf{u}^h, p^h\right) \mathrm{d}\Omega$$

$$- \int_{(\Gamma_t)_{\mathrm{h}}} \mathbf{w}^h \cdot \mathbf{h}^h \, \mathrm{d}\Gamma + \int_{\Omega_t} q^h \boldsymbol{\nabla} \cdot \mathbf{u}^h \, \mathrm{d}\Omega$$

Multiscale

SUPG/PSPG (SUPS)

$$+ \sum_{e=1}^{n_{\mathrm{el}}} \int_{\Omega_t^e} \tau_{\mathrm{SUPS}} \left(\left(\mathbf{u}^h - \hat{\mathbf{u}}^h \right) \cdot \boldsymbol{\nabla} \mathbf{w}^h + \frac{\boldsymbol{\nabla} q^h}{\rho} \right) \cdot \mathbf{r}_{\mathrm{M}} \left(\mathbf{u}^h, p^h \right) \mathrm{d}\Omega$$

$$+ \sum_{e=1}^{n_{\mathrm{el}}} \int_{\Omega_t^e} \rho \nu_{\mathrm{LSIC}} \boldsymbol{\nabla} \cdot \mathbf{w}^h r_{\mathrm{C}}(\mathbf{u}^h, p^h) \, \mathrm{d}\Omega$$

$$- \sum_{e=1}^{n_{\mathrm{el}}} \int_{\Omega_t^e} \tau_{\mathrm{SUPS}} \mathbf{w}^h \cdot \left(\mathbf{r}_{\mathrm{M}} \left(\mathbf{u}^h, p^h \right) \cdot \boldsymbol{\nabla} \mathbf{u}^h \right) \mathrm{d}\Omega$$

$$- \sum_{e=1}^{n_{\mathrm{el}}} \int_{\Omega_t^e} \frac{\boldsymbol{\nabla} \mathbf{w}^h}{\rho} : \left(\tau_{\mathrm{SUPS}} \mathbf{r}_{\mathrm{M}} \left(\mathbf{u}^h, p^h \right) \right) \otimes \left(\tau_{\mathrm{SUPS}} \mathbf{r}_{\mathrm{M}} \left(\mathbf{u}^h, p^h \right) \right) \mathrm{d}\Omega = 0$$

Weak Enforcement of Essential BCs

Consistency

$$-\sum_{b=1}^{n_{\text{eb}}} \int_{\Gamma^b \bigcap (\Gamma_t)_{\text{g}}} \mathbf{w}^h \cdot \boldsymbol{\sigma}\left(\mathbf{u}^h, p^h\right) \mathbf{n} \, \mathrm{d}\Gamma$$

Adjoint Consistency

$$-\sum_{b=1}^{n_{\text{eb}}} \int_{\Gamma^b \bigcap (\Gamma_t)_{\text{g}}} \left(2\mu\boldsymbol{\varepsilon}\left(\mathbf{w}^h\right)\mathbf{n} + q^h\mathbf{n}\right) \cdot \left(\mathbf{u}^h - \mathbf{g}^h\right) \, \mathrm{d}\Gamma$$

Convective Stabilization

$$-\sum_{b=1}^{n_{\text{eb}}} \int_{\Gamma^b \bigcap (\Gamma_t)_{\text{g}}^-} \mathbf{w}^h \cdot \rho\left(\left(\mathbf{u}^h - \hat{\mathbf{u}}^h\right) \cdot \mathbf{n}\right)\left(\mathbf{u}^h - \mathbf{g}^h\right) \, \mathrm{d}\Gamma$$

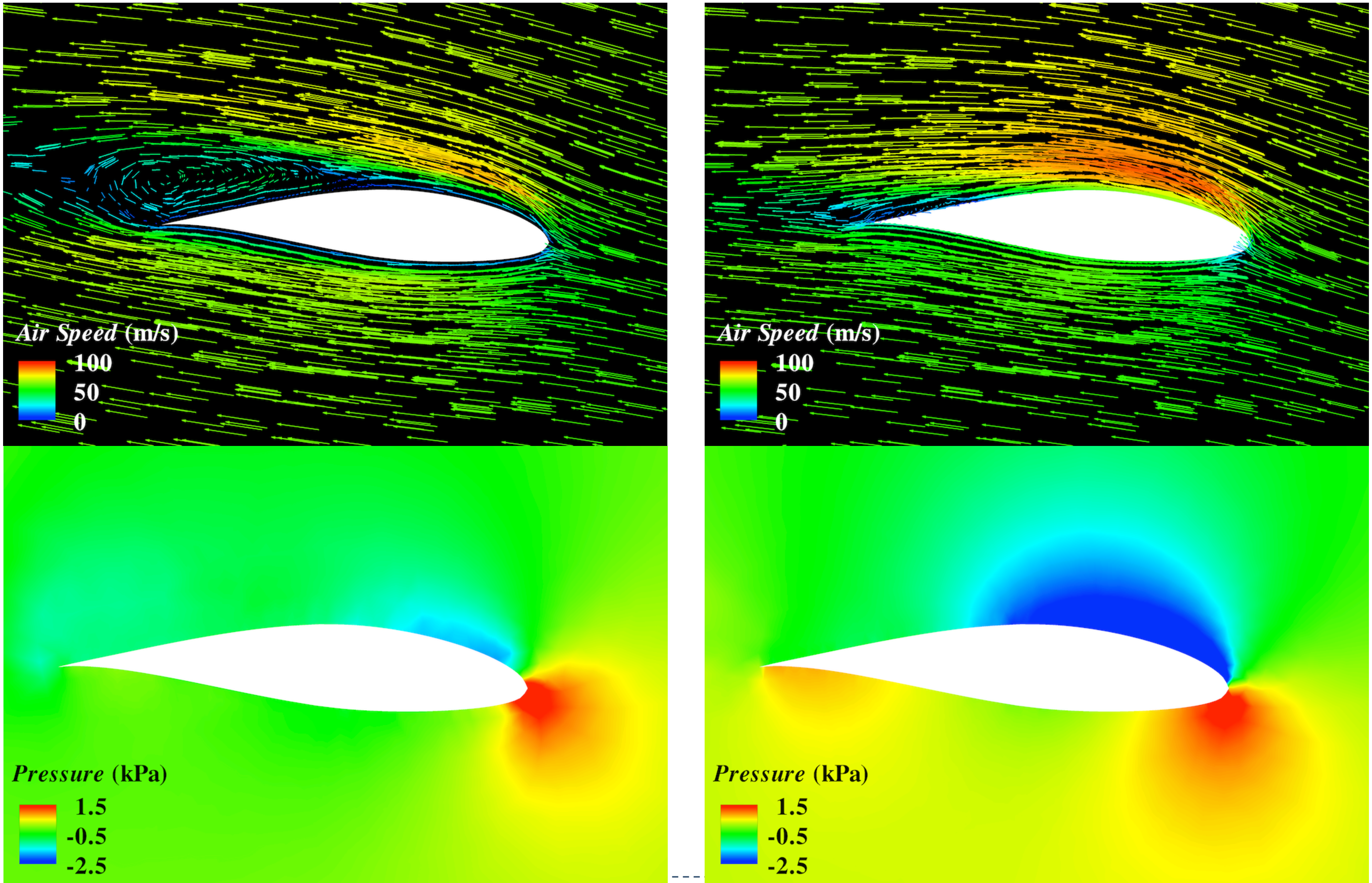
Viscous Stabilization

$$+\sum_{b=1}^{n_{\text{eb}}} \int_{\Gamma^b \bigcap (\Gamma_t)_{\text{g}}} \tau_{\text{TAN}}^B \left(\mathbf{w}^h - \left(\mathbf{w}^h \cdot \mathbf{n}\right)\mathbf{n}\right) \cdot \left(\left(\mathbf{u}^h - \mathbf{g}^h\right) - \left(\left(\mathbf{u}^h - \mathbf{g}^h\right) \cdot \mathbf{n}\right)\mathbf{n}\right) \, \mathrm{d}\Gamma$$

$$+\sum_{b=1}^{n_{\text{eb}}} \int_{\Gamma^b \bigcap (\Gamma_t)_{\text{g}}} \tau_{\text{NOR}}^B \left(\mathbf{w}^h \cdot \mathbf{n}\right)\left(\left(\mathbf{u}^h - \mathbf{g}^h\right) \cdot \mathbf{n}\right) \, \mathrm{d}\Gamma$$

Nitsche's method, or DG method at the solid (moving) wall

Strong vs. Weak Enforcement of Essential Boundary Conditions
Example from a Wind Turbine Rotor Simulation



LHS/RHS structure

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{G} \\ -\boldsymbol{G}^t & \boldsymbol{L} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{U} \\ \Delta \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_m \\ \boldsymbol{R}_c \end{bmatrix}$$

Factorization

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{G} \\ -\boldsymbol{G}^t & \boldsymbol{L} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{U} \\ \Delta \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_m \\ \boldsymbol{R}_c \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{K}^{-1} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{K} & \boldsymbol{G} \\ -\boldsymbol{G}^t & \boldsymbol{L} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{U} \\ \Delta \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}^{-1} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_m \\ \boldsymbol{R}_c \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{K}^{-1}\boldsymbol{G} \\ -\boldsymbol{G}^t & \boldsymbol{L} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{U} \\ \Delta \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}^{-1}\boldsymbol{R}_m \\ \boldsymbol{R}_c \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{G}^t & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{K}^{-1}\boldsymbol{G} \\ -\boldsymbol{G}^t & \boldsymbol{L} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{U} \\ \Delta \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{G}^t & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{K}^{-1}\boldsymbol{R}_m \\ \boldsymbol{R}_c \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \boldsymbol{I} & \boldsymbol{K}^{-1}\boldsymbol{G} \\ 0 & \boldsymbol{S} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{U} \\ \Delta \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}^{-1}\boldsymbol{R}_m \\ \boldsymbol{R}_c + \boldsymbol{G}^t\boldsymbol{K}^{-1}\boldsymbol{R}_m \end{bmatrix}}$$

$$\boldsymbol{S} = \boldsymbol{L} + \boldsymbol{G}^t\boldsymbol{K}^{-1}\boldsymbol{G}$$

Implementation

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{K}^{-1}\boldsymbol{G} \\ 0 & \boldsymbol{S} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{U} \\ \Delta \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}^{-1}\boldsymbol{R}_m \\ \boldsymbol{R}_c + \boldsymbol{G}^t\boldsymbol{K}^{-1}\boldsymbol{R}_m \end{bmatrix}$$

GMRES

$$\Delta \boldsymbol{U} \leftarrow \boldsymbol{K}^{-1}\boldsymbol{R}_m$$

$$\tilde{\boldsymbol{R}}_c \leftarrow \boldsymbol{G}^t \Delta \boldsymbol{U}$$

$$\tilde{\boldsymbol{R}}_c \leftarrow \boldsymbol{R}_c + \tilde{\boldsymbol{R}}_c$$

CG

$$\Delta \boldsymbol{P} = \boldsymbol{S}^{-1}\tilde{\boldsymbol{R}}_c$$

$$\tilde{\boldsymbol{R}}_m \leftarrow \boldsymbol{G}\Delta \boldsymbol{P}$$

GMRES

$$\Delta \tilde{\boldsymbol{U}} \leftarrow \boldsymbol{K}^{-1}\tilde{\boldsymbol{R}}_m$$

$$\Delta \boldsymbol{U} \leftarrow \Delta \boldsymbol{U} - \Delta \tilde{\boldsymbol{U}}$$

New Bi-Partitioned Solver

Approximate Schur complement:

$$\boldsymbol{S} \approx \boldsymbol{L} + \boldsymbol{G}^{\mathrm{T}} \boldsymbol{K}_{\mathrm{d}}^{-1} \boldsymbol{G}$$

Define **separate sets** for velocity and pressure as:

$$\boldsymbol{Y}_{\mathrm{u}} = \left\{ \boldsymbol{y}_{\mathrm{u}}^{1}, \boldsymbol{y}_{\mathrm{u}}^{2}, \cdots, \boldsymbol{y}_{\mathrm{u}}^{n} \right\}$$
$$\boldsymbol{Y}_{\mathrm{p}} = \left\{ \boldsymbol{y}_{\mathrm{p}}^{1}, \boldsymbol{y}_{\mathrm{p}}^{2}, \cdots, \boldsymbol{y}_{\mathrm{p}}^{n} \right\}$$

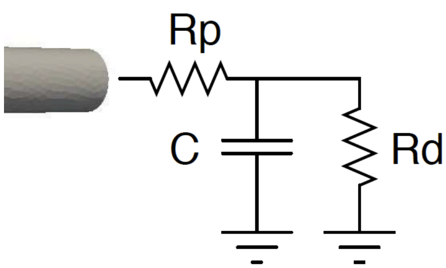
"Krylov" spaces from inexact factorization solution !!!

Construct linear solver algorithm based on:

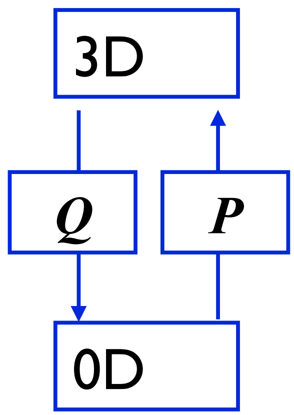
$$\min_{\{\boldsymbol{\alpha}_{\mathrm{u}}, \boldsymbol{\alpha}_{\mathrm{p}}\} \in \mathbb{R}^n} \left(\left\| \boldsymbol{R}_{\mathrm{m}} - \boldsymbol{\alpha}_{\mathrm{u}}^{\mathrm{T}} \boldsymbol{Y}_{\mathrm{u}} \boldsymbol{K} - \boldsymbol{\alpha}_{\mathrm{p}}^{\mathrm{T}} \boldsymbol{Y}_{\mathrm{p}} \boldsymbol{G} \right\|_{l_2}^{2} + \left\| \boldsymbol{R}_{\mathrm{c}} - \boldsymbol{\alpha}_{\mathrm{u}}^{\mathrm{T}} \boldsymbol{Y}_{\mathrm{u}} \boldsymbol{D} - \boldsymbol{\alpha}_{\mathrm{p}}^{\mathrm{T}} \boldsymbol{Y}_{\mathrm{p}} \boldsymbol{L} \right\|_{l_2}^{2} \right)^{\frac{1}{2}}$$

Esmaily-Moghadam, Bazilevs, and Marsden, CMAME, 2015

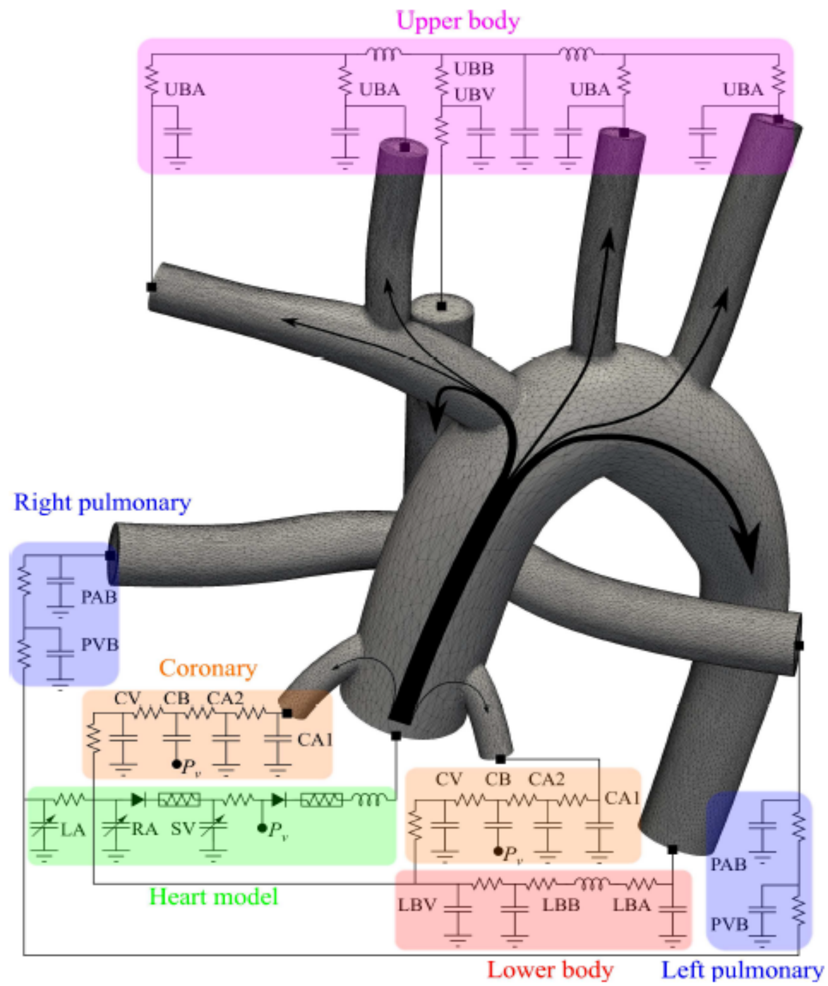
Outlet BCs in Cardiovascular FSI



$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{Q}{C} - \frac{P}{CR_d}$$



$$P = P_0 \exp\left(-\frac{t}{CR_d}\right) + \frac{1}{C}\int_0^t Q(\tau)\mathrm{d}\tau$$



LHS/RHS Modification

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{G} \\ -\boldsymbol{G}^t & \boldsymbol{L} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{U} \\ \Delta \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_m \\ \boldsymbol{R}_c \end{bmatrix}$$

$$\boldsymbol{K} = \widehat{\boldsymbol{K}} + \boldsymbol{K}^{BC}$$

$$K_{AiBj} = \widehat{K}_{AiBj} + \gamma \Delta t M_{kl} \int_{\Gamma_k} N_A n_i \mathrm{d}\Gamma \int_{\Gamma_l} N_B n_j \mathrm{d}\Gamma$$

Outlet coupling matrix
Contains "large" values due to resistance-like BCs

Nonstandard entries
Coupling of DOFs on ALL outlets
"No room" in sparse data structure

Preconditioner Design

Rank-one update

$$K^{BC}_{AiBj} = \sum_n R^n S^n_{Ai} S^n_{Bj}.$$

We have:

$$\boldsymbol{K} = \hat{\boldsymbol{K}} + \boldsymbol{K}^{BC}.$$

We want:

$$\boldsymbol{H} \simeq \boldsymbol{K}^{-1}$$

Instead, we find ***H***,

$$\boldsymbol{H}\boldsymbol{K}_d = \boldsymbol{I}$$

where

$$\boldsymbol{K}_d \equiv \mathcal{D}(\hat{\boldsymbol{K}}) + \boldsymbol{K}^{BC}.$$

Preconditioner

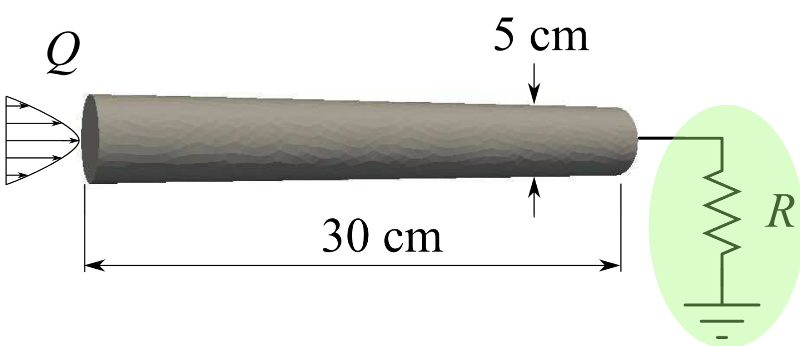
$$\boldsymbol{H} = \hat{\boldsymbol{K}}_d^{-1} - \sum_m \left[\frac{R^m (\hat{\boldsymbol{K}}_d^{-1} \boldsymbol{S}^m) \otimes (\hat{\boldsymbol{K}}_d^{-1} \boldsymbol{S}^m)}{1 + R^m \parallel \hat{\boldsymbol{K}}_d^{-\frac{1}{2}} \boldsymbol{S}^m \parallel^2} \right]$$

Sherman-Morrison formula

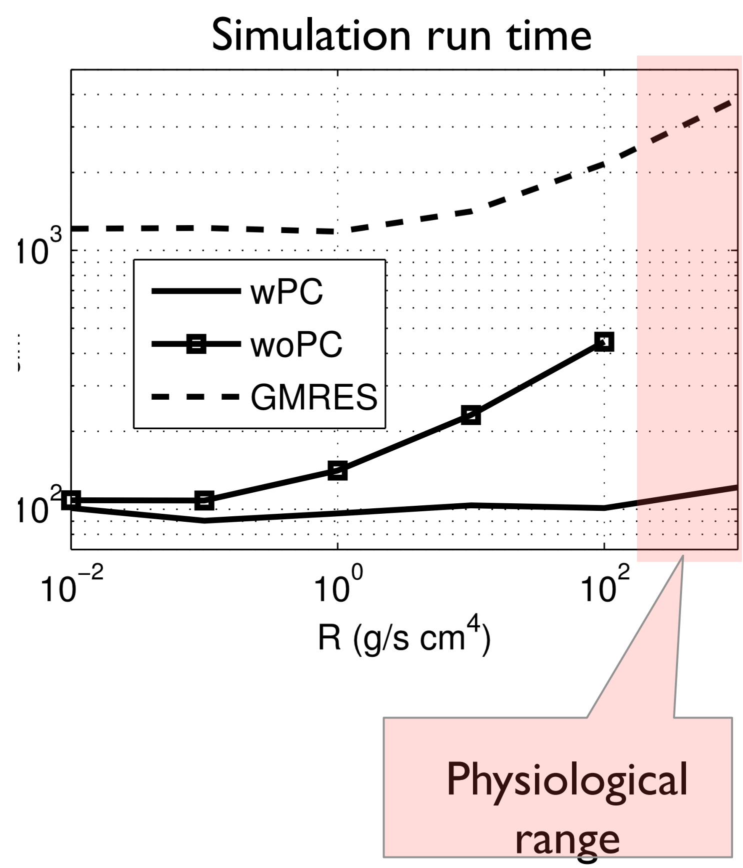
Use in the approximate Shur complement

Esmaily-Moghadam, Marsden, and Bazilevs, Comp. Mech., 2013

PC results: Cylinder



Re = 1,325
Number of nodes: 64k
Number of non-zeros in the LHS: 14M



BP + PC vs. GMRES:

Speedup: 32X

PC results: Patient-Specific Aorta

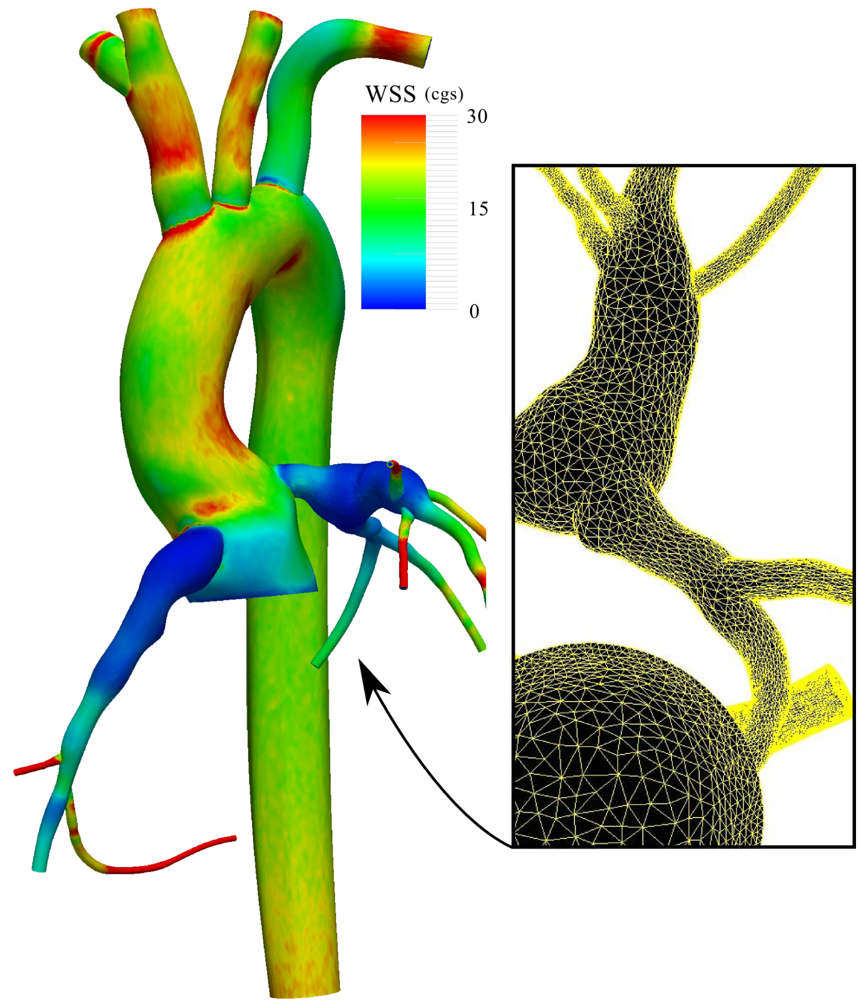
Number of processors: 64
Number of nodes: 510k
Number of non-zeros in the LHS: 115M

BP + PC vs. GMRES (D*t* = 1ms):

Speedup: 15X

BP + PC vs. GMRES (D*t* = 0.2ms):

Speedup: 8X



FSI: Coupling

Discrete Equations of
Fluid Mechanics (1),
Structural Mechanics (2),
and Mesh Moving (3)

$$\begin{cases} \mathbf{N}_1\left(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\right) = \mathbf{0} \\ \mathbf{N}_2\left(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\right) = \mathbf{0} \\ \mathbf{N}_3\left(\underbrace{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3}\right) = \mathbf{0} \end{cases}$$

Discrete Unknowns of
Fluid Mechanics (1),
Structural Mechanics (2),
and Mesh Moving (3)

FSI: Coupling

If all three sets of equations are satisfied (within a time step), then the FSI technique is referred to as **strongly-coupled.**

Everything else is called **loosely-coupled**, **weakly-coupled**, or **staggered** FSI technique. *Note that in this case the true FSI coupling (compatibility of kinematics and tractions) is not guaranteed.*

FSI: Block-Iterative Coupling

$$\left.\frac{\partial \mathbf{N}_1}{\partial \mathbf{d}_1}\right|_{(\mathbf{d}_1^i,\ \mathbf{d}_2^i,\ \mathbf{d}_1^i)} \Delta \mathbf{d}_1^i = -\mathbf{N}_1\left(\mathbf{d}_1^i,\ \mathbf{d}_2^i,\ \mathbf{d}_3^i\right),$$

$$\mathbf{d}_1^{i+1} = \mathbf{d}_1^i + \Delta \mathbf{d}_1^i,$$

$$\left.\frac{\partial \mathbf{N}_2}{\partial \mathbf{d}_2}\right|_{(\mathbf{d}_1^{i+1},\ \mathbf{d}_2^i,\ \mathbf{d}_1^i)} \Delta \mathbf{d}_2^i = -\mathbf{N}_2\left(\mathbf{d}_1^{i+1},\ \mathbf{d}_2^i,\ \mathbf{d}_3^i\right),$$

$$\mathbf{d}_2^{i+1} = \mathbf{d}_2^i + \Delta \mathbf{d}_2^i,$$

$$\left.\frac{\partial \mathbf{N}_3}{\partial \mathbf{d}_3}\right|_{(\mathbf{d}_1^{i+1},\ \mathbf{d}_2^{i+1},\ \mathbf{d}_1^i)} \Delta \mathbf{d}_3^i = -\mathbf{N}_3\left(\mathbf{d}_1^{i+1},\ \mathbf{d}_2^{i+1},\ \mathbf{d}_3^i\right),$$

$$\mathbf{d}_3^{i+1} = \mathbf{d}_3^i + \Delta \mathbf{d}_3^i.$$

FSI: Quasi-Direct Coupling

$$\left.\frac{\partial \mathbf{N}_1}{\partial \mathbf{d}_1}\right|_{(\mathbf{d}_1^i,\ \mathbf{d}_2^i,\ \mathbf{d}_3^i)} \Delta \mathbf{d}_1^i + \left.\frac{\partial \mathbf{N}_1}{\partial \mathbf{d}_2}\right|_{(\mathbf{d}_1^i,\ \mathbf{d}_2^i,\ \mathbf{d}_3^i)} \Delta \mathbf{d}_2^i = -\mathbf{N}_1\left(\mathbf{d}_1^i,\ \mathbf{d}_2^i,\ \mathbf{d}_3^i\right),$$

$$\left.\frac{\partial \mathbf{N}_2}{\partial \mathbf{d}_1}\right|_{(\mathbf{d}_1^i,\ \mathbf{d}_2^i,\ \mathbf{d}_3^i)} \Delta \mathbf{d}_1^i + \left.\frac{\partial \mathbf{N}_2}{\partial \mathbf{d}_2}\right|_{(\mathbf{d}_1^i,\ \mathbf{d}_2^i,\ \mathbf{d}_3^i)} \Delta \mathbf{d}_2^i = -\mathbf{N}_2\left(\mathbf{d}_1^i,\ \mathbf{d}_2^i,\ \mathbf{d}_3^i\right),$$

$$\mathbf{d}_1^{i+1} = \mathbf{d}_1^i + \Delta \mathbf{d}_1^i,$$

$$\mathbf{d}_2^{i+1} = \mathbf{d}_2^i + \Delta \mathbf{d}_2^i,$$

$$\left.\frac{\partial \mathbf{N}_3}{\partial \mathbf{d}_3}\right|_{(\mathbf{d}_1^{i+1},\ \mathbf{d}_2^{i+1},\ \mathbf{d}_3^i)} \Delta \mathbf{d}_3^i = -\mathbf{N}_3\left(\mathbf{d}_1^{i+1},\ \mathbf{d}_2^{i+1},\ \mathbf{d}_3^i\right),$$

$$\mathbf{d}_3^{i+1} = \mathbf{d}_3^i + \Delta \mathbf{d}_3^i.$$

Typically require special implementation, especially for nonmatching interfaces

Direct Coupling: All three subsystems linearized and solved simultaneously. Rarely used as quasi-direct technique works well.

FSI: Matrix-Free Matvecs for Off-Diagonal Blocks!

$$\frac{\partial \mathbf{N}_1}{\partial \mathbf{d}_1}\Delta\mathbf{d}_1 \text{ and } \frac{\partial \mathbf{N}_2}{\partial \mathbf{d}_2}\Delta\mathbf{d}_2 \text{ - Standard, sparse product}$$

$$\frac{\partial \mathbf{N}_1}{\partial \mathbf{d}_2}\Delta\mathbf{d}_2 \approx \frac{\mathbf{N}_1\left(\mathbf{d}_1,\mathbf{d}_2+\varepsilon_1\Delta\mathbf{d}_2,\mathbf{d}_3\right)-\mathbf{N}_1\left(\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\right)}{\varepsilon_1}$$

$$\frac{\partial \mathbf{N}_2}{\partial \mathbf{d}_1}\Delta\mathbf{d}_1 \approx \frac{\mathbf{N}_1\left(\mathbf{d}_1+\varepsilon_2\Delta\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\right)-\mathbf{N}_1\left(\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\right)}{\varepsilon_2}$$

Moving Sliding-Interface Formulation

Extract structure angular velocity

$$\omega = \boldsymbol{I}^{-1} \int_{\Omega_t} \boldsymbol{r} \times \rho \boldsymbol{u} \, d\Omega.$$

Remove the "normal/axial" component

$$\omega_\tau = \omega - (\omega \cdot \boldsymbol{n})\boldsymbol{n}.$$

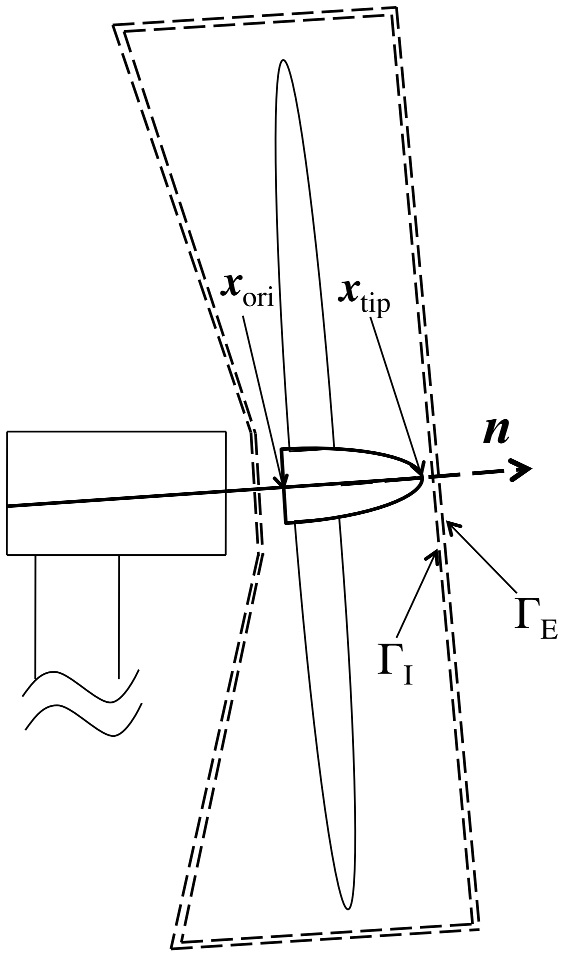
Compute rotation matrix of the "spinning" cylinder

$$\frac{d\boldsymbol{R}}{dt} - \Omega \boldsymbol{R} = \boldsymbol{0}.$$

Compute rotation matrix of the "non-spinning" cylinder

$$\frac{d\boldsymbol{R}_\tau}{dt} - \Omega_\tau \boldsymbol{R}_\tau = \boldsymbol{0}.$$

Use the rotation matrices to update the positions of the sliding-interface meshes and recompute "closest points".



Sliding-Interface Coupling (DG)

Consistency

$$-\sum_{b=1}^{n_{\mathrm{eb}}} \int_{\Gamma^b \bigcap (\Gamma_{\mathrm{I}})_t} (\mathbf{w}_1^h - \mathbf{w}_2^h) \cdot \frac{1}{2}(\boldsymbol{\sigma}\left(\mathbf{u}_1^h, p_1^h\right) \mathbf{n}_1 - \boldsymbol{\sigma}\left(\mathbf{u}_2^h, p_2^h\right) \mathbf{n}_2)\ \mathrm{d}\Gamma$$

Adjoint Consistency

$$-\sum_{b=1}^{n_{\mathrm{eb}}} \int_{\Gamma^b \bigcap (\Gamma_{\mathrm{I}})_t} \frac{1}{2}(\delta\boldsymbol{\sigma}\left(\mathbf{w}_1^h, q_1^h\right) \mathbf{n}_1 - \delta\boldsymbol{\sigma}\left(\mathbf{w}_2^h, q_2^h\right) \mathbf{n}_2) \cdot \left(\mathbf{u}_1^h - \mathbf{u}_2^h\right)\ \mathrm{d}\Gamma$$

Convective Stabilization

$$-\sum_{b=1}^{n_{\mathrm{eb}}} \int_{\Gamma^b \bigcap (\Gamma_{\mathrm{I}})_t} \mathbf{w}_1^h \cdot \rho\left(\left(\mathbf{u}_1^h - \hat{\mathbf{u}}_1^h\right) \cdot \mathbf{n}_1\right)_- \left(\mathbf{u}_1^h - \mathbf{u}_2^h\right)\ \mathrm{d}\Gamma$$

$$-\sum_{b=1}^{n_{\mathrm{eb}}} \int_{\Gamma^b \bigcap (\Gamma_{\mathrm{I}})_t} \mathbf{w}_2^h \cdot \rho\left(\left(\mathbf{u}_2^h - \hat{\mathbf{u}}_2^h\right) \cdot \mathbf{n}_2\right)_- \left(\mathbf{u}_2^h - \mathbf{u}_1^h\right)\ \mathrm{d}\Gamma$$

Viscous Stabilization

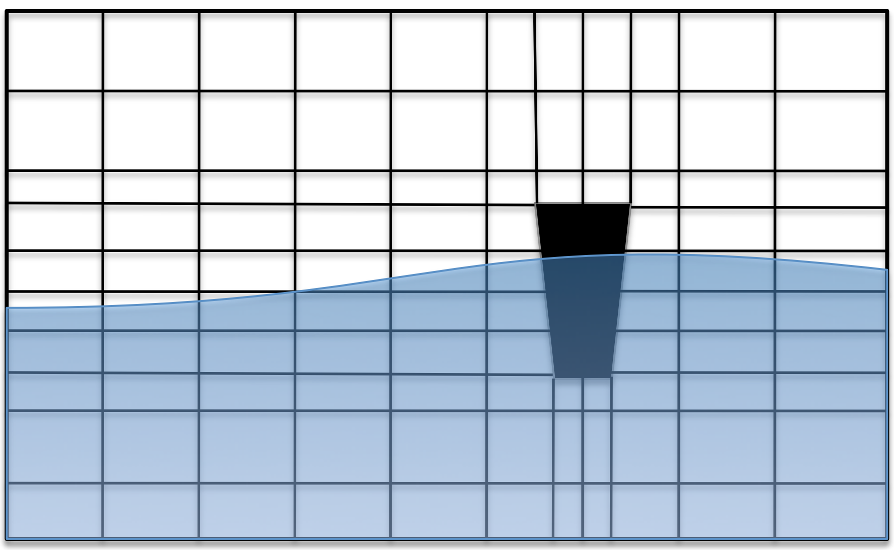
$$+\sum_{b=1}^{n_{\mathrm{eb}}} \int_{\Gamma^b \bigcap (\Gamma_{\mathrm{I}})_t} \frac{C\mu}{h} \left(\mathbf{w}_1^h - \mathbf{w}_2^h\right) \cdot \left(\mathbf{u}_1^h - \mathbf{u}_2^h\right)\ \mathrm{d}\Gamma$$

Valid for moving sliding interfaces!

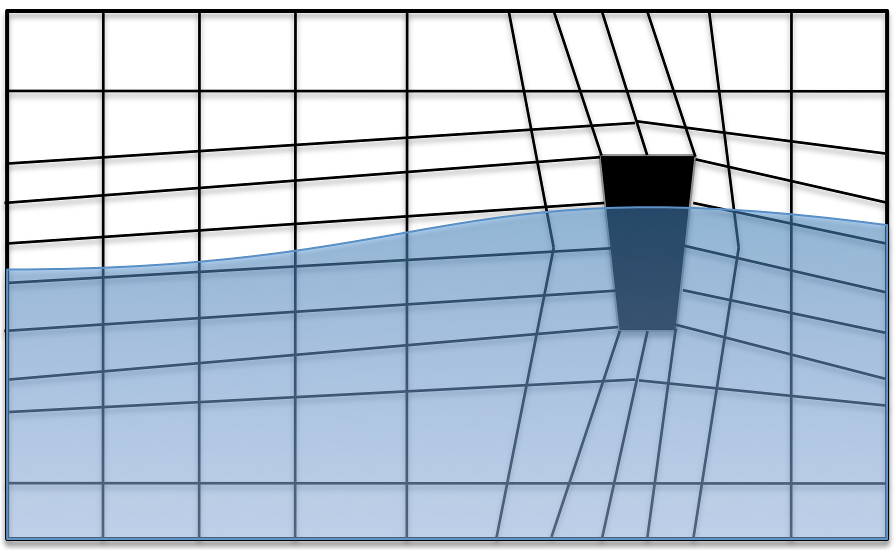
□

Free-Surface Flow and FSI: Methodology

- Air-water interface: “Interface capturing” – (Level set or VOF)
- Fluid-structure interface: “Interface tracking” – (ALE or ST)
- Approach was termed “MITICT” by T.E. Tezduyar
- **Quasi-direct FSI coupling (Fluid, Structure, Level set)**



Problem Reference Configuration and Mesh



Problem Current Configuration and Mesh

Free-Surface Flow: Theory

$$\rho_\varepsilon \left.\frac{\partial \boldsymbol{u}}{\partial t}\right|_{\hat{y}} + \rho_\varepsilon (\boldsymbol{u} - \hat{\boldsymbol{u}}) \cdot \nabla \boldsymbol{u} + \nabla p - \nabla \cdot 2\mu_\varepsilon \nabla^s \boldsymbol{u} - \rho_\varepsilon \boldsymbol{f} = \boldsymbol{0}$$

$$\left.\frac{\partial \varphi}{\partial t}\right|_{\hat{y}} + (\boldsymbol{u} - \hat{\boldsymbol{u}}) \cdot \nabla \varphi = 0$$

$$\rho_\varepsilon = \rho_w H_\varepsilon(\varphi) + \rho_a \left(1 - H_\varepsilon(\varphi)\right)$$

$$\mu_\varepsilon = \mu_w H_\varepsilon(\varphi) + \mu_a \left(1 - H_\varepsilon(\varphi)\right)$$

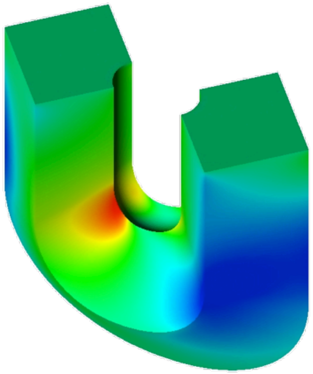
$$H_\varepsilon(\varphi) = \begin{cases} 0 \\ 1/2\left(1 + \varphi/\varepsilon + 1/\pi \sin(\varphi\pi/\varepsilon)\right) \\ 1 \end{cases}$$

Regularization requires that φ is a distance function.

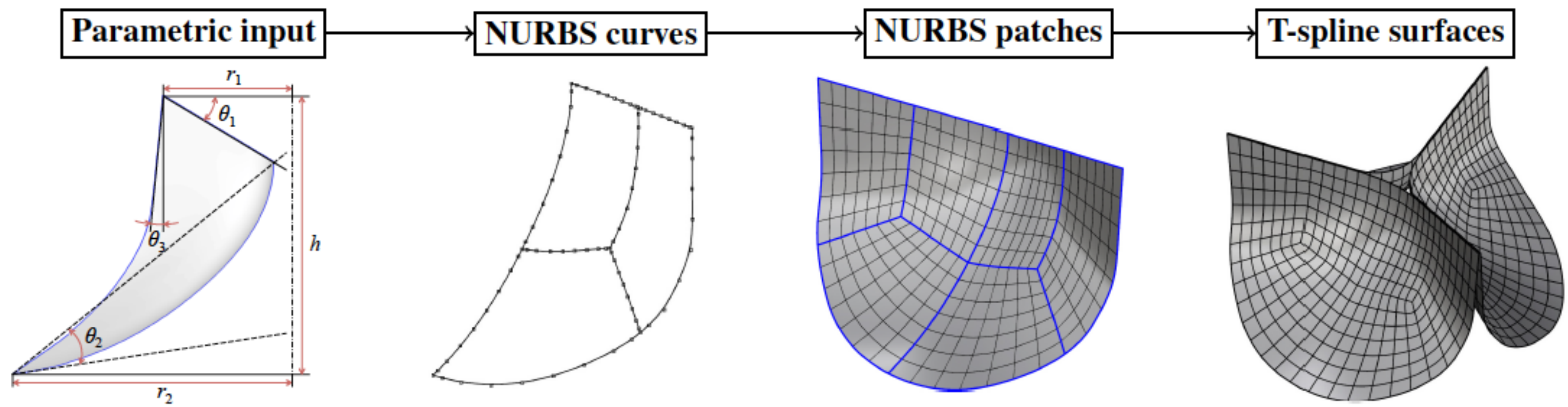
Isogeometric Analysis (IGA)

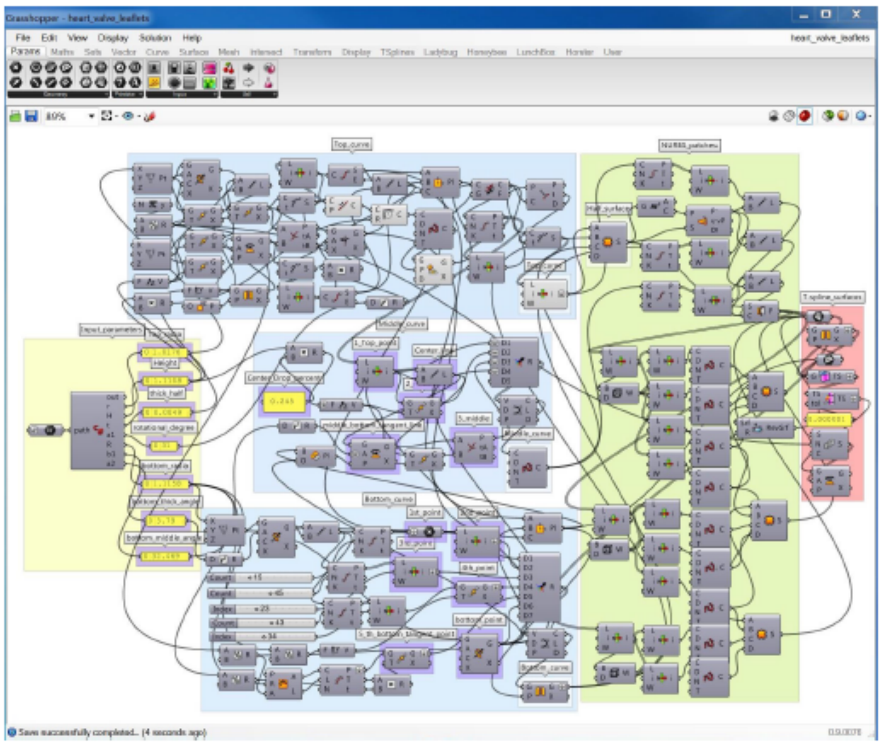
- Hughes, Cottrell, and YB. First paper appeared in the Fall 2005
- Based on technologies (e.g., NURBS, T-splines) from *computational geometry* used in:
  - Design (CAD)
  - Animation (CG)
  - Visualization (CG)

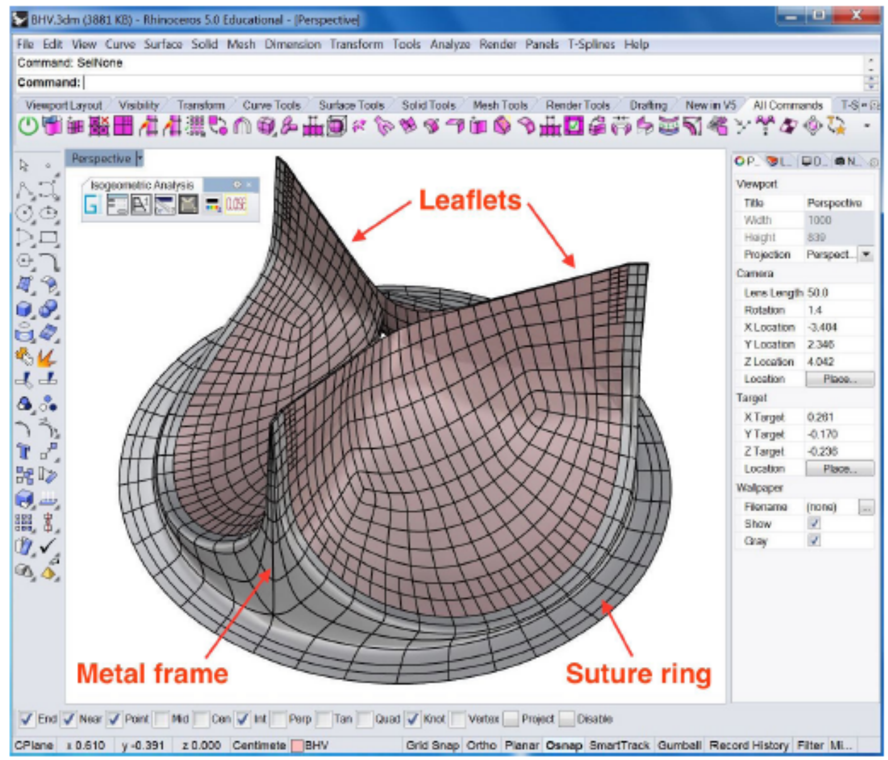
- IGA = "Exact" geometry + *the isoparametric concept* in FEM
- Includes standard FEA as a special case, but offers other possibilities:
  - Precise and efficient geometric modeling
  - Simplified mesh refinement
  - Superior approximation properties
  - Smooth basis functions
  - ***Integration* of design and analysis: "modeling platforms"**



IGA Modeling Platform

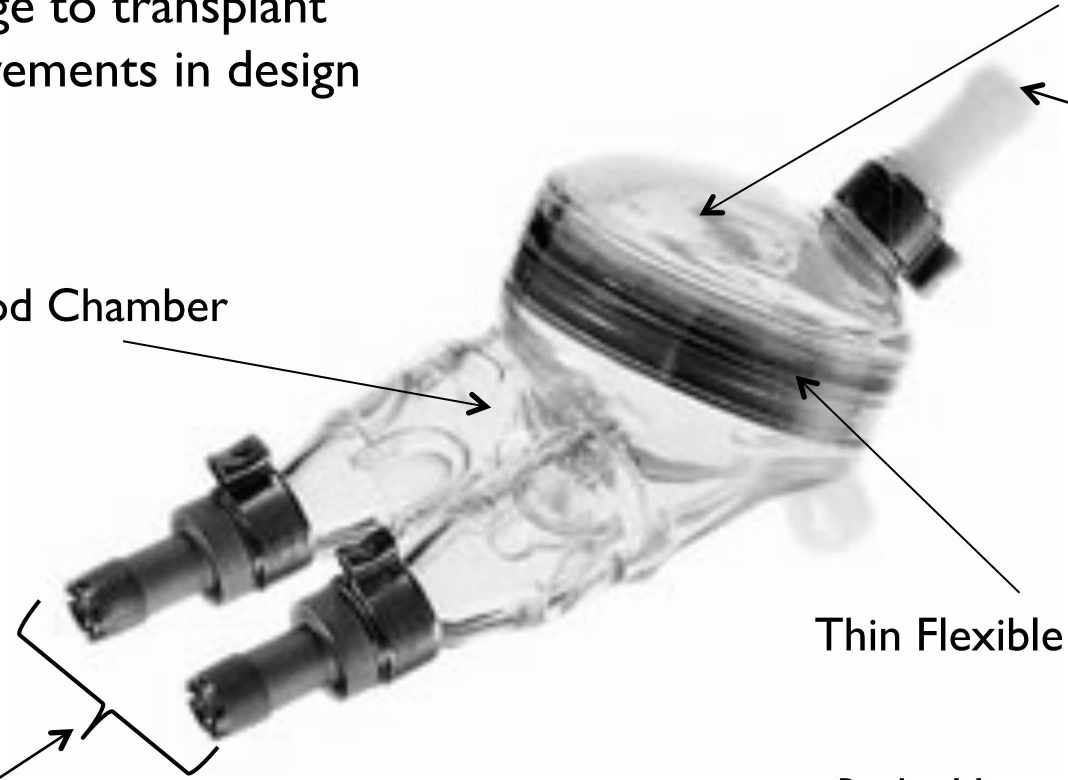




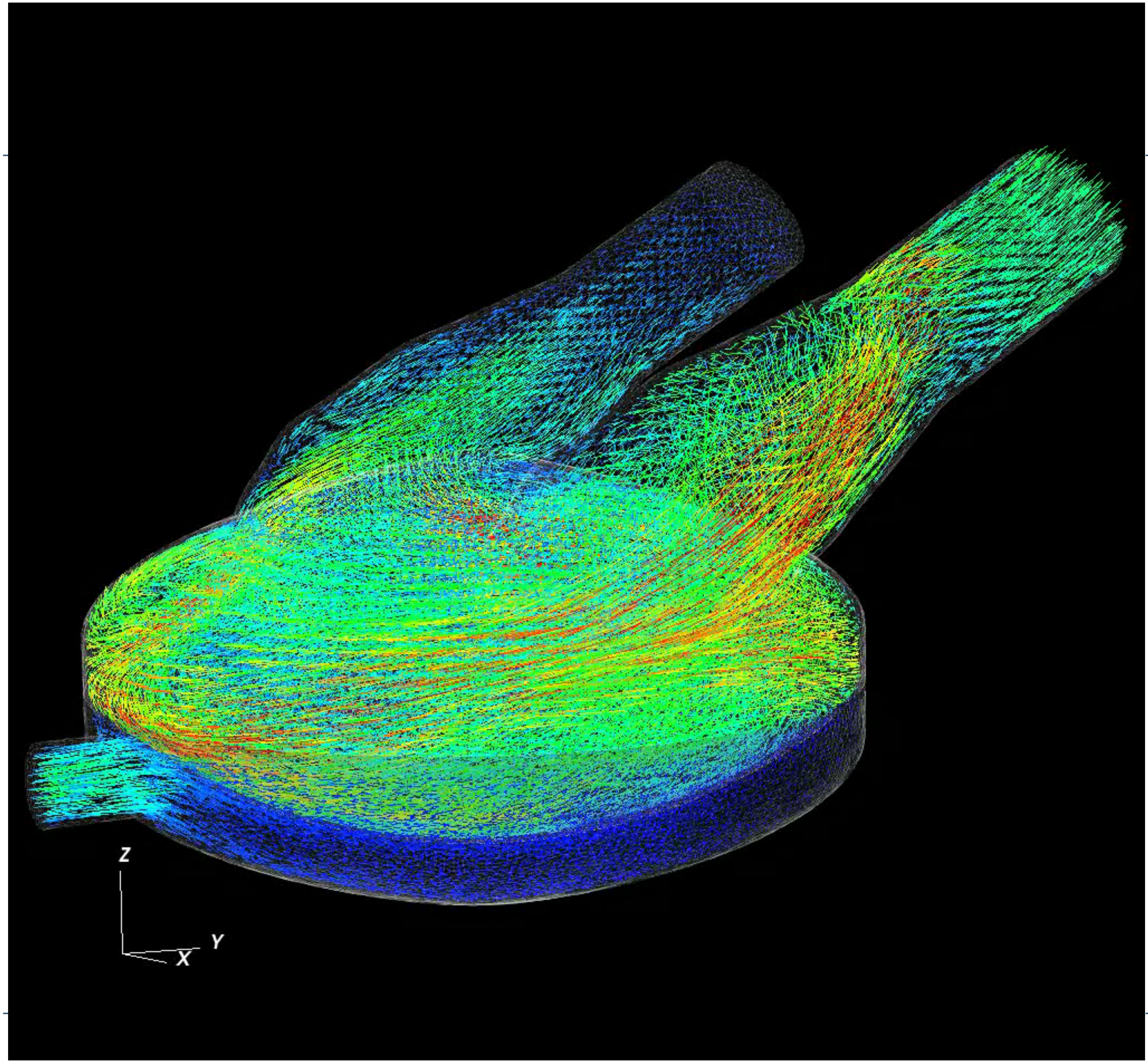


FSI of PVADs

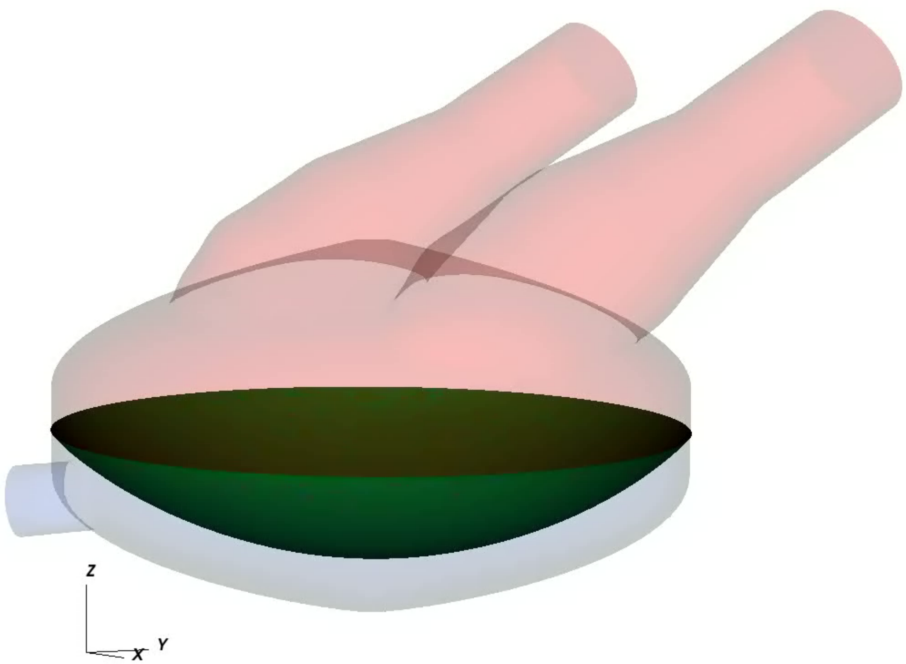
- Blood clots in up to 40% of pediatric patients
- Too risky as destination therapy
- Used as bridge to transplant
- Need improvements in design

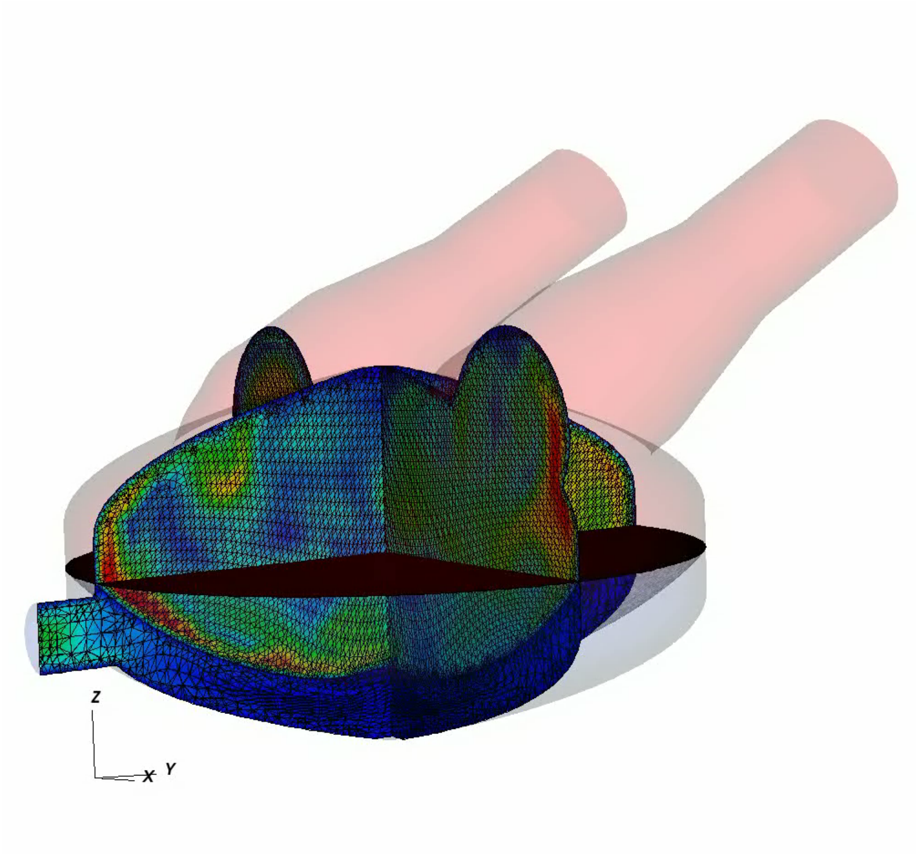


Berlin Heart PVAD
Membrane properties of the ***Penn State Device***
(Private communication with K. Manning, Bioengineering Department, Penn State)

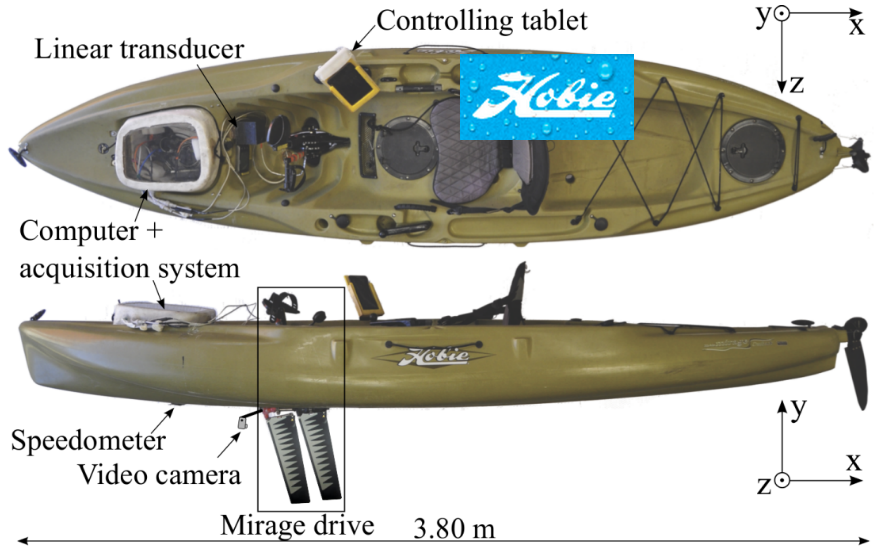

Z
Y
X

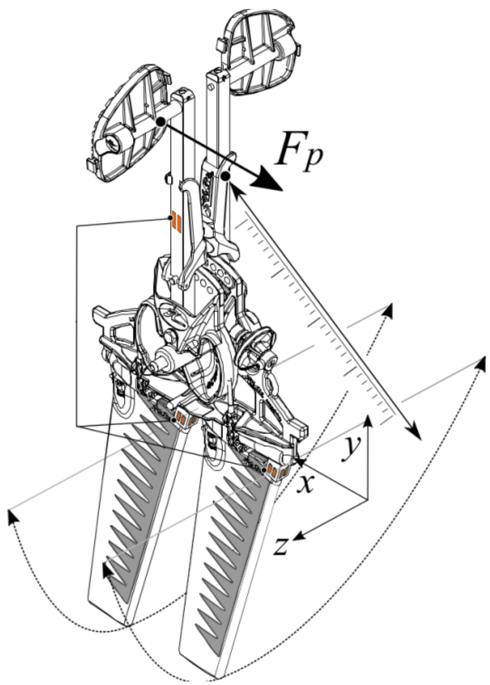
FSI of PVADs

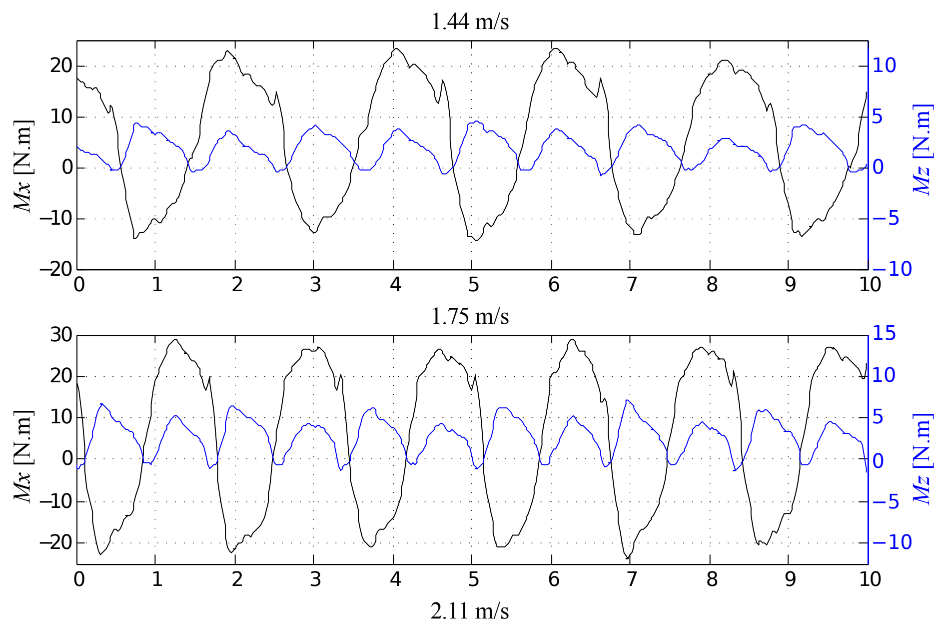




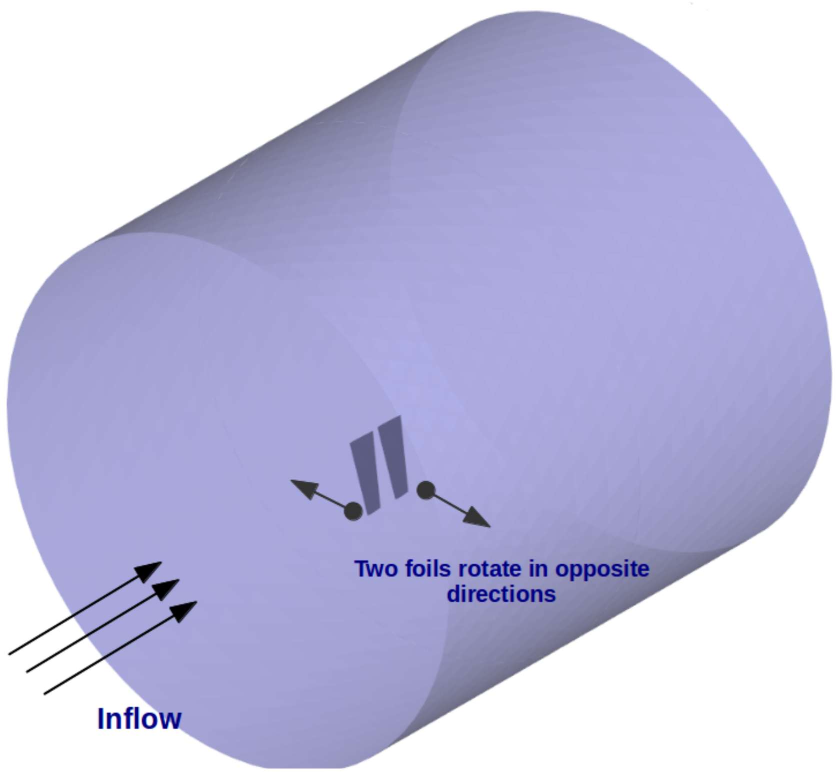
Mirage Drive by Hobie Cat Co.



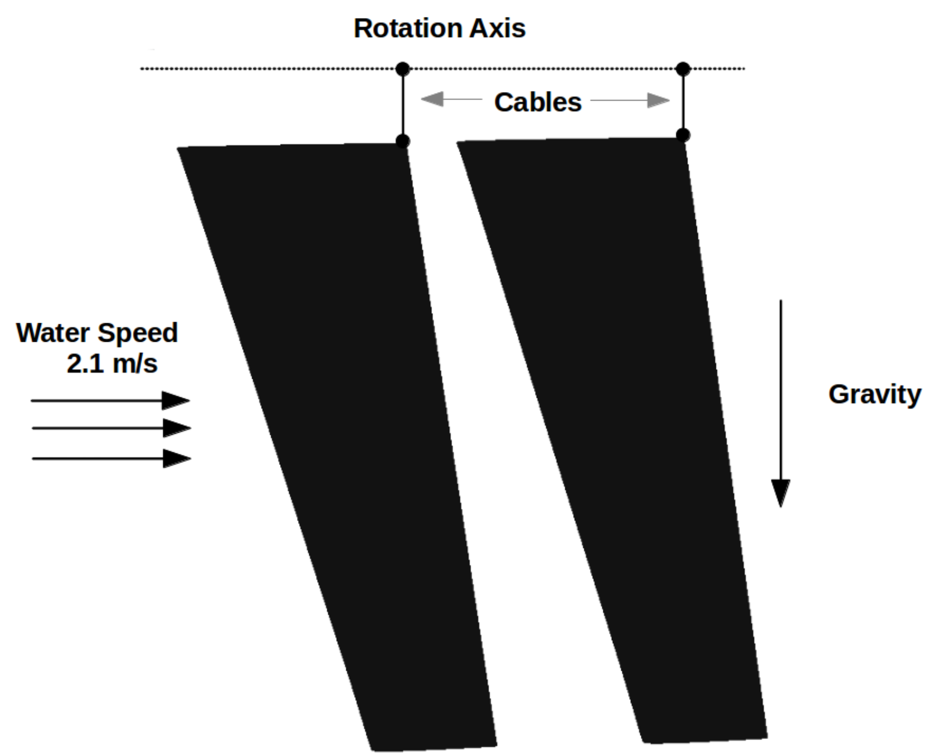




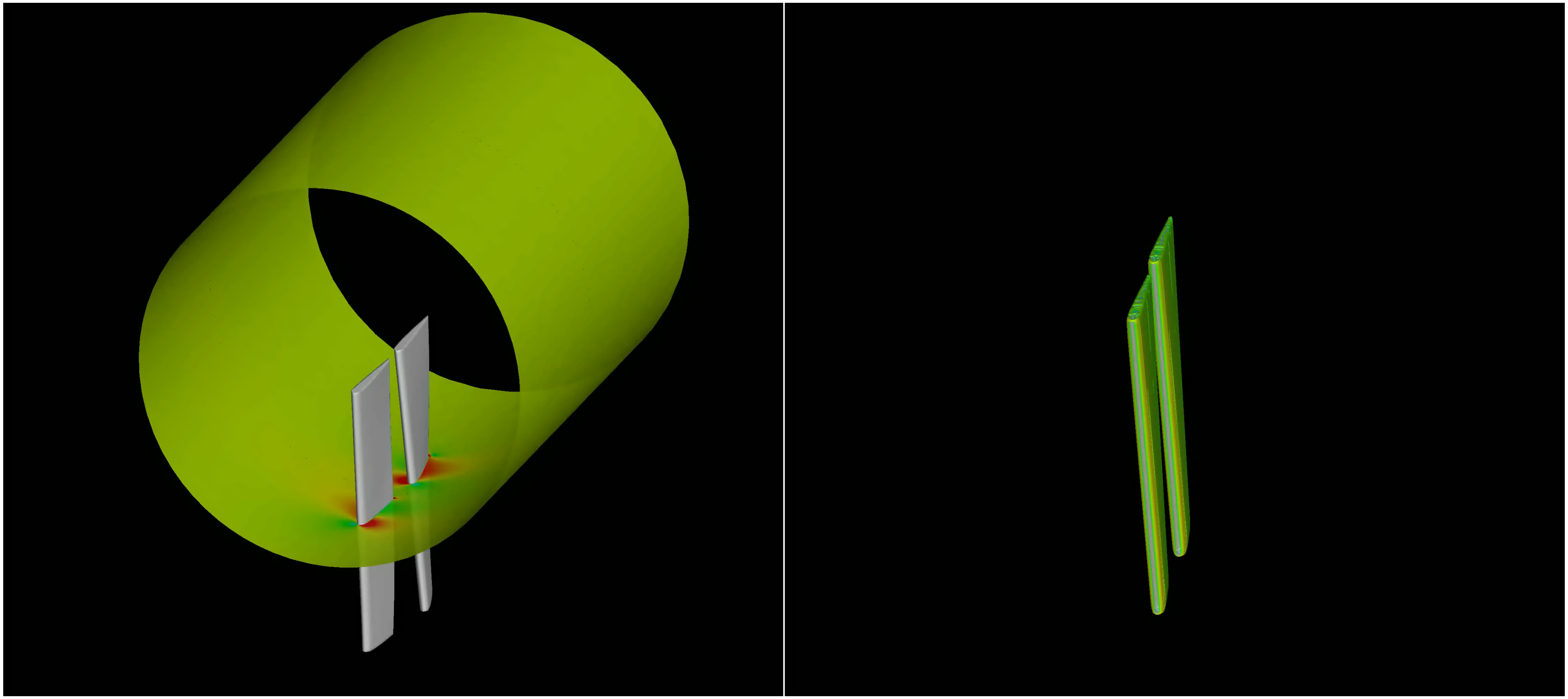
FSI Setup



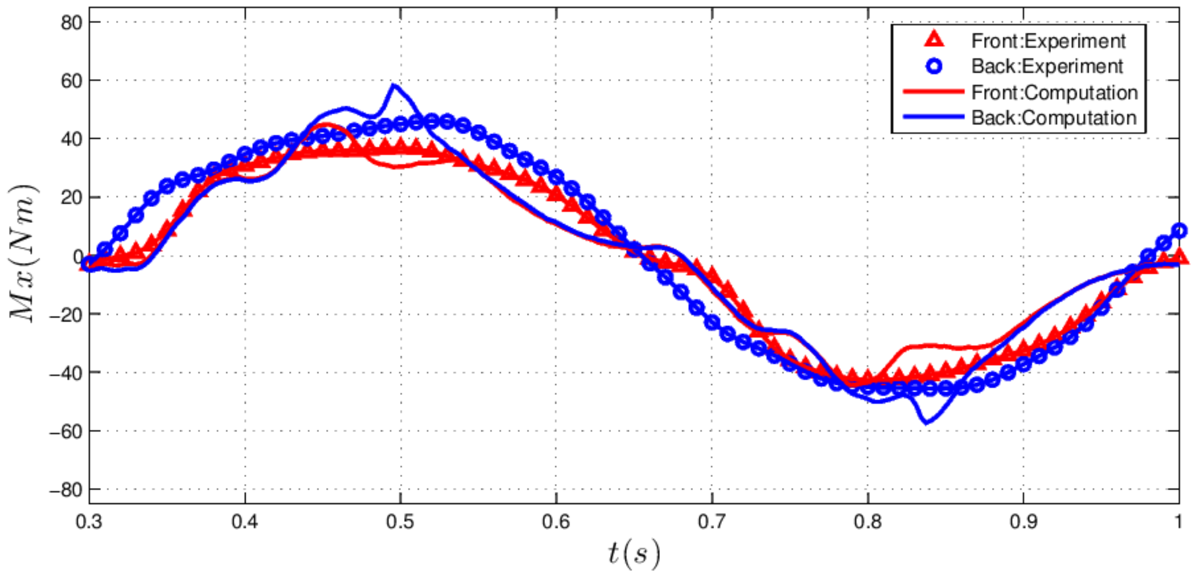
Two foils rotate in opposite directions
Inflow

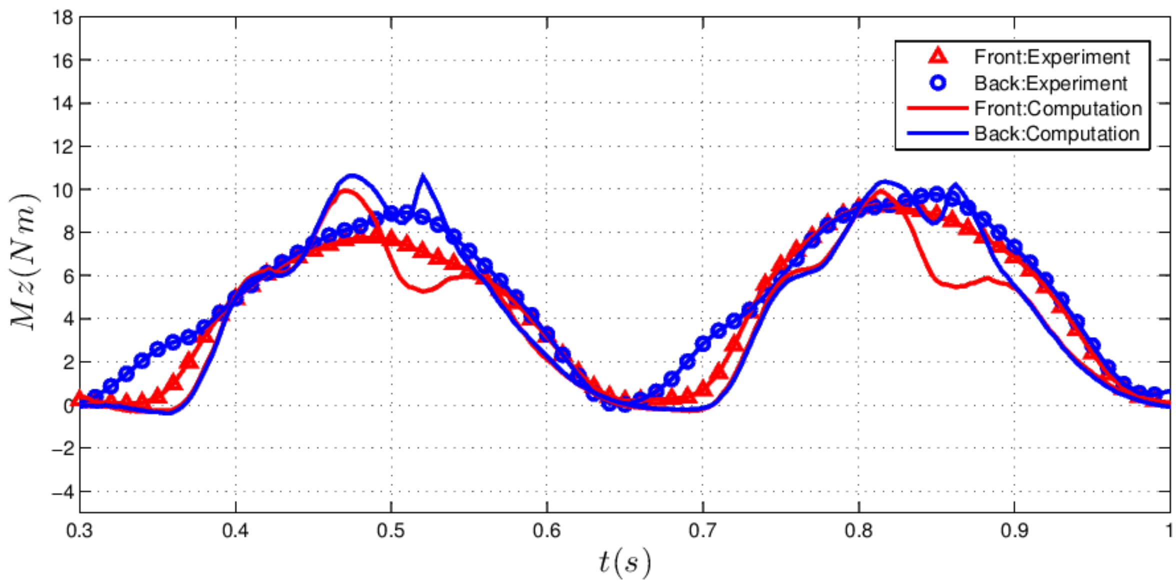


Rotation Axis
Cables
Water Speed
2.1 m/s
Gravity

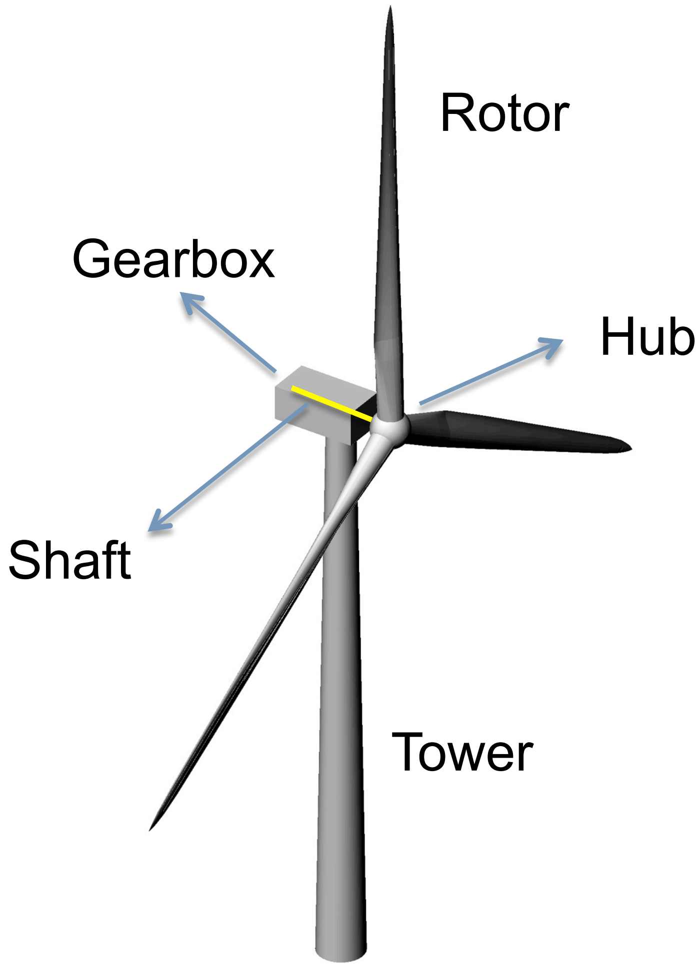


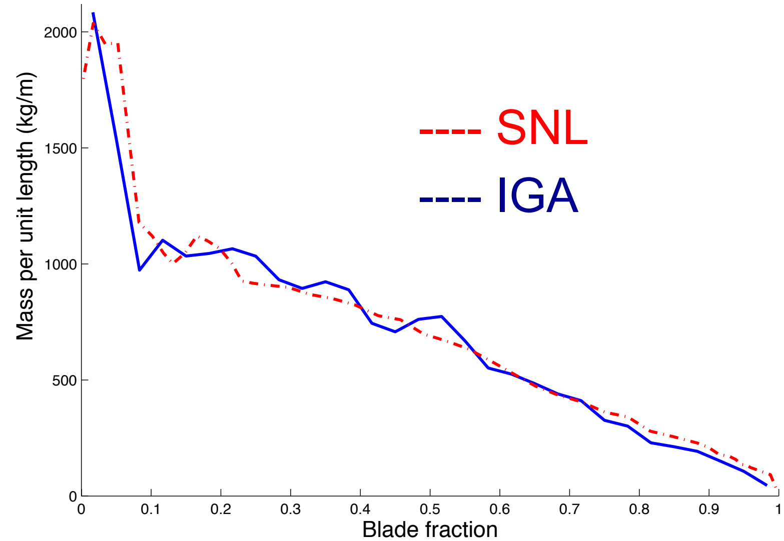
Hydrodynamic Moment Comparison



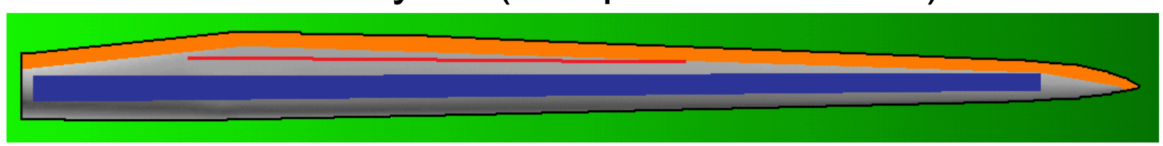


Offshore Wind Turbine (62m Blade)

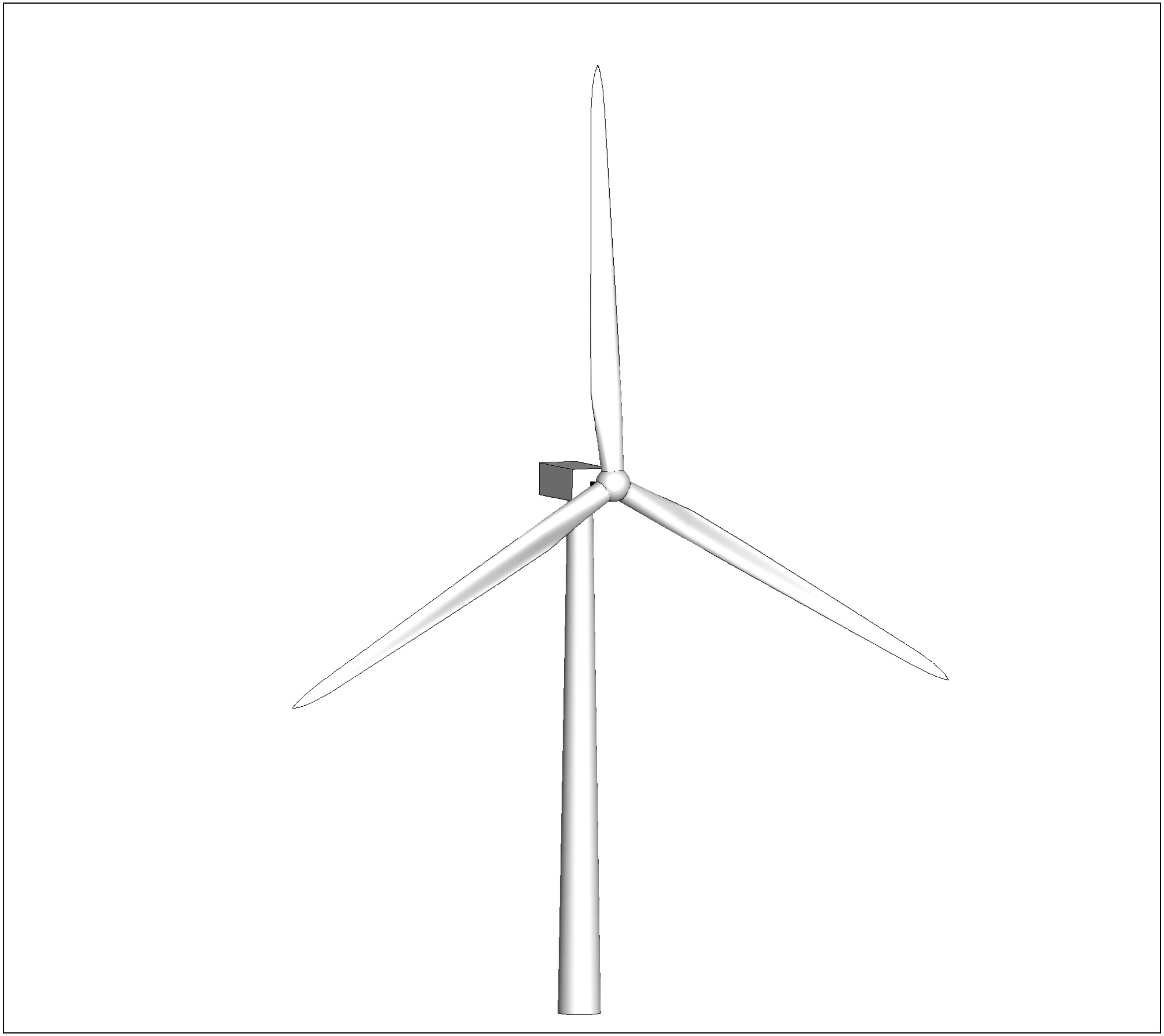


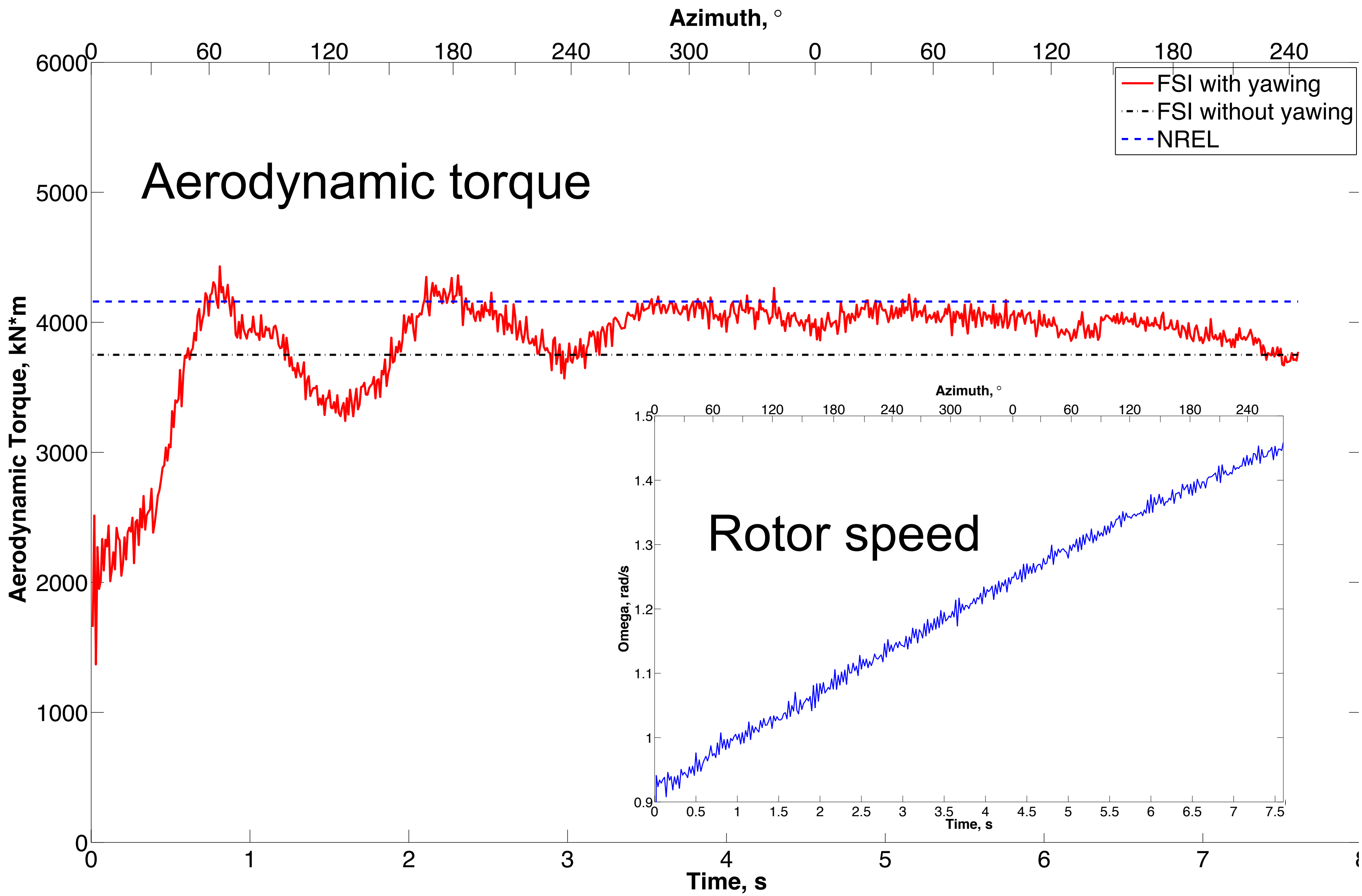


Blade layout (composite laminate)

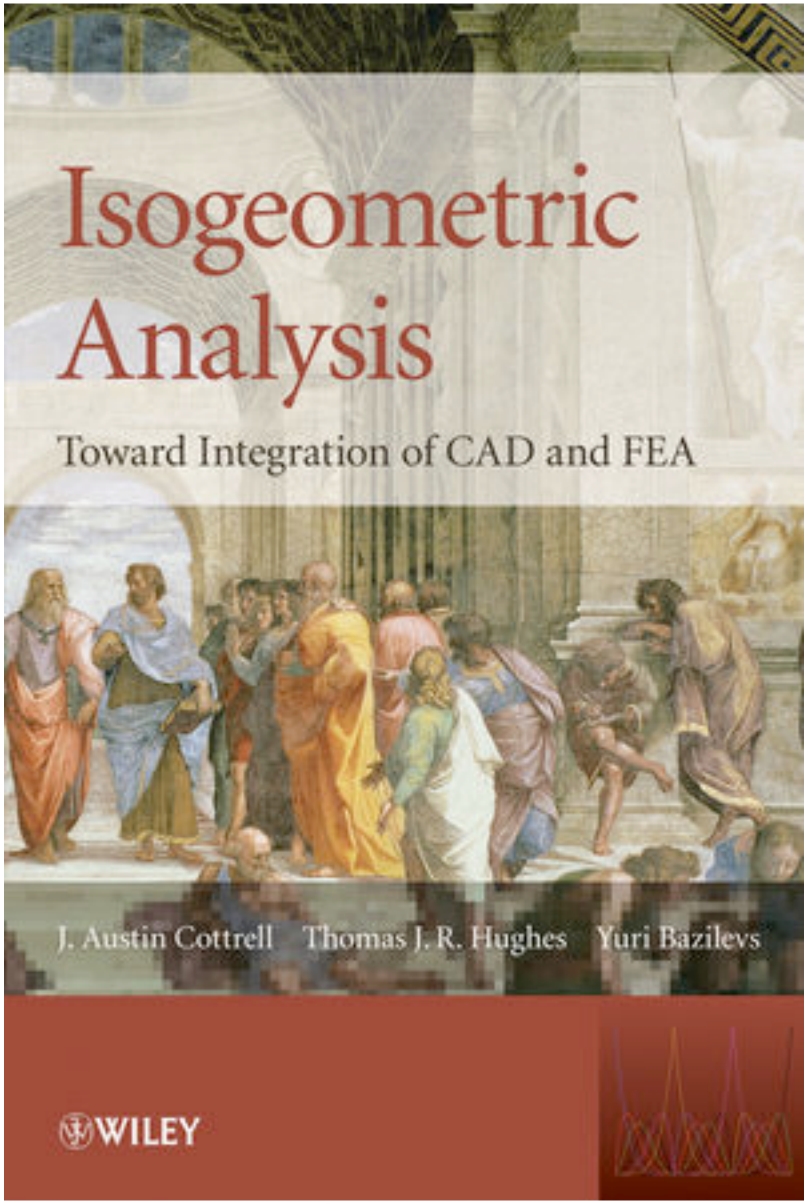


Mode (Hz)	IGA: M1/M2/M3	Ref.
1st Flapwise Bending	0.456/0.454/0.453	0.42
1st Edgewise Bending	0.681/0.679/0.678	0.69





Azimuth, °
0 60 120 180 240 300 0 60 120 180 240
FSI with yawing
FSI without yawing
NREL
Aerodynamic torque
Aerodynamic Torque, kN*m
0 1000 2000 3000 4000 5000 6000
Rotor speed
Omega, rad/s
0.9 1 1.1 1.2 1.3 1.4 1.5
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5 5.5 6 6.5 7 7.5
Time, s
0 1 2 3 4 5 6 7 8
Time, s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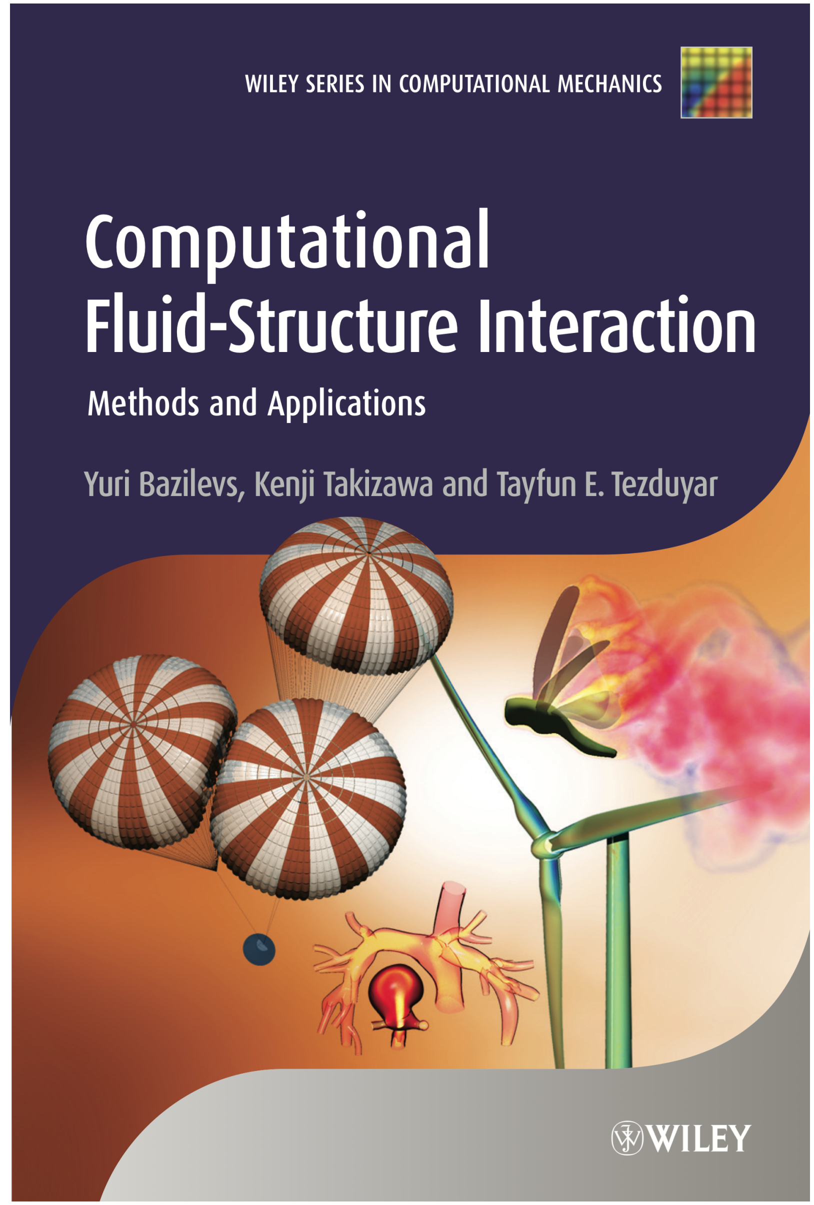
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San Diego, CA
July 26-30, 2015

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