Global convergence rates of some multilevel methods for variational and quasi-variational inequalities

Lori Badea[1]

1 Introduction

The first multilevel method for variational inequalities has been proposed in Mandel [1984a] for complementarity problems. An upper bound of the asymptotic convergence rate of this method is derived in Mandel [1984b]. The method has been studied later in Kornhuber [1994] in two variants, standard monotone multigrid method and truncated monotone multigrid method. These methods have been extended to variational inequalities of the second kind in Kornhuber [1996] and Kornhuber [2002]. Also, versions of this method have been applied to Signorini's problem in elasticity in Kornhuber and Krause [2001]. In Badea [2003] and Badea [2006] global convergence rates of some projected multilevel relaxation methods of multiplicative type are given. Also, a global convergence rate was derived in Badea [2008] for a two-level additive method. Two-level methods for variational inequalities of the second kind and for some quasi variational inequalities have been analyzed in Badea and Krause [2012]. In Badea [2014], it was theoretically justified the global convergence rate of the standard monotone multigrid methods and, in Badea [2015], this result has been extended to the hybrid algorithms, where the type of the iterations on the levels is different from the type of the iterations over the levels. Finally, a multigrid method for inequalities containing a term given by a Lipschitz operator is analyzed in Badea [2016]. Evidently, the above list of citations is not exhaustive and, for further information, we can see the review article Gräser and Kornhuber [2009].

This is a review paper regarding the convergence rate of some multilevel methods for variational inequalities and also, for more complicated problems such as variational inequalities of the second kind, quasi-variational inequalities and inequalities with a term containing a Lipschitz operator. The meth-

Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania `lori.badea@imar.ro`

ods are first introduced as some subspace correction algorithms in a reflexive Banach space and, under some assumptions, general convergence results (error estimations, included) are given. In the finite element spaces, we prove that these assumptions are satisfied and that the introduced algorithms are in fact one-, two-, multilevel or multigrid methods. The constants in the error estimations are explicitly written in functions of the overlapping and mesh parameters for the one- and two-level methods and in function of the number of levels for the multigrid methods.

In this paper, we denote by V a reflexive Banach space and $K \subset V$ is a non empty closed convex subset. Also, $F : K \to \mathbf{R}$ is a Gâteaux differentiable functional and we assume that there exist two real numbers $p, q > 1$ such that for any $M > 0$ there exist $\alpha_M,\ \beta_M > 0$ for which

$$\begin{aligned}&\alpha_M ||v-u||^p \leq < F'(v) - F'(u), v-u > \\ &\text{and } ||F'(v) - F'(u)||_{V'} \leq \beta_M ||v-u||^{q-1},\end{aligned}$$

for any $u, v \in K$, $||u||, ||v|| \leq M$. In view of these properties, we can prove that F is a convex functional and $1 < q \leq 2 \leq p$.

2 One- and two-level methods

In this section we introduce one- and two-level methods of multiplicative type, first as a general subspace correction algorithm. Details concerning the proof of its global convergence can be found in Badea [2003]. The one- and two-level methods are derived from this algorithm by the introduction of the finite element spaces and details are given in Badea [2006]. Similar results can be proved for the additive variant of the methods (see Badea [2008]).
We consider the variational inequality

$$u \in K : \ < F'(u), v-u > \geq 0, \text{ for any } v \in K, \tag{1}$$

and if K is not bounded, we suppose that F is coercive, i.e. $F(v) \to \infty$ as $||v|| \to \infty$. Then, problem (1) has an unique solution. Let $V_1, \cdots, V_m$ be some closed subspaces of V for which we make the following

Assumption 1 *There exists a constant $C_0 > 0$ such that for any $w, v \in K$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying*

$$w+\sum_{j=1}^{i-1} w_j + v_i \in K,\ v-w = \sum_{i=1}^{m} v_i,\ \sum_{i=1}^{m} ||v_i||^p \leq C_0^p \left(||v-w||^p + \sum_{i=1}^{m} ||w_i||^p \right).$$

For linear problems, the last condition has a more simple form and is named the stability condition of the space decomposition. To solve problem (1), we introduce the following subspace correction algorithm.

Algorithm 1 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we sequentially compute for $i = 1, \cdots, m$,*

$$w_i^{n+1} \in V_i,\ u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K : \langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle \geq 0,$$

for any $v_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$, and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

The following result proves the global convergence of this algorithm (see Theorem 2 in Badea [2003]).

Theorem 1. *On the above conditions on the spaces and the functional F, if Assumption 1 holds, then there exists an $M > 0$ such that $||u^n|| \leq M$, for any $n \geq 0$, and we have the following error estimations:*
(i) if $p = q = 2$ we have $||u^n - u||^2 \leq \frac{2}{\alpha_M}\left(\frac{\tilde{C}_1}{\tilde{C}_1+1}\right)^n \left[F(u^0) - F(u)\right]$.
(ii) if $p > q$ we have $||u - u^n||^p \leq \frac{p}{\alpha_M} \frac{F(u^0)-F(u)}{\left[1+n\tilde{C}_2(F(u^0)-F(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}$,

where

$\tilde{C}_1 = \beta_M(\frac{p}{\alpha_M})^{\frac{q}{p}} m^{2-\frac{q}{p}} \Big[(1+2C_0)\left(F(u^0) - F(u)\right)^{\frac{p-q}{p(p-1)}} + \left(\beta_M(\frac{p}{\alpha_M})^{\frac{q}{p}} m^{2-\frac{q}{p}}\right)^{\frac{1}{p-1}} C_0^{\frac{p}{p-1}}/\eta^{\frac{1}{p-1}}\Big]/(1-\eta)$ *and*
$\tilde{C}_2 = \frac{p-q}{(p-1)(F(u^0)-F(u))^{\frac{p-q}{q-1}} + (q-1)\hat{C}^{\frac{p-1}{q-1}}}$.
The value of η in the the expression of $\tilde{C}_1$ can be arbitrary in $(0,1)$, but we can also chose a $\eta_0 \in (0,1)$ such that $\tilde{C}_1(\eta_0) \leq \tilde{C}_1(\eta)$ for any $\eta \in (0,1)$.

One-level methods are obtained from Algorithm 1 by using the finite element spaces. To this end, we consider a simplicial regular mesh partition $\mathcal{T}_h$ of mesh size h over $\Omega \subset \mathbf{R}^d$. Also, let $\Omega = \cup_{i=1}^m \Omega_i$ be a domain decomposition of Ω, the overlapping parameter being δ, and we assume that $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$. In Ω, we use the linear finite element space V_h whose functions vanish on the boundary of Ω and, for each $i = 1, \ldots, m$, we consider the linear finite element space $V_h^i \subset V_h$ whose functions vanish outside Ω_i. Spaces V_h and V_h^i, $i = 1, \ldots, m$, are considered as subspaces of $W^{1,\sigma}$, $1 \leq \sigma \leq \infty$, and let $K_h \subset V_h$ be a convex set satisfying

Property 1. If $v, w \in K_h$, and if $\theta \in C^0(\bar{\Omega})$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \leq \theta \leq 1$, then $L_h(\theta v + (1-\theta)w) \in K_h$, where L_h is the P_1-Lagrangian interpolation.

We see that the convex sets of obstacle type satisfy this property, and we have (see Proposition 3.1 in Badea [2006] for the proof)

Proposition 1. *Assumption 1 holds for the linear finite element spaces, $V = V_h$ and $V_i = V_h^i$, $i = 1, \ldots, m$, and for any convex set $K = K_h \subset V_h$ having Property 1. The constant C_0 in Assumption 1 can be written as $C_0 = C(m+1)(1 + \frac{m-1}{\delta})$, where C is independent of the mesh parameter and the domain decomposition.*

In the case of the two-level methods, we consider two regular simplicial mesh partitions $\mathcal{T}_h$ and $\mathcal{T}_H$ on $\Omega \subset \mathbf{R}^d$, $\mathcal{T}_h$ being a refinement of $\mathcal{T}_H$. Besides the finite element spaces V_h, V_h^i, $i = 1, \ldots, m$ and the convex set K_h, defined for the one-level methods, we introduce the linear finite element space V_H^0 corresponding to the H-level, whose functions vanish on the boundary of Ω. The two-level method is obtained from the general subspace correction Algorithm 1 for $V = V_h$, $K = K_h$, and the subspaces $V_0 = V_H^0$, $V_1 = V_h^1$, $V_2 = V_h^2$, ..., $V_m = V_h^m$. Also, these spaces are considered as subspaces of $W^{1,\sigma}$, $1 \le \sigma \le \infty$, and we have the following (see Proposition 4.1 in Badea [2006] for the proof)

Proposition 2. *Assumption 1 is satisfied for the linear finite element spaces $V = V_h$ and $V_0 = V_H^0$, $V_i = V_h^i$, $i = 1, \ldots, m$, and any convex set $K = K_h$ having Property 1. The constant C_0 can be taken of the form $C_0 = Cm\left(1 + (m-1)\frac{H}{\delta}\right) C_{d,\sigma}(H,h)$, where C is independent of the mesh and domain decomposition parameters, and*

$$C_{d,\sigma}(H,h) = \begin{cases} 1 & \text{if } d = \sigma = 1 \text{ or } 1 \le d < \sigma \le \infty \\ \left(\ln \frac{H}{h} + 1\right)^{\frac{d-1}{d}} & \text{if } 1 < d = \sigma < \infty \\ \left(\frac{H}{h}\right)^{\frac{d-\sigma}{\sigma}} & \text{if } 1 \le \sigma < d < \infty. \end{cases}$$

Some numerical results have been given in Badea [2009] to compare the convergence of the one-level and two-level methods. They concern the two-obstacle problem of a nonlinear elastic membrane,

$$u \in [a,b] : \int_\Omega |\nabla u|^{\sigma-2} \nabla u \nabla (v-u) \ge 0, \text{ for any } v \in [a,b] \tag{2}$$

where $\Omega \subset \mathbf{R}^2$, $K = [a,b]$, $a \le b$, $a, b \in W_0^{1,\sigma}(\Omega)$, $1 < \sigma < \infty$. These numerical experiments have confirmed the previous theoretical results.

3 Multilevel and multigrid methods

Details concerning the results in this section can be found in Badea [2014] and Badea [2015]. As in the case of the one- and two-level methods, we consider problem (1). Let V_j, $j = 1, \ldots, J$, be closed subspaces of $V = V_J$ which will be associated with the level discretizations, and V_{ji}, $i = 1, \ldots, I_j$, be closed subspaces of V_j which will be associated with the domain decompositions on the levels. We consider $K \subset V$ a non empty closed convex subset and write $I = \max_{j=J,\ldots,1} I_j$.

To get sharper error estimations in the case of the multigrid method, we consider some constants $0 < \beta_{jk} \le 1$, $\beta_{jk} = \beta_{kj}$, j, $k = J, \ldots, 1$, for which $\langle F'(v + v_{ji}) - F'(v), v_{kl} \rangle \le \beta_M \beta_{jk} ||v_{ji}||^{q-1} ||v_{kl}||$, for any $v \in V$, $v_{ji} \in V_{ji}$, $v_{kl} \in V_{kl}$ with $||v||$, $||v+v_{ji}||$, $||v_{kl}|| \le M$, $i = 1, \ldots, I_j$ and $l = 1, \ldots, I_l$. Also, we fix a constant $\frac{p}{p-q+1} \leq \sigma \leq p$ and assume that there exists a constant C_1 such that $||\sum_{j=1}^{J}\sum_{i=1}^{I_j} w_{ji}|| \leq C_1(\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||w_{ji}||^{\sigma})^{\frac{1}{\sigma}}$, for any $w_{ji} \in V_{ji}$, $j = J, \dots, 1$, $i = 1, \dots, I_j$. Evidently, in general, we can take $\beta_{jk} = 1,\ j,\ k = J, \dots, 1$ and $C_1 = (IJ)^{\frac{\sigma-1}{\sigma}}$. In the multigrid methods, the convex sets where we look for the corrections are iteratively constructed from a level to another during the iterations in function of the current approximation. In this general background we make the following

Assumption 2 *For a given $w \in K$, we recursively introduce the level convex sets $\mathcal{K}_j$, $j = J,\ J-1,\ \dots, 1$, satisfying*

- at level J: we assume that $0 \in \mathcal{K}_J$, $\mathcal{K}_J \subset \{v_J \in V_J : w + v_J \in K\}$ and consider a $w_J \in \mathcal{K}_J$,

- at a level $J-1 \geq j \geq 1$: we assume that $0 \in \mathcal{K}_j$, $\mathcal{K}_j \subset \{v_j \in V_j : w + w_J + \ldots + w_{j+1} + v_j \in K\}$ and consider a $w_j \in \mathcal{K}_j$.

Also, we make a similar assumption with that in the case of the -one and two-level methods,

Assumption 3 *There exists two constants $C_2,\ C_3 > 0$ such that for any $w \in K$, $w_{ji} \in V_{ji}$, $w_{j1} + \ldots + w_{ji} \in \mathcal{K}_j$, $j = J, \dots, 1$, $i = 1, \dots, I_j$, and $u \in K$, there exist $u_{ji} \in V_{ji}$, $j = J, \dots, 1$, $i = 1, \dots, I_j$, which satisfy*

$$u_{j1} \in \mathcal{K}_j \text{ and } w_{j1} + \ldots + w_{ji-1} + u_{ji} \in \mathcal{K}_j,\ i = 2, \dots, I_j,\ j = J, \dots, 1,$$
$$u - w = \sum_{j=1}^{J}\sum_{i=1}^{I_j} u_{ji},\ \sum_{j=1}^{J}\sum_{i=1}^{I_j} ||u_{ji}||^{\sigma} \leq C_2^{\sigma}||u-w||^{\sigma} + C_3^{\sigma}\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||w_{ji}||^{\sigma}$$

The convex sets $\mathcal{K}_j$, $j = J, \dots, 1$, are constructed as in Assumption 2 with the above w and $w_j = \sum_{i=1}^{I_j} w_{ji}$, $j = J, \dots, 1$.

The general subspace correction algorithm corresponding to the multigrid method is written as (see Algorithm 2.2 in Badea [2014] or Algorithm 1.1 in Badea [2015]),

Algorithm 2 *We start with an arbitrary $u^0 \in K$. At iteration $n+1$ we have $u^n \in K$, $n \geq 0$, and successively perform:*

- at level J: as in Assumption 2, with $w = u^n$, we construct $\mathcal{K}_J$.
Then, we write $w_J^n = 0$, and, for $i = 1, \dots, I_J$, we successively calculate $w_{Ji}^{n+1} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1} \in \mathcal{K}_J$,

$$\langle F'(u^n + w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}), v_{Ji} - w_{Ji}^{n+1}\rangle \geq 0$$

for any $v_{Ji} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + v_{Ji} \in \mathcal{K}_J$, and write $w_J^{n+\frac{i}{I_J}} = w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}$.

- at a level $J-1 \geq j \geq 1$: as in Assumption 2, we construct $\mathcal{K}_j$ with $w = u^n$ and $w_J = w_J^{n+1}, \dots, w_{j+1} = w_{j+1}^{n+1}$.

Then, we write $w_j^n = 0$*, and for* $i = 1, \ldots, I_j$*, we successively calculate* $w_{ji}^{n+1} \in V_{ji}$*,* $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in \mathcal{K}_j$*,*

$$\langle F'(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1}\rangle \geq 0$$

for any $v_{ji} \in V_{ji}$*,* $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in \mathcal{K}_j$*, and write* $w_j^{n+\frac{i}{I_j}} = w_J^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$.

- *we write* $u^{n+1} = u^n + \sum_{j=1}^{J} w_j^{n+1}$.

Convergence of this algorithm is given by (see Theorem 1.1 in Badea [2015])

Theorem 2. *Under the above conditions on the spaces and the functional* F*, if Assumptions 2 and 3 hold, then there exists an* $M > 0$ *such that* $||u^n|| \leq M$*, for any* $n \geq 0$*, and we have the following error estimations:*

(i) if $p = q = 2$ *we have* $||u^n - u||^2 \leq \frac{2}{\alpha_M}(\frac{\tilde{C}_1}{\tilde{C}_1+1})^n[F(u^0) - F(u)]$,

(ii) if $p > q$ *we have* $||u - u^n||^p \leq \frac{p}{\alpha_M}\frac{F(u^0)-F(u)}{[1+n\tilde{C}_2(F(u^0)-F(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}}$,

where

$$\tilde{C}_1 = \frac{1}{C_2\varepsilon}\left[\frac{C_2}{\varepsilon} + 1 + C_1C_2 + C_3\right],$$

$$\tilde{C}_2 = \frac{p-q}{(p-1)(F(u^0)-F(u))^{\frac{p-q}{q-1}} + (q-1)\tilde{C}_3^{\frac{p-1}{q-1}}} \text{ with}$$

$$\tilde{C}_3 = \frac{\frac{\alpha_M}{p}}{C_2\varepsilon}\left[\frac{C_2}{\varepsilon^{\frac{1}{p-1}}(\frac{\alpha_M}{p})^{\frac{q-1}{p-1}}} + \frac{(1 + C_1C_2 + C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}}(F(u^0) - F(u))^{\frac{p-q}{p(p-1)}}\right]$$

$$\varepsilon = \frac{\alpha_M}{p}\frac{1}{2C_2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}} J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}}(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})}.$$

To get the multilevel method corresponding to Algorithm 2, we consider a family of regular meshes $\mathcal{T}_{h_j}$ of mesh sizes h_j, $j = 1, \ldots, J$, over the domain $\Omega \subset \mathbf{R}^d$ and assume that $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$. Let, at each level $j = 1, \ldots, J$, $\{\Omega_j^i\}_{1\leq i\leq I_j}$ be an overlapping decomposition of Ω, of overlapping size δ_j. We also assume that, for $1 \leq i \leq I_j$, the mesh partition $\mathcal{T}_{h_j}$ of Ω supplies a mesh partition for each Ω_j^i, $\text{diam}(\Omega_{j+1}^i) \leq Ch_j$ and $I_1 = 1$.

We introduce the linear finite element spaces, $V_{h_j} = \{v \in C(\bar{\Omega}_j) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_{h_j},\ v = 0 \text{ on } \partial\Omega_j\}$, $j = 1, \ldots, J$, corresponding to the level meshes, and $V_{h_j}^i = \{v \in V_{h_j} : v = 0 \text{ in } \Omega_j \backslash \Omega_j^i\}$, $i = 1, \ldots, I_j$, associated with the level decompositions. Spaces V_{h_j} $j = 1, \ldots, J-1$, will be considered as subspaces of $W^{1,\sigma}$, $1 \leq \sigma \leq \infty$.

The multilevel and multigrid methods will be obtained from Algorithm 2 for a two sided obstacle problem (1), i.e. the convex set is of the form $K = \{v \in V_{h_J} : \varphi \leq v \leq \psi\}$, with $\varphi,\ \psi \in V_{h_J}$, $\varphi \leq \psi$. Concerning the construction of the level convex sets, we have (Proposition 3.1 in Badea [2014])

Proposition 3. *Assumption 2 holds for the convex sets* $\mathcal{K}_j$*,* $j = J, \ldots, 1$*, defined as,*

- for $w \in K$*, at the level* J*, we take* $\varphi_J = \varphi - w$*,* $\psi_J = \psi - w$*,* $\mathcal{K}_J = [\varphi_J, \psi_J]$*, and consider an* $w_J \in \mathcal{K}_J$*,*

- at a level $j = J-1, \ldots, 1$*, we define* $\varphi_j = I_{h_j}(\varphi_{j+1} - w_{j+1})$*,* $\psi_j = I_{h_j}(\psi_{j+1} - w_{j+1})$*,* $\mathcal{K}_j = [\varphi_j, \psi_j]$*, and consider an* $w_j \in \mathcal{K}_j$*,* $I_{h_j} : V_{h_{j+1}} \to V_{h_j}$*,* $j = 1, \ldots, J-1$*, being some nonlinear interpolation operators between two consecutive levels.*

Also, our second assumption holds (see Proposition 2 in Badea [2015]),

Proposition 4. *Assumption 3 holds for the convex sets* $\mathcal{K}_j$*,* $j = J, \ldots, 1$*, defined in Proposition 3. The constants* C_2 *and* C_3 *are written as*

$$C_2 = CI^{\frac{\sigma+1}{\sigma}}(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\textstyle\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}}$$
$$C_3 = CI^{2}(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\textstyle\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}}$$

We proved that Assumptions 2 and 3 hold, and have explicitly written constants C_2 and C_3 in function of the mesh and overlapping parameters. We can then conclude from Theorem 2 that Algorithm 2 is globally convergent. Convergence rates given in Theorem 2 depend on the functional F, the maximum number of the subdomains on each level, I, and the number of levels J. Since the number of subdomains on levels can be associated with the number of colors needed to mark the subdomains such that the subdomains with the same color do not intersect with each other, we can conclude that the convergence rate essentially depends on the number of levels J.

In the general framework of multilevel methods we take $C_1 = CJ^{\frac{\sigma-1}{\sigma}}$ $\max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj} = J$ and, as functions depending only of J, we have

$$C_2 = C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J) \text{ and } C_3 = C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J) \text{ where}$$

$$S_{d,\sigma}(J) = \left[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}\right]^{\frac{1}{\sigma}} = \begin{cases} (J-1)^{\frac{1}{\sigma}} & \text{if } d = \sigma = 1 \\ & \text{or } 1 \le d < \sigma < \infty \\ CJ & \text{if } 1 < d = \sigma < \infty \\ C^{J} & \text{if } 1 \le \sigma < d < \infty. \end{cases}$$

In the above multilevel methods a mesh is the refinement of that one on the previous level, but the domain decompositions are almost independent from one level to another. We obtain similar multigrid methods by decomposing the domain by the supports of the nodal basis functions of each level. Consequently, the subspaces $V_{h_j}^{i}$, $i = 1, \ldots, I_j$, are one-dimensional spaces generated by the nodal basis functions associated with the nodes of $\mathcal{T}_{h_j}$, $j = J, \ldots, 1$. In the case of the multigrid methods, we can take $C_1 = C$ and $\max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj} = C$. Now we can write the convergence rate of the multigrid method corresponding to Algorithm 2 in function of the number of levels J for a given particular problem. In Badea [2014], the convergence rate of the multigrid method for the example in (2) has been written.

Remark 1. (see also Badea [2014])

1. The above results referred to problems in $W^{1,\sigma}$ with Dirichlet boundary conditions, but they also hold for Neumann or mixed boundary conditions.

2. Similar convergence results can be obtained for problems in $(W^{1,\sigma})^d$.

3. The analysis and the estimations of the global convergence rate which are given above refers to two sided obstacle problems which arise from the minimization of functionals defined on $W^{1,\sigma}$, $1 < \sigma < \infty$.

4. We can compare the convergence rates we have obtained with similar ones in the literature in the case of H^1 ($p = q = 2$) and $d = 2$. In this case, we get that the global convergence rate of Algorithm 2 is $1 - \frac{1}{1+CJ^3}$. The same estimate, of $1 - \frac{1}{1+CJ^3}$, is obtained by R. Kornhuber for the asymptotic convergence rate of the standard monotone multigrid methods for the complementarity problems.

Algorithm 2 is of multiplicative type over the levels as well as on each level, i.e. the current correction is found in function of all corrections on both the previous levels and the current level. We can also imagine hybrid algorithms where the type of the iteration over the levels is different from the type of the iteration on the levels. This idea can be also found in Smith et al. [1996]. In Badea [2015], such hybrid algorithms (multiplicative over the levels - additive on levels, additive over the levels - multiplicative on levels and additive over the levels as well as on levels) have been introduced and analyzed in a similar manner with that of Algorithm 2. The following remark contains some conclusions withdrawn in Badea [2015] concerning the convergence rate (expressed only in function of J) of these hybrid algorithms for problem (2).

Remark 2. 1. Regardless of the iteration type on levels, algorithms having the same type of iterations over the levels have the same convergence rate, provided that additive iterations on levels are parallelized.

2. The algorithms which are of multiplicative type over the levels converge better, by a factor of between $1/J$ and 1 (depending on σ), than their additive similar variants.

4 One- and two-level methods for variational inequalities of the second kind and quasi-variational inequalities

The results in this section are detailed in Badea and Krause [2012] where one- and two-level methods have been introduced and analyzed for the second kind and quasi-variational inequalities. In the case of the variational inequalities of the second kind, let $\varphi : K \to \mathbf{R}$ be a convex, lower semicontinuous, not differentiable functional and, if K is not bounded, we assume that $F + \varphi$ is coercive, i.e. $F(v) + \varphi(v) \to \infty$, as $\|v\| \to \infty$, $v \in K$. We consider the variational of the second kind

$$u \in K : \langle F'(u), v-u\rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for any } v \in K \tag{3}$$

which, in view of the properties of F and φ, has a unique solution. An example of such a problem is given by the contact problems with Tresca friction. To solve problem (3), we introduce

Algorithm 3 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, the local corrections $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$ as the solution of the variational inequality*

$$\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}} + v_i) - \varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0,$$

for any $v_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$, and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

To prove the convergence of the algorithm, we introduce a technical assumption,

$$\sum_{i=1}^{m}[\varphi(w + \sum_{j=1}^{i-1} w_j + v_i) - \varphi(w + \sum_{j=1}^{i-1} w_j + w_i)] \leq \varphi(v) - \varphi(w + \sum_{i=1}^{m} w_i)$$

for $v, w \in K$, and $v_i, w_i \in V_i$, $i = 1, \ldots, m$, in Assumption 1. In general, φ has not such a property and to show that this assumption holds when the finite element spaces are used, we have to take a numerical approximation of φ. The convergence of Algorithm 3 is proved by the following

Theorem 3. *Under the above assumptions on V, F and φ, let u be the solution of the problem and u^n, $n \geq 0$, be its approximations obtained from Algorithm 3. If Assumption 1 holds, then there exists $M > 0$ such that such that $\|u^{n+\frac{i}{m}}\| \leq M$, $n \geq 0, 1 \leq i \leq m$, and we have the following error estimations:*

(i) $\|u^n - u\|^2 \leq \frac{p}{\alpha_M}\left(\frac{\tilde{C}_1}{\tilde{C}_1+1}\right)^n \left[F(u^0) + \varphi(u^0) - F(u) - \varphi(u)\right]$ *if* $p = q = 2$,

(ii) $\|u - u^n\|^p \leq \frac{p}{\alpha_M} \frac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{\left[1+n\tilde{C}_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}$ *if* $p > q$,

where

$$\tilde{C}_1 = \beta_M(1+2C_0)m^{2-\frac{q}{p}}(\frac{p}{\alpha_M})^{\frac{q}{p}}\left(F(u^0) - F(u) + \varphi(u^0) - \varphi(u)\right)^{\frac{p-q}{p(p-1)}} + \beta_M C_0 m^{\frac{p-q+1}{p}} \frac{1}{\varepsilon^{\frac{1}{p-1}}}(\frac{p}{\alpha_M})^{\frac{q-1}{p-1}} \text{ with } \varepsilon = \alpha_M / \left(p\beta_M C_0 m^{\frac{p-q+1}{p}}\right),$$

$$\tilde{C}_2 = \frac{p-q}{(p-1)(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}} + (q-1)C_1^{\frac{p-1}{q-1}}}$$

In the case of the quasivariational inequalities, we consider only the case of $p = q = 2$ and let $\varphi : K \times K \to \mathbf{R}$ be a functional such that, for any $u \in K$, $\varphi(u, \cdot) : K \to \mathbf{R}$ is convex, lower semicontinuous and, if K is not bounded, $F(\cdot) + \varphi(u, \cdot)$ is coercive, i.e. $F(v) + \varphi(u, v) \to \infty$ as $\|v\| \to \infty, v \in K$. We assume that for any $M > 0$ there exists a constant $c_M > 0$ such that

$$|\varphi(v_1, w_2) + \varphi(v_2, w_1) - \varphi(v_1, w_1) - \varphi(v_2, w_2)| \leq c_M ||v_1 - v_2|| ||w_1 - w_2||$$

for any $v_1,\ v_2,\ w_1\ w_2 \in K$, $||v_1||,\ ||v_2||,\ ||w_1||\ ||w_2|| \le M$. If φ has the above property, the quasi-variational inequality

$$u \in K : \langle F'(u), v-u\rangle + \varphi(u,v) - \varphi(u,u) \ge 0, \text{ for any } v \in K$$

has a unique solution. An example of such a problem is given by the contact problems with non-local Coulomb friction. We can write three algorithms depending on the first argument of φ.

Algorithm 4 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \ge 0$, we compute sequentially for $i = 1, \cdots, m$, the local corrections $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, satisfying*

$$\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(v_i^{n+1}, u^{n+\frac{i-1}{m}} + v_i)$$
$$-\varphi(v_i^{n+1}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \ge 0,$$

for any $v_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$, and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

Above, the first argument v_i^{n+1} of φ can be taken either $u^{n+\frac{i-1}{m}} + w_i^{n+1}$ or $u^{n+\frac{i-1}{m}}$ or even u^n. As we shall see in the next convergence theorem, the three variants of the algorithm are convergent. Similarly with the case of the inequalities of the second kind, we introduce the technical assumption

$$\sum_{i=1}^{m}[\varphi(u, w + \sum_{j=1}^{i-1} w_j + v_i) - \varphi(u, w + \sum_{j=1}^{i} w_j)] \le \varphi(u, v) - \varphi(u, w + \sum_{i=1}^{m} w_i)$$

for any $u \in K$ and for $v, w \in K$ and $v_i, w_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$, $i = 1, \ldots, m$, in Assumption 1. Also, in the finite element spaces, φ of the continuous problem is numerically approximated in order to get the above assumption satisfied. Convergence of the three algorithms is proved by

Theorem 4. *Under the above assumptions on V, F and φ, let u be the solution of the problem and u^n, $n \ge 0$, be its approximations obtained from one of the variants of Algorithm 4. If Assumption 1 holds, and if $\frac{\alpha_M}{2} \ge mc_M + \sqrt{2m(25C_0 + 8)\beta_M c_M}$, for any $M > 0$, then there exists an $M > 0$ such that $\|u^{n+\frac{i}{m}}\| \le M$, $n \ge 0, 1 \le i \le m$, and we have the following error estimation*

$$\|u^n - u\|^2 \le \frac{2}{\alpha_M}\left(\frac{\tilde{C}_1}{\tilde{C}_1+1}\right)^n \left[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u, u)\right].$$

where

$$\tilde{C}_1 = \tilde{C}_2/\tilde{C}_3 \text{ with } \tilde{C}_2 = \beta_M m(1 + 2C_0 + \tfrac{C_0}{\varepsilon_1}) + c_M m(1 + 2C_0 + \tfrac{1+3C_0}{\varepsilon_2}),$$
$$\tilde{C}_3 = \tfrac{\alpha_M}{2} - c_M(1+\varepsilon_3)m \text{ and } \varepsilon_1 = \varepsilon_2 = \frac{2c_M m}{\frac{\alpha_M}{2} - c_M m}, \quad \varepsilon_3 = \frac{\frac{\alpha_M}{2} - c_M m}{2c_M m}.$$

Remark 3. 1. Extension of the previous methods (given for variational inequalities of the second kind and quasi-variational inequalities) to methods with more than two levels, having an optimal rate of convergence, is not very evident because of the technical conditions we have introduced, which are not satisfied when the domain decompositions on the coarse levels are considered.

2. By using Newton linearizations of φ, R. Kornhuber introduced multigrid methods for complementarity problems and estimated the asymptotic convergence rates.

5 Multigrid methods for inequalities with a term given by a Lipschitz operator

In this section, we estimate the global convergence rate of a multigrid method for the particular case of quasi-variational inequalities when the inequality contains a term given by a Lipschitz operator. Details concerning the results of this section can be found in Badea [2016]. As in the previous section, we consider the case when $p = q = 2$ and $\alpha_M = \alpha$, $\beta_M = \beta$, i.e. they not depend on M. Let $T : V \to V'$ be a Lipschitz continuous operator $||T(v) - T(u)||_{V'} \leq \gamma||v - u||$ for any $v, u \in V$, and we consider the problem

$$u \in K : \langle F'(u), v - u\rangle + \langle T(u), v - u\rangle \geq 0 \text{ for any } v \in K.$$

In the following algorithm, each iteration contains κ intermediate iterations in which the argument of T is kept unchanged.

Algorithm 5 *We start the algorithm with an arbitrary $u^0 \in K$. Assuming that at iteration $n+1$ we have $u^n \in K$, $n \geq 0$, we write $\tilde{u}^n = u^n$ and carry out the following two steps:*
1. We perform $\kappa \geq 1$ iterations of Algorithm 2 starting with $\tilde{u}^n$ and keeping the argument of T equal with u^n, i.e. we apply Algorithm 2 to the inequality

$$\tilde{u} \in K : \langle F'(\tilde{u}), v - \tilde{u}\rangle + \langle T(u^n), v - \tilde{u}\rangle \geq 0 \textit{ for any } v \in K$$

After the κ iterations we get the approximation $\tilde{u}^{n+\kappa}$ of $\tilde{u}$.
2. We write $u^{n+1} = \tilde{u}^{n+\kappa}$.

Convergence condition of Theorem 4 depends on the number m of the subspaces in the one- or two-level methods. We will see in the next theorem that if the Lipschitz constant of the operator T is small enough, the convergence condition of the above algorithm is independent of the number of levels and the number of subdomains on the levels.

Theorem 5. *We assume that V, F and T satisfy the above conditions and that Assumptions 2-3 hold. Then, if $\gamma/\alpha < 1/2$ and κ satisfies $(\frac{\tilde{C}}{\tilde{C}+1})^\kappa < \frac{1-2\frac{\gamma}{\alpha}}{1+3\frac{\gamma}{\alpha}+4\frac{\gamma^2}{\alpha^2}+\frac{\gamma^3}{\alpha^3}}$, Algorithm 5 is convergent and we have the following error estimation*

$$\|u^n - u\|^2 \leq \frac{2}{\alpha}[2\frac{\gamma}{\alpha} + (\frac{\tilde{C}}{\tilde{C}+1})^\kappa(1 + 3\frac{\gamma}{\alpha} + 4\frac{\gamma^2}{\alpha^2} + \frac{\gamma^3}{\alpha^3})]^n$$
$$\cdot[F(u^0) + \langle T(u), u^0\rangle - F(u) - \langle T(u), u\rangle],$$

where $\tilde{C} = \dfrac{1}{C_2\varepsilon}\left[1 + C_2 + C_1C_2 + \dfrac{C_2}{\varepsilon}\right]$, $\varepsilon = \dfrac{\alpha}{2\beta I(\max_{k=1,\cdots,J}\sum_{j=1}^J \beta_{kj})C_2}$.

References

L. Badea. Convergence rate of a multiplicative Schwarz method for strongly nonlinear variational inequalities. In *V. Barbu et al. (eds.), Analysis and Optimization of Differential Systems*, pages 31–42. Kluwer Academic Publishers, 2003.

L. Badea. Convergence rate of a Schwarz multilevel method for the constrained minimization of non-quadratic functionals. *SIAM J. Numer. Anal.*, 44(2):449–477, 2006.

L. Badea. Additive Schwarz method for the constrained minimization of functionals in reflexive banach spaces. In *U. Langer et al. (eds.), Domain decomposition methods in science and engineering XVII, LNSE 60*, pages 427–434. Springer, 2008.

L. Badea. One- and two-level domain decomposition methods for nonlinear problems. In *B.H.V. Topping, P. Iványi (eds.), Proceedings of the First International Conference on Parallel, Distributed and Grid Computing for Engineering*, page Paper 6. Civil-Comp Press, Stirlingshire, UK, 2009.

L. Badea. Global convergence rate of a standard multigrid method for variational inequalities. *IMA J. Numer. Anal.*, 34(1):197–216, 2014.

L. Badea. Convergence rate of some hybrid multigrid methods for variational inequalities. *Journal of Numerical Mathematics*, 23(3):195–210, 2015.

L. Badea. Globally convergent multigrid method for variational inequalities with a nonlinear term. In *T. Dickopf et al. (eds.) Domain Decomposition Methods in Science and Engineering XXII, LNCSE 104*, pages 427–435. Springer, 2016.

L. Badea and R. Krause. One- and two-level Schwarz methods for inequalities of the second kind and their application to frictional contact. *Numer. Math.*, 120(4):573–599, 2012.

C. Gräser and R. Kornhuber. Multigrid methods for obstacle problems. *J. Comput. Math.*, 27(1):1–44, 2009.

R. Kornhuber. Monotone multigrid methods for elliptic variational inequalities I. *Numer. Math.*, 69:167–184, 1994.

R. Kornhuber. Monotone multigrid methods for elliptic variational inequalities II. *Numer. Math.*, 72:481–499, 1996.

R. Kornhuber. On constrained Newton linearization and multigrid for variational inequalities. *Numer. Math.*, 91:699–721, 2002.

R. Kornhuber and R. Krause. Adaptive multigrid methods for signorini's problem in linear elasticity. *Comp. Visual. Sci.*, 4:9–20, 2001.

J. Mandel. A multilevel iterative method for symmetric, positive definite linear complementarity problems. *Appl. Math. Opt.*, 11:77–95, 1984a.

J. Mandel. Etude algébrique d'une méthode multigrille pour quelques problèmes de frontière libre. *C. R. Acad. Sci. Ser. I*, 298:469–472, 1984b.

B. F. Smith, P. E. Bjørstad, and W. Gropp. *Domain Decomposition. Parallel multilevel methods for elliptic partial differential equations.* Cambridge University Press, 1996.