Development of Nonlinear Structural Analysis using Co-rotational Finite Elements with Improved Domain Decomposition Method

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1 Introduction

Recent advances in computational science and technologies induce increasing size of the engineering problems, and impact the fields of computational fluids and structural dynamics as well as multi-physics problems, such as fluid-structure interactions. At the same time, structural components used in many engineering applications show geometrically nonlinear characteristics. Therefore, development of effective solution methodologies for large-size nonlinear structural problems is required seriously in the fields of the mechanical and aerospace engineering. Especially, general finite element methods require a large number of elements in order to predict precise stress or deformation, resulting in increased computational costs due to enlarged computational time and memory requirement. Therefore, careful selection of grid size and solution methodology becomes important.

One of the most successful approaches for large-size finite element analysis is the finite element tearing and interconnecting (FETI) method proposed by Farhat and Roux [1]. The basic idea of FETI is to decompose the computational domain into non-overlapping sub-domains. Lagrange multipliers are used to enforce compatibility of the degrees of freedom along the interfaces between the sub-domains. The manner of handling such interfaces can distinguish the interface problem. Recently, the dual-primal FETI (FETI-DP) method [2] was proposed; it is a dual sub-structuring method, which introduces Lagrange multipliers and a small number of coarse mesh nodes to enforce the continuity at sub-domain interfaces. The resulting dual problem is then solved by seeking a saddle-point of the relevant Lagrangian functional.

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The FETI-DP method is a standard preconditioned conjugate algorithm, which may use an arbitrary initial guess. Thus, the solution of the interface problem is obtained using an iterative process, which requires an adequate pre-conditioner. Therefore, to improve solution convergence, iterative solvers rely on various types of preconditioning techniques. By observing such limitation, the combination of domain decomposition methods with the direct solvers was significantly investigated, an approach that seems to have received little attention thus far [3]. Bauchau [4] suggested the use of an augmented Lagrangian formulation (ALF) in conjunction with both global and local Lagrange multipliers. The use of augmented Lagrangian terms was considered to improve the conditioning of the flexibility matrix, thereby increasing the convergence performance of the iterative procedure used to solve the interface problem. As a preliminary step to the present effort, the authors proposed an improved domain decomposition approach, the FETI-Local, and the FETI algorithm was developed for multibody type structures [5]. Moreover, in order to improve the computational efficiency, a parallel version of the column solver was employed to deal with the interface problem [6].

On the other hand, a co-rotational (CR) formulation has been developed and improved in accordance with an increased amount of interest during the last few decades to analyze the geometrical nonlinearity of structures [7]. The main advantage of the CR framework is that it leads to an artificial separation between the material and any geometrical nonlinearity. This concept was originally developed by Rankin et al. during the formulating procedure of what is known as the element-independent co-rotational (EICR) description [8]. In addition, Felippa et al. concluded that the CR formulation would be extremely useful for elements of a simple geometry; they were able to provide a reasonable solution to the localized failure problem as well [7]. However, such nonlinear structural analysis would be confronted with the significant computational problem with increasing computational costs due to enlarged computational time and memory requirement, followed by prediction of precise stress and large deformation. Thus, an effective solution methodology for large-size nonlinear structural problem would be suggested through an extension of the CR framework into the FETI-Local method.

This manu script is organi zedas follo ws. Formulation pro cedure of the FE TI-Local method will be d escrib ed. Aft erthat, d erivation of the CR fr ame-work will be introdu ced. Then, unifi ed computational algorithm of the FE TI-Lo caland the CRfr amework will be d escrib ed. Finally, computational cost and scalabil ity r esults obtain edby the prop osedapproach will be pr esent ed.

2 Domain decomposition method: FETI-Local

Consider a planar solid depicted in Fig. 1. To develop a parallel solution algorithm for this problem, the solid is partitioned into N_s non-overlapping

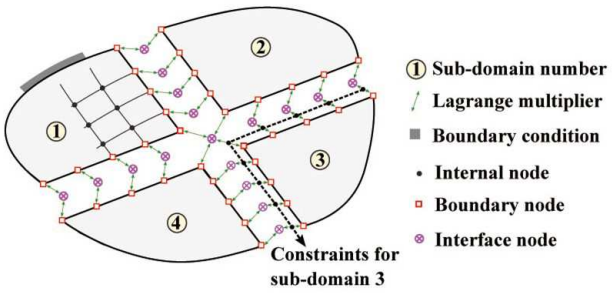


Fig. 1: Planar solid separated into four non-overlapping sub-domains by following the FETI-Local.

sub-domains. Each of these sub-domains could themselves be multibody systems comprising both elastic elements and nonlinear kinematic constraints. The FETI-Local uses local Lagrange multipliers to impose continuity of displacements at the nodes corresponding to adjacent sub-domains with those corresponding to the coarse mesh nodes. At corner nodes, i.e., at sub-domain cross-points, a single interface node is defined, and Lagrange multipliers are used to enforce equality of the displacements at the coarse mesh with those corresponding to all the adjacent nodes. Because four sub-domains are associated at this node, four boundary nodes would be created, one for each sub-domain. Note that for multiple connections, constraints and Lagrange multipliers remain localized, *i.e.*, each associated with a single sub-domain. In finite element formulations, this approach has been used to enforce the continuity of displacement fields between adjacent incompatible elements [9]. The same approach, called "localized version of the method of Lagrange multipliers," has been advocated by Park *et al.* [10].

In the FETI-Local method, the kinematic continuity conditions between sub-domain interfaces is enforced via the localized Lagrange multiplier technique. Let $\underline{u}_b^{[j]}$ and $\underline{c}^{[j]}$ denote the arrays of dofs at a boundary node and at an interface node, respectively. Kinematic constraint j is written as $\underline{\mathcal{C}}^{[j]} = \underline{u}_b^{[j]} - \underline{c}^{[j]} = \underline{0}$ and the associated potential is

$$V_c^{[j]} = s\underline{\lambda}^{[j]T}\underline{\mathcal{C}}^{[j]} + \frac{p}{2}\underline{\mathcal{C}}^{[j]T}\underline{\mathcal{C}}^{[j]}, \tag{1}$$

where $\underline{\lambda}^{[j]}$ is the array of Lagrange multipliers used to enforce the constraint, and s the scaling factor for those multipliers. The second term of the potential is a penalty term and p is the penalty coefficient. The potential defined by eq. (1) combines the localized Lagrange multiplier technique with the penalty method. This combination is known as the augmented Lagrangian formulation and has been examined extensively [11]. It is an effective approach for

the enforcement of kinematic constraints in multibody dynamics, as proposed by Bayo *et al.* [12].

A variation of the potential defined by eq. (1) is obtained easily.

$$\begin{aligned}\delta V_c^{[j]} =& \delta \underline{u}_b^{[j]T} \left[s\underline{\lambda}^{[j]} + p\underline{\mathcal{C}}^{[j]} \right] + \delta \underline{\lambda}^{[j]T} \left[s\underline{\mathcal{C}}^{[j]} \right] \\ &+ \delta \underline{c}^{[j]T} \left[-s\underline{\lambda}^{[j]} - p\underline{\mathcal{C}}^{[j]} \right],\end{aligned} \tag{2}$$

The Lagrange multipliers become localized in the formulation, *i.e.*, Lagrange multipliers are associated with one sub-domain unequivocally. The potential of kinematic constraint involves two types of dofs, the sub-domain dofs, $\underline{u}_b^{[j]}$ and $\underline{\lambda}^{[j]}$, and the interface dofs, $\underline{c}^{[j]}$. The constraint forces and stiffness matrix are partitioned to reflect this fact

$$\underline{f}^{[j]} = \begin{Bmatrix} \underline{f}_b^{[j]} \\ \underline{f}_c^{[j]} \end{Bmatrix}, \quad \underline{\underline{k}}^{[j]} = \begin{bmatrix} \underline{\underline{k}}_{bb}^{[j]} & \underline{\underline{k}}_{bc}^{[j]} \\ \underline{\underline{k}}_{bc}^{[j]T} & \underline{\underline{k}}_{cc}^{[j]} \end{bmatrix}. \tag{3}$$

Subscripts $(\cdot)_b$ and $(\cdot)_c$ denote dofs associated with boundary and interface nodes, respectively. Partitioning the constraint forces can be defined as follows.

$$\underline{f}_b^{[j]} = \begin{Bmatrix} s\underline{\lambda}^{[j]} + p\underline{\mathcal{C}}^{[j]} \\ s\underline{\mathcal{C}}^{[j]} \end{Bmatrix}, \quad \underline{f}_c^{[j]} = -\left\{ s\underline{\lambda}^{[j]} + p\underline{\mathcal{C}}^{[j]} \right\}. \tag{4}$$

A similar operation for the constraint stiffness matrix leads to

$$\underline{\underline{k}}_{bb}^{[j]} = \begin{bmatrix} p\underline{\underline{I}} & s\underline{\underline{I}} \\ s\underline{\underline{I}} & \underline{\underline{0}} \end{bmatrix}, \quad \underline{\underline{k}}_{cc}^{[j]} = \left[p\underline{\underline{I}} \right], \quad \underline{\underline{k}}_{bc}^{[j]} = \begin{bmatrix} -p\underline{\underline{I}} \\ -s\underline{\underline{I}} \end{bmatrix}. \tag{5}$$

Each constraint element contributes constraint forces and stiffness matrices defined by eqs. (4) and (5), respectively. Using the standard assembly procedure used in the finite element method, the force arrays and stiffness matrices generated by all the constraint elements associated with sub-domain i are assembled into the following sub-domain arrays and matrices

$$\check{\underline{F}}_b^{(i)} = \sum_{j=1}^{N_b^{(i)}} \underline{\underline{B}}_b^{[j]T} \underline{f}_b^{[j]}, \quad \check{\underline{\underline{K}}}_{bb}^{(i)} = \sum_{j=1}^{N_b^{(i)}} \underline{\underline{B}}_b^{[j]T} \underline{\underline{k}}_{bb}^{[j]} \underline{\underline{B}}_b^{[j]}, \tag{6}$$

where $\underline{\underline{B}}_b^{[j]}$ is the Boolean matrices used for the assembly process, *i.e.*, $\underline{u}_b^{[j]} = \underline{\underline{B}}_b^{[j]} \check{\underline{u}}^{(i)}$. Of course, the assembly procedure can be performed in parallel for all sub-domains. Similarly, the constraint elements contribute force arrays and stiffness matrices to the interface problem,

$$\underline{F}_c^{(i)} = \sum_{j=1}^{N_b^{(i)}} \underline{\underline{B}}_c^{[j]T} \underline{f}_c^{[j]}, \quad \underline{\underline{K}}_{cc}^{(i)} = \sum_{j=1}^{N_b^{(i)}} \underline{\underline{B}}_c^{[j]T} \underline{\underline{k}}_{cc}^{[j]} \underline{\underline{B}}_c^{[j]}, \tag{7}$$

where $\underline{\underline{B}}_c^{[j]}$ is the Boolean matrices used for the assembly process, *i.e.*, $\underline{c}^{[j]} = \underline{\underline{B}}_c^{[j]}\underline{c}$. Finally, the constraint coupling stiffness is assembled to find

$$\underline{\underline{K}}_{bc}^{(i)} = \sum_{j=1}^{N_b^{(i)}} \underline{\underline{B}}_b^{[j]T}\underline{\underline{k}}_{bc}^{[j]}\underline{\underline{B}}_c^{[j]}. \tag{8}$$

By considering the potential energy of the system composed of the strain energy (A)/the work done by external force(Φ)/additional energy induced by Lagrange multipliers(V_c), $\Pi = A + \Phi + V_c$, and the principle of minimum total potential energy, the governing equations can be expressed as

$$\begin{bmatrix} \mathrm{diag}(\check{\underline{\underline{K}}}^{(\alpha)} + \check{\underline{\underline{K}}}_{bb}^{(\alpha)}) & \underline{\underline{K}}_{bc} \\ \underline{\underline{K}}_{bc}^T & \underline{\underline{K}}_{cc} \end{bmatrix} \begin{Bmatrix} \check{\underline{u}} \\ \underline{c} \end{Bmatrix} = \begin{Bmatrix} \check{\underline{Q}} - \check{\underline{F}}_b \\ -\underline{F}_c \end{Bmatrix}, \tag{9}$$

where $\check{\underline{Q}}^T = [\underline{Q}^T, 0]$ and $\check{\underline{u}}$ is the displacement of the sub-domain. The sub-domain stiffness matrix $\check{\underline{\underline{K}}}^{(\alpha)}$ is now

$$\check{\underline{\underline{K}}}^{(\alpha)} = \begin{bmatrix} \underline{\underline{K}}^{(\alpha)} & \underline{\underline{0}} \\ \underline{0} & \underline{\underline{0}} \end{bmatrix}. \tag{10}$$

Arrays $\check{\underline{F}}_b$ and $\underline{F}_c$ are the assembly of their sub-domain counterparts, $\check{\underline{F}}_b^{(i)}$ and $\underline{F}_c^{(i)}$, respectively, $\underline{\underline{K}}_{cc} = \sum_{i=1}^{N_s} \underline{\underline{K}}_{cc}^{(i)}$ and

$$\underline{\underline{K}}_{bc}^T = \left[\underline{\underline{K}}_{bc}^{(1)T}, \underline{\underline{K}}_{bc}^{(2)T}, \ldots, \underline{\underline{K}}_{bc}^{(N_s)T}\right]. \tag{11}$$

The block-diagonal nature of the leading entry of the system matrix makes this approach amenable to parallel solution algorithms.

3 Co-rotational (CR) Finite Elements

Figure 2 shows the coordinates defined in the present CR framework and rotational transformations when obeying the elemental kinematics. Beginning with the fixed frame, a rotational operator, $\underline{\underline{R}}_o$, can be defined by tracking the elemental initial state. The rotational operator, $\underline{\underline{R}}_G$, can be defined by elemental rotational displacement referring to an undeformed configuration. The complete behavior included in this case can be decomposed into rigid body rotation and elastic deformational rotation. According to such kinematics, the origin of each coordinate is taken at the centroid of the triangle.

In the CR formulation, the existing linearized formulation is selected for the local system matrices, i.e., the stiffness matrix and the internal load

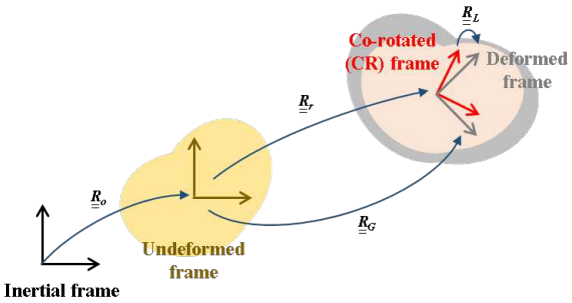


Fig. 2: Coordinate in the CR framework.

vector. These physical variables is re-expressed between the local and global quantities by the introduction of a transformation matrix. The virtual work with respect to the local and global systems can be obtained in terms of the local and global internal load vectors and displacements.

$$V = \delta \underline{q}_G^T \underline{f}_G = \delta \underline{q}_L^T \underline{f}_L = \delta \underline{q}_G^T \underline{\underline{B}}^T \underline{f}_L \tag{12}$$

Hence the global internal load vector is obtained with Eq. (12) by taking the transformation matrix, $\underline{\underline{B}}$, into account.

$$\underline{f}_G = \underline{\underline{B}}^T \underline{f}_L, \quad \underline{f}_L = \left\{ \underline{f}_L^i \right\}^T \quad i = 1, 2, \ldots, N_e, \tag{13a}$$

$$\underline{f}_L^i = \left\{ n_1^i, n_2^i, m^i \right\}^T \quad i = 1, 2, \ldots, N_e. \tag{13b}$$

By the differentiation of Eq. (12) with respect to the displacements, the internal load vector can then be

$$\delta \underline{f}_G = \underline{\underline{K}}_G \delta \underline{q}_G \tag{14}$$

In addition, by Eqs. (12) and (14) the global stiffness matrix $\underline{\underline{K}}_G$ can be derived as shown below.

$$\underline{\underline{K}}_G = \underline{\underline{B}}^T \underline{\underline{K}}_L \underline{\underline{B}} + \underline{\underline{K}}_T, \quad \underline{\underline{K}}_T = \frac{\delta \underline{f}_G}{\delta \underline{q}_G} = \frac{\delta (\underline{\underline{B}}^T \underline{f}_L)}{\delta \underline{q}_G} \tag{15}$$

In the present transformation procedure regarding the load vector and stiffness matrix, the computed local elemental loads can naturally be related to the CR frame rather than to the final deformed frame. Thus, the local internal load can not be a self-equilibrating set of loads under the deformed frame. Introducing the projector matrix $\underline{\underline{P}}$, resolves this problem [8]. The projector matrix $\underline{\underline{P}}$ can be considered as a type of 3×3 block matrix related to the elemental nodes, $\underline{\underline{P}}^{ij}$. The derivative form of $\underline{\underline{P}}$ is obtained as follows.

$$\underline{\underline{P}}_{ij} = \begin{bmatrix} \frac{\partial \underline{u}_L^i}{\partial \underline{u}_G^j} & \frac{\partial \underline{u}_L^i}{\partial \underline{\theta}_G^j} \\ \frac{\partial \underline{\theta}_L^i}{\partial \underline{u}_G^j} & \frac{\partial \underline{\theta}_L^i}{\partial \underline{\theta}_G^j} \end{bmatrix} \tag{16}$$

Using the differentiation of the local translational and rotational components, it can be

$$\underline{\underline{P}}_{ij} = \underline{\underline{I}}_3 \delta_{ij} - \underline{\Xi}^i \underline{\Gamma}^{jT} \tag{17}$$

where δ_{ij} is Kronecker's delta. Let $\underline{r}_o^i = \underline{r}_G^i + \underline{u}_L^i$ and then $\underline{\Xi}^i$, $\underline{\Gamma}^j$ can be

$$\underline{\Xi}^i = \left\{ -r_{o,2}^i, r_{o,1}^i, 1 \right\}^T \tag{18a}$$

$$\underline{\Gamma}^j = s_r^{-1} \left\{ -r_{G,2}^j, r_{G,1}^j, 0 \right\}^T \tag{18b}$$

After the projector matrix for the element is constructed, the transformation matrix between the local and global internal load vectors can be expressed in terms of the projector matrix.

$$\underline{f}_G = \underline{\underline{B}}^T \underline{f}_L = \underline{\underline{E}}\,\underline{\underline{P}}^T \underline{f}_L \tag{19}$$

Here, the matrix $\underline{\underline{E}} = \mathrm{diag}(\underline{\underline{R}}_r, \underline{\underline{R}}_r, \underline{\underline{R}}_r)$. Taking the variation of $\underline{f}_G$, the resulting global stiffness matrix $\underline{\underline{K}}_G$ can be

$$\underline{\underline{K}}_G = \underline{\underline{E}}\,\underline{\underline{P}}^T \underline{\underline{K}}_L \underline{\underline{P}}\,\underline{\underline{E}}^T + \underline{\underline{E}} \left[-\underline{\Gamma}\,\underline{F}_1^T \underline{\underline{P}} - \underline{F}_2 \underline{\Gamma}^T \right] \underline{\underline{E}}^T \tag{20}$$

where the vectors $\underline{F}_1$ and $\underline{F}_2$ are expressed in terms of $\underline{F}_t = \underline{\underline{P}}^T \underline{f}_L$.

4 Unified Computational Algorithm

The FETI-Local proceeds in the three computational steps as follows. Step I sets up the structural interface problem, Step II evaluates the solution of the structural interface problem, and Step III recovers the solution in each sub-domain. In order to involve nonlinear structural analysis, iterative computational algorithm is developed. A load incremental Newton-Rhapson iterative scheme is employed. The unified computational algorithm is depicted in Fig. 3. The purpose of Step I is to set up the interface problem. For each sub-domain, this involves the evaluation and assembly of the stiffness matrix, the factorization of the stiffness matrix, and the assembly of the interface stiffness matrix. In Step II, the solution of the interface problem is computed first. In this step, the stiffness matrix corresponding to the interface nodes existing in the individual sub-domains needs to be distributed to each processor. Using the MPI_REDUCE routine, the matrix data are collected to a root process. In Step III, the final solution for each sub-domain is obtained by the linear solver. From Step II, array $\underline{c}$, degrees of freedom at the interface nodes, is

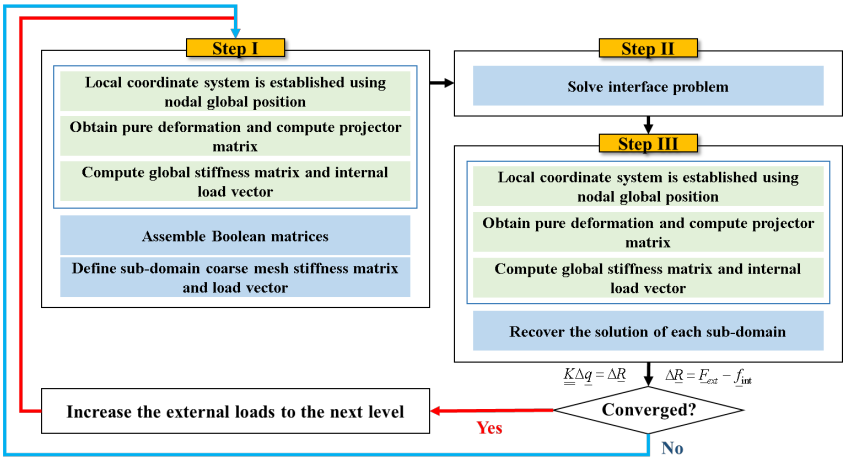


Fig. 3: Unified computational algorithm.

obtained. Thus, the displacement of each sub-domain is obtained easily. The MPI_BCAST routine sends the value of array to all the other processes first, and then, the solution of a linear equation for each sub-domain. In order to handle the sparsity of the system matrix generated in each computational step, i.e. Eq. (9), the sparse linear solver, PARDISO, is implemented. Such process is illustrated in Fig. 4.

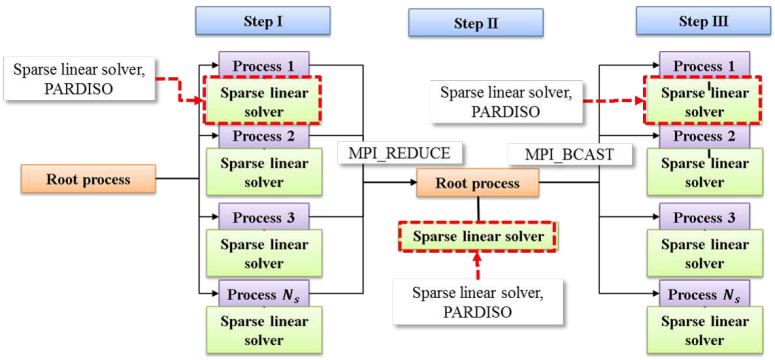


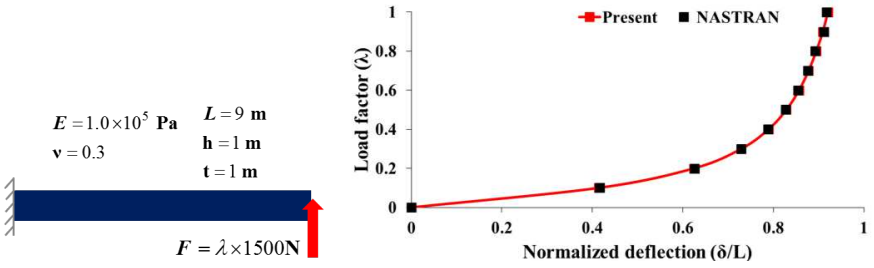
Fig. 4: Parallel implementation of the FETI-Local.

5 Numerical Investigation regarding Nonlinear Problems

Numerical assessment of the present FETI-Local method was performed by comparing with the standard FETI method by iterative solvers in the previous studies conducted by the present authors [5, 6]. The present approach developed herein is applied to the solution of a static, two-dimensional nonlinear problems. The parallel computations were executed in the TACHYON system [13], which is one of the supercomputers operated by Korea Institute of Science and Technology Information. Section 5.1 will discuss the results for the two-dimensional configuration: the computational cost and scalability in a parallel environment are examined. Section 5.2 will examine an application for nonlinear flexible multi-body dynamics.

5.1 Computational Efficiency for Nonlinear Problem

Before the examination of computational efficiency for the analysis of the CR finite element with the FETI-Local method, geometrically nonlinear characteristic of a cantilevered plate discretized by the CR finite element is evaluated. The geometry and operating condition are described in Fig. 5a. The resulting tip deflection is compared with those predicted by MSC.NASTRAN. Comparison shows excellent correlation between the CR planar element and MSC.NASTRAN prediction and it is illustrated in Fig. 5b. Then, the analysis



(a) Analysis condition (b) Comparison of tip deflection

Fig. 5: Nonlinear analysis regarding a cantilevered plate using the CR finite element

of the CR finite element with FETI-Local method is performed by using the same condition (Fig. 5a). However, the tip load is chosen to be 150N. The number of the sub-domains is increased from 8 to 60, but the number of DOFs

is kept to a total of 39,864. Figure 6 shows benign scalability characteristics exhibited by the CR finite element with FETI-Local method.

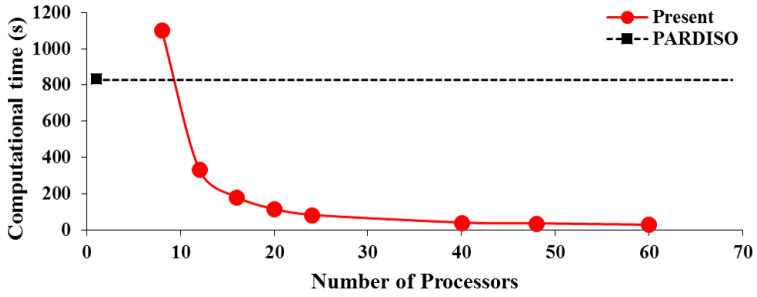


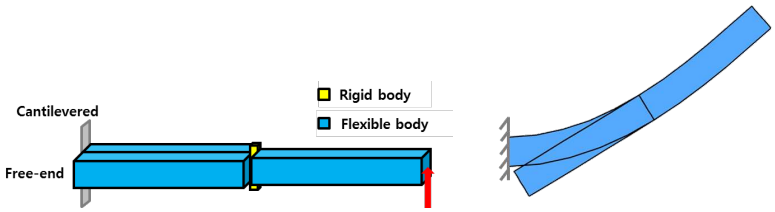
Fig. 6: Computational time and trend of the nonlinear analysis regarding a cantilevered plate.

5.2 Application for Nonlinear Flexible Multi-body Dynamics

In this section, the analysis of the CR finite element with the FETI-Local method is applied to the large scale multi-body system. Analysis condition and resulting deformed configuration is depicted in Fig 7. In parallel computation, the number of the sub-domains is increased from 9 to 36, but the number of DOFs is kept to a total of 32,400. To verify an efficiency of the FETI-Local method in nonlinear flexible multi-body system, equivalent serial analysis employing the classical Lagrange multiplier and PARDISO, is conducted and compared. As the number of processors is increased, the computational time is varied from 2081.09 to 177.03 (sec). Figure 8 shows benign scalability characteristics possessed and exhibited by the analysis of the CR finite element with the FETI-Local method.

6 Conclusion

The development of a nonlinear structural analysis using CR finite element finite element with a domain decomposition algorithm relying on direct solvers only was described. While the FETI-Local method uses the domain decomposition concept that characterizes classical FETI methods, The continuity of the displacement field within sub-domain interfaces is enforced by using



(a) Analysis condition (b) Deformed configuration

Fig. 7: Analysis condition and deformed configuration of multi-body system.

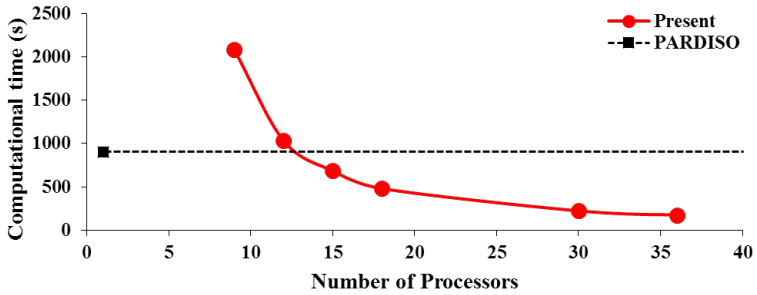


Fig. 8: Computational time and trend of multi-body analysis.

a combination of the localized Lagrange multiplier and of the augmented Lagrangian formulation. Therefore, well-conditioned stiffness matrices is derived. Moreover, direct solvers can be used for both sub-domain and interface problems. The FETI-Local method was further improved by employing the sparse matrix solver to handle the sparsity within the governing equation. The computational cost and scalability of the analysis of the CR finite element with the FETI-Local method was compared to those of the sparse linear equation solver, PARDISO. Good scalability characteristics of the analysis of the CR finite element with the FETI-Local method were demonstrated for a general nonlinear analysis and flexible multi-body dynamic analysis.

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