Volume locking phenomena arising in a hybrid symmetric interior penalty method with continuous numerical traces

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1 Introduction

When we compute numerical solutions of linear elasticity problems for nearly incompressible materials by using the P_1 conforming finite element method, we need to use sufficiently fine meshes in order to get numerical solutions with accuracy. This is referred to as *volume locking* Babuška and Suri [1992]. It is well-known that discontinuous Galerkin (DG) methods are effective in eliminating locking (see, e.g., Hansbo and Larson [2002]).

We investigate locking effects in a hybrid version of a symmetric interior penalty (SIP) method, which is one of DG methods, and is called the *HSIP* method in this paper. Unknowns in the HSIP method are approximations to the displacement of the elastic body and to the trace of the displacement on the skeleton. The latter is called the *numerical trace*. We consider two formulations of the HSIP method: the HSIP methods using discontinuous numerical traces (HSIP-D) and using continuous ones (HSIP-C). The degrees of freedom of the continuous numerical traces are less than those of the discontinuous ones. This gives the HSIP-C method an advantage over the HSIP-D method in practical computations. However, in Kikuchi [2015], it is numerically demonstrated that the HSIP-C method using P_1 elements for both the two unknowns causes volume locking phenomena. On the other hand, in Koyama and Kikuchi [2016], it is established that the HSIP-D is free from locking. In this paper, we mathematically prove that the HSIP-C method shows locking in the case when P_1 elements are employed to approximate displacement and its trace on the skeleton.

We close this section with the introduction of several notations which will be used throughout this paper. For an arbitrary open subset Ω of $\mathbb{R}^2$, we

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denote by $L^2(\Omega)$ and by $H^s(\Omega)$ $(s > 0)$ the usual space of real-valued square integrable functions on Ω and the real Sobolev space on Ω, respectively (see, e.g., Brenner and Scott [2008]). We denote by $(\cdot, \cdot)_\Omega$ and by $\|\cdot\|_\Omega$ the inner product of $L^2(\Omega)$ and the associated norm, respectively. We equip $H^s(\Omega)$ with the usual norm denoted by $\|\cdot\|_{s,\Omega}$. We denote by $|\cdot|_{s,\Omega}$ the usual semi-norm of $H^s(\Omega)$. For the union Γ of arbitrary line segments in $\mathbb{R}^2$, we denote by $\langle\cdot, \cdot\rangle_\Gamma$ and by $|\cdot|_\Gamma$ the inner product of $L^2(\Gamma)$ and the associated norm, respectively. We use the same notations of the norm, the semi-norm, and the inner product for vector valued functions as well. In addition, C denotes a generic positive constant, and can be a different value at each of different places.

2 Linear plane strain problem

For the two-dimensional displacement $\underline{u} = [u_1, u_2]^T$ of an elastic body, the strain tensor is given by $\underline{\underline{\varepsilon}}(\underline{u}) = \left[\frac{1}{2}\left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)\right]_{1\le i,j\le 2}$. We use an underline (resp. double underlines) to denote two dimensional vector (resp. 2×2 matrix) valued functions, operators, and their associated spaces. The isotropic linear elastic stress-strain relation is written by

$$\underline{\underline{\sigma}}(\underline{u}) = 2\mu\,\underline{\underline{\varepsilon}}(\underline{u}) + \lambda(\operatorname{div}\underline{u})\,\underline{\underline{\delta}},$$

where λ (> 0) and μ (> 0) are the Lamé parameters, and $\underline{\underline{\delta}}$ is the identity matrix. We consider the following linear plane strain problem:

$$\begin{cases} -\dfrac{\partial\sigma_{11}(\underline{u})}{\partial x_1} - \dfrac{\partial\sigma_{12}(\underline{u})}{\partial x_2} = f_1 \text{ in } \Omega, \\ -\dfrac{\partial\sigma_{21}(\underline{u})}{\partial x_1} - \dfrac{\partial\sigma_{22}(\underline{u})}{\partial x_2} = f_2 \text{ in } \Omega, \\ \underline{u} = \underline{0} \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\underline{\underline{\sigma}}(\underline{u}) = [\sigma_{ij}(\underline{u})]_{1\le i,j\le 2}$, and $\underline{f} = [f_1, f_2]^T$ is a distributed external body force per unit in-plane area. We assume that Ω is a bounded polygonal domain of $\mathbb{R}^2$. In addition we fix $\mu > 0$.

3 The HSIP-D method

Let $\mathcal{T}^h$ be a triangulation of Ω. We assume that $\mathcal{T}^h$ has no hanging nodes. The set of edges of $\mathcal{T}^h$ is denoted by $\mathcal{E}^h$. For each $K \in \mathcal{T}^h$, we define $\mathcal{E}^K := \left\{e \in \mathcal{E}^h \mid e \subset \partial K\right\}$. We define the skeleton Γ^h of $\mathcal{T}^h$ by $\Gamma^h := \bigcup_{e\in\mathcal{E}^h} \overline{e}$. The diameter of K is denoted by h_K, and the length of an edge $e \in \mathcal{E}^K$ by $|e|$.

In addition, we set $h := \max_{K \in \mathcal{T}^h} h_K$. Assume that a family $\{\mathcal{T}^h\}_{h \in (0, \bar{h}]}$ of triangulations is regular.

The HSIP-D method seeks approximations to the solution $\underline{u}$ of (1) and to the trace of $\underline{u}$ on Γ^h by using functions belonging to

$$U^h := \prod_{K \in \mathcal{T}^h} P_k(K) \quad \text{and} \quad \widehat{U}^h := \prod_{e \in \mathcal{E}^h} P_k(e),$$

respectively, where P_k denotes the set of polynomial functions of order at most $k \geq 1$. So we consider their product space: $\underline{\boldsymbol{U}}^h := \underline{U}^h \times \underline{\widehat{U}}^h \subset \underline{H}^1(\mathcal{T}^h) \times \underline{L}^2(\Gamma^h)$, where $H^s(\mathcal{T}^h) := \{v \in L^2(\Omega) \mid v|_K \in H^s(K)\ \forall K \in \mathcal{T}^h\}$ $(s > 0)$. We will denote the first and the second components of $\underline{\boldsymbol{v}} \in \underline{H}^1(\mathcal{T}^h) \times \underline{L}^2(\Gamma^h)$ by $\underline{v}$ and $\underline{\hat{v}}$, i.e., $\underline{\boldsymbol{v}} = \{\underline{v}, \underline{\hat{v}}\}$, unless specifically stated otherwise.

For each $K \in \mathcal{T}^h$ and for each $i = 1, 2$, we define local lifting operator $R_i^K : L^2(\partial K) \longrightarrow Q^K$ by $(R_i^K g, \varphi)_K = \langle g, \varphi n_i \rangle_{\partial K}$ for all $g \in L^2(\partial K)$ and for all $\varphi \in Q^K$, where $Q^K := P_{k-1}(K)$ and n_i is the ith component of the outward unit normal $\underline{n}$ on ∂K. We further define lifting operators $R_{\mathrm{div}}^K : \underline{L}^2(\partial K) \longrightarrow Q^K$ and $\underline{\underline{R_\varepsilon^K}}(\underline{g}) : \underline{L}^2(\partial K) \longrightarrow \underline{\underline{Q}}^K$ as follows Kikuchi [2015]: $R_{\mathrm{div}}^K \underline{g} := \sum_{i=1}^2 R_i^K g_i$ and $\underline{\underline{R_\varepsilon^K}}(\underline{g}) := \left[\frac{1}{2}\left(R_i^K g_j + R_j^K g_i\right)\right]_{1 \leq i,\, j \leq 2}$ for $\underline{g} = [g_1, g_2]^T \in \underline{L}^2(\partial K)$.

We introduce the following three bilinear forms: for $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \underline{H}^2(\mathcal{T}^h) \times \underline{L}^2(\Gamma^h)$,

$$\begin{aligned}
\tilde{a}_\eta^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) &:= 2\mu \sum_{K \in \mathcal{T}^h} \Bigg[\Big(\underline{\underline{\varepsilon}}(\underline{u}), \underline{\underline{\varepsilon}}(\underline{v})\Big)_K + \Big\langle \underline{\underline{\varepsilon}}(\underline{u})\underline{n}, \underline{\hat{v}} - \underline{v} \Big\rangle_{\partial K} \\
&\qquad + \Big\langle \underline{\hat{u}} - \underline{u}, \underline{\underline{\varepsilon}}(\underline{v})\underline{n} \Big\rangle_{\partial K} + \Big(\underline{\underline{R_\varepsilon^K}}(\underline{\hat{u}} - \underline{u}), \underline{\underline{R_\varepsilon^K}}(\underline{\hat{v}} - \underline{v}) \Big)_K \Bigg] \\
&\qquad + \eta \sum_{K \in \mathcal{T}^h} \sum_{e \in \mathcal{E}^K} \frac{1}{|e|} \langle \underline{\hat{u}} - \underline{u}, \underline{\hat{v}} - \underline{v} \rangle_e, \\
l^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) &:= \sum_{K \in \mathcal{T}^h} \Big[(\mathrm{div}\, \underline{u}, \mathrm{div}\, \underline{v})_K + \langle (\mathrm{div}\, \underline{u})\underline{n}, \underline{\hat{v}} - \underline{v} \rangle_{\partial K} \\
&\qquad + \langle \underline{\hat{u}} - \underline{u}, (\mathrm{div}\, \underline{v})\underline{n} \rangle_{\partial K} + \big(R_{\mathrm{div}}^K(\underline{\hat{u}} - \underline{u}), R_{\mathrm{div}}^K(\underline{\hat{v}} - \underline{v}) \big)_K \Big], \\
a_\eta^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) &:= \tilde{a}_\eta^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) + \lambda l^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}),
\end{aligned} \tag{2}$$

where η is an interior penalty parameter ≥ 0, and $(\underline{\underline{\sigma}}, \underline{\underline{\tau}})_K := \sum_{i,j=1}^2 \int_K \sigma_{ij} \tau_{ij}\, dx$ for $\underline{\underline{\sigma}} = [\sigma_{ij}]_{1 \leq i,j \leq 2}$, $\underline{\underline{\tau}} = [\tau_{ij}]_{1 \leq i,j \leq 2} \in \underline{\underline{L}}^2(K)$.

We are now in a position to present a discrete problem, which provides the HSIP-D method: find $\underline{\boldsymbol{u}}^h \in \underline{\boldsymbol{V}}^h$ such that

$$a^h_\eta(\underline{\boldsymbol{u}}^h, \underline{\boldsymbol{v}}^h) = \left(\underline{f}, \underline{v}^h\right)_\Omega \quad \forall \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h, \tag{3}$$

where $L^2_D(\Gamma^h) := \left\{\hat{v} \in L^2(\Gamma^h) \mid \hat{v} = 0 \text{ on } \partial\Omega\right\}$, $\widehat{V}^h := \widehat{U}^h \cap L^2_D(\Gamma^h)$, and $\underline{\boldsymbol{V}}^h := \underline{U}^h \times \underline{\widehat{V}}^h$.

Problem (3) has a unique solution for every $\underline{f} \in \underline{L}^2(\Omega)$ and for every $\eta > 0$ (see Koyama and Kikuchi [2016]). Moreover the HSIP-D method is free from locking with respect to the solution set B_λ and the norm $|||\cdot|||_h$ in the sense of Babuška and Suri [1992] (see Koyama and Kikuchi [2016]), where $B_\lambda := \left\{\underline{v} \in \underline{H}^2(\Omega) \cap \underline{H}^1_D(\Omega) \mid \|\underline{v}\|_{2,\Omega} + \lambda \|\operatorname{div} \underline{v}\|_{1,\Omega} \le 1\right\}$, $H^1_D(\Omega) := \left\{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\right\}$, and

$$|||\underline{\boldsymbol{v}}|||^2_h := \sum_{K \in \mathcal{T}^h} \left[|\underline{v}|^2_{1,K} + \sum_{e \in \mathcal{E}^K} \left(\frac{1}{|e|} |\underline{\hat{v}} - \underline{v}|^2_e + |e| \sum_{i,j=1}^{2} \left| \frac{\partial v_i}{\partial x_j} \right|^2_e \right) \right].$$

We now introduce a semi-norm on $\underline{H}^1(\mathcal{T}^h) \times \underline{L}^2(\Gamma^h)$ as follows:

$$|\underline{\boldsymbol{v}}|^2_h := \sum_{K \in \mathcal{T}^h} \left(|\underline{v}|^2_{1,K} + \sum_{e \in \mathcal{E}^K} \frac{1}{|e|} |\underline{\hat{v}} - \underline{v}|^2_e \right) \quad \forall \underline{\boldsymbol{v}} \in \underline{H}^1(\mathcal{T}^h) \times \underline{L}^2(\Gamma^h).$$

This semi-norm can be a norm on $\underline{\boldsymbol{V}}^h$ equivalent to $|||\cdot|||_h$, that is, there exists a positive constant C such that for all $h \in (0, \bar{h}]$ and for all $\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h$,

$$C\left|\left|\left|\underline{\boldsymbol{v}}^h\right|\right|\right|_h \le |\underline{\boldsymbol{v}}^h|_h \le \left|\left|\left|\underline{\boldsymbol{v}}^h\right|\right|\right|_h. \tag{4}$$

We define $\underline{\underline{\boldsymbol{\varepsilon}}}^h : \underline{\boldsymbol{U}}^h \longrightarrow \underline{\underline{L}}^2(\Omega)$ and $\mathbf{div}^h : \underline{\boldsymbol{U}}^h \longrightarrow L^2(\Omega)$ as follows Kikuchi [2015]: for every $\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{U}}^h$ and for every $K \in \mathcal{T}^h$,

$$\begin{aligned} \underline{\underline{\boldsymbol{\varepsilon}}}^h(\underline{\boldsymbol{v}}^h)|_K &:= \underline{\underline{\varepsilon}}(\underline{\boldsymbol{v}}^h|_K) + \underline{\underline{R^K_\varepsilon}}(\underline{\hat{v}}^h - \underline{v}^h), \\ \left(\mathbf{div}^h\, \underline{\boldsymbol{v}}^h\right)|_K &:= \operatorname{div}(\underline{v}^h|_K) + R^K_{\operatorname{div}}(\underline{\hat{v}}^h - \underline{v}^h). \end{aligned} \tag{5}$$

For all $\underline{\boldsymbol{u}}^h$, $\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{U}}^h$, we have

$$\tilde{a}^h_0(\underline{\boldsymbol{u}}^h, \underline{\boldsymbol{v}}^h) = 2\mu \left(\underline{\underline{\boldsymbol{\varepsilon}}}^h(\underline{\boldsymbol{u}}^h), \underline{\underline{\boldsymbol{\varepsilon}}}^h(\underline{\boldsymbol{v}}^h)\right)_\Omega, \tag{6}$$

$$l^h(\underline{\boldsymbol{u}}^h, \underline{\boldsymbol{v}}^h) = \left(\mathbf{div}^h\, \underline{\boldsymbol{u}}^h, \mathbf{div}^h\, \underline{\boldsymbol{v}}^h\right)_\Omega \quad \text{(see Kikuchi [2015]).} \tag{7}$$

For all $\lambda > 0$, for all $\eta > 0$, for all $h \in (0, \bar{h}]$, and for all $\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h$,

$$a^h_\eta(\underline{\boldsymbol{v}}^h, \underline{\boldsymbol{v}}^h) \ge \alpha \min\{1, \eta\} \left|\left|\left|\underline{\boldsymbol{v}}^h\right|\right|\right|^2_h, \tag{8}$$

where α is a positive constant independent of λ, η, h, and $\underline{\boldsymbol{v}}^h$ (see Koyama and Kikuchi [2016]). Note that (8) holds for all $\eta > 0$ because bilinear form a_η^h includes the terms defined by lifting operators $\underline{\underline{R}}_\varepsilon^K$ and R_{div}^K.

4 Volume locking phenomena in the HSIP-C method

In this section, we fix η and assume that $k = 1$.

We introduce finite element spaces:

$$\begin{aligned} U_c^h &:= U^h \cap H^1(\Omega), & V_c^h &:= U^h \cap H_D^1(\Omega), \\ \widehat{U}_c^h &:= \widehat{U}^h \cap C^0(\Gamma^h), & \widehat{V}_c^h &:= \widehat{U}_c^h \cap L_D^2(\Gamma^h), \\ \underline{\boldsymbol{U}}_c^h &:= \underline{U}^h \times \underline{\widehat{U}}_c^h, & \underline{\boldsymbol{V}}_c^h &:= \underline{U}^h \times \underline{\widehat{V}}_c^h. \end{aligned}$$

Replacing $\underline{\boldsymbol{V}}^h$ by $\underline{\boldsymbol{V}}_c^h$ in (3), we can obtain the HSIP-C method.

We mathematically demonstrate that the HSIP-C method shows locking by following the method of proof due to Brenner and Scott [2008].

We can naturally identify $\underline{\widehat{U}}_c^h$ with $\underline{U}_c^h$, that is, there uniquely exists a linear operator $\mathcal{J}$ from $\underline{\widehat{U}}_c^h$ onto $\underline{U}_c^h$ such that $\mathcal{J}\underline{\hat{v}}_c^h = \underline{\hat{v}}_c^h$ on ∂K for every $\underline{\hat{v}}_c^h \in \underline{\widehat{U}}_c^h$ and for every $K \in \mathcal{T}^h$.

Lemma 1. *There exists a positive constant C such that for all $h \in (0, \bar{h}]$, for all $\underline{v} \in \underline{H}^1(\Omega)$, and for all $\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{U}}_c^h$,*

$$|\underline{v} - \mathcal{J}\underline{\hat{v}}^h|_{1,\Omega} \le C|\underline{\boldsymbol{v}} - \underline{\boldsymbol{v}}^h|_h, \tag{9}$$

where $\underline{\boldsymbol{v}} = \{\underline{v}, \underline{v}|_{\Gamma^h}\}$, and C is independent of h, $\underline{v}$, and $\underline{\boldsymbol{v}}^h$.

Proof. The usual scaling argument leads to that there exists a positive constant C such that for all $h \in (0, \bar{h}]$, for all $K \in \mathcal{T}^h$, and for all $v \in P_1(K)$,

$$\|v\|_{1,K} \le C \left(\sum_{e \in \mathcal{E}^K} \frac{1}{|e|} |v|_e^2 \right)^{1/2}, \tag{10}$$

where C is independent of h, K, and v. For all $\underline{v} \in \underline{H}^1(\Omega)$ and for all $\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{U}}_c^h$,

$$\begin{aligned} |\underline{v} - \mathcal{J}\underline{\hat{v}}^h|_{1,\Omega}^2 &\le 2 \sum_{K \in \mathcal{T}^h} \left(|\underline{v} - \underline{v}^h|_{1,K}^2 + |\underline{v}^h - \mathcal{J}\underline{\hat{v}}^h|_{1,K}^2 \right) \\ &\qquad \text{(by the triangle and the Schwarz inequalities)} \\ &\le C \sum_{K \in \mathcal{T}^h} \left(|\underline{v} - \underline{v}^h|_{1,K}^2 + \sum_{e \in \mathcal{E}^K} \frac{1}{|e|} |\underline{v}^h - \underline{\hat{v}}^h|_e^2 \right) \quad \text{(by (10))}. \end{aligned}$$

This yields (9). □

We now pose a hypothesis:

$$\left\{\underline{v}^h \in \underline{V}_c^h \mid \operatorname{div}\underline{v}^h = 0\right\} = \{\underline{0}\}. \tag{L}$$

We understand from the following lemma that many triangulations satisfy (L) (cf. [Brenner and Scott, 2008, Exercise 11.x.14]).

Lemma 2. *Let K_1 and K_2 be triangular elements whose vertices are $\{A, B, C\}$ and $\{B, C, D\}$, respectively. Let v_j^h $(j = 1, 2)$ be continuous piecewise linear functions on $\overline{K_1 \cup K_2}$. Set $\underline{v}^h := [v_1^h, v_2^h]^T$. Assume that $\operatorname{div}\underline{v}^h = 0$ and that $\underline{v}^h = \underline{0}$ on the sides AB and BD. If A, B, and D are not collinear, then $\underline{v}^h \equiv \underline{0}$ on $\overline{K_1 \cup K_2}$.*

We leave the proof to readers.

Lemma 3. *If (L) holds, then*

$$\operatorname{Ker}(\mathbf{div}^h|_{\underline{\boldsymbol{V}}_c^h}) = \left\{\{\underline{v}^h, \underline{0}\} \in \underline{\boldsymbol{V}}_c^h \mid \underline{v}^h \in \underline{U}^h\right\}, \tag{11}$$

where $\mathbf{div}^h|_{\underline{\boldsymbol{V}}_c^h}$ denotes the restriction of $\mathbf{div}^h$ to $\underline{\boldsymbol{V}}_c^h$.

Proof. We see from the Green formula that for every $\underline{v} \in \underline{P}_1(K)$,

$$\operatorname{div}\underline{v} = R_{\mathrm{div}}^K(\underline{v}) \quad \text{in } \mathbb{R}. \tag{12}$$

It follows from (5) and (12) that for all $\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{U}}^h$,

$$\left(\mathbf{div}^h\,\underline{\boldsymbol{v}}^h\right)\Big|_K = R_{\mathrm{div}}^K(\hat{\underline{v}}^h) \quad \forall K \in \mathcal{T}^h. \tag{13}$$

This implies that $\mathbf{div}^h\left(\{\underline{v}^h, \underline{0}\}\right) = 0$ for every $\underline{v}^h \in \underline{U}^h$. Thus the right-hand side of (11) is included in $\operatorname{Ker}(\mathbf{div}^h|_{\underline{\boldsymbol{V}}_c^h})$.

Conversely, we suppose that $\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}_c^h$ satisfies $\mathbf{div}^h\,\underline{\boldsymbol{v}}^h = 0$. We find from (13) and (12) that for each $K \in \mathcal{T}^h$,

$$0 = \left(\mathbf{div}^h\,\underline{\boldsymbol{v}}^h\right)\Big|_K = R_{\mathrm{div}}^K(\hat{\underline{v}}^h) = R_{\mathrm{div}}^K\left((\mathcal{J}\hat{\underline{v}}^h)|_{\partial K}\right) = \operatorname{div}\left((\mathcal{J}\hat{\underline{v}}^h)|_K\right),$$

and hence $\operatorname{div}\left(\mathcal{J}\hat{\underline{v}}^h\right) = 0$ in Ω. Since $\mathcal{J}\hat{\underline{v}}^h \in \underline{V}_c^h$, it follows from hypothesis (L) that $\mathcal{J}\hat{\underline{v}}^h = \underline{0}$ in Ω. This implies that $\hat{\underline{v}}^h = \underline{0}$ on Γ^h. Thus $\underline{\boldsymbol{v}}^h$ belongs to the right-hand side of (11). □

We now define mapping $\mathbf{div}_1^h : \underline{\boldsymbol{V}}_c^h / \operatorname{Ker}(\mathbf{div}^h|_{\underline{\boldsymbol{V}}_c^h}) \longrightarrow L^2(\Omega)$ by

$$\mathbf{div}_1^h[\underline{\boldsymbol{v}}^h] := \mathbf{div}^h\,\underline{\boldsymbol{v}}^h \quad \forall \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}_c^h,$$

where $[\underline{\boldsymbol{v}}^h]$ is the set of equivalence class of $\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}_c^h$. Since $\mathbf{div}_1^h$ is injective and $\underline{\boldsymbol{V}}_c^h / \operatorname{Ker}(\mathbf{div}^h|_{\underline{\boldsymbol{V}}_c^h})$ is finite dimensional, there exists a positive constant

$C(h)$ such that for all $\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}_c^h$,

$$\inf_{\underline{\chi}^h \in \underline{U}^h} \left|\left|\left| \underline{\boldsymbol{v}}^h + \{\underline{\chi}^h, \underline{0}\} \right|\right|\right|_h \le C(h) \left\| \mathbf{div}^h \, \underline{\boldsymbol{v}}^h \right\|_{\Omega}. \tag{14}$$

Using (9) with $\underline{v} \equiv \underline{0}$ and (4), we get

$$\left| \mathcal{J} \hat{\underline{v}}^h \right|_{1,\Omega} \le C \inf_{\underline{\chi}^h \in \underline{U}^h} \left|\left|\left| \underline{\boldsymbol{v}}^h + \{\underline{\chi}^h, \underline{0}\} \right|\right|\right|_h \quad \forall \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}_c^h. \tag{15}$$

Combining (14) and (15) gives us

$$\left| \mathcal{J} \hat{\underline{v}}^h \right|_{1,\Omega} \le C(h) \left\| \mathbf{div}^h \, \underline{\boldsymbol{v}}^h \right\|_{\Omega} \quad \forall \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}_c^h. \tag{16}$$

Proposition 1. *Let* $\underline{u} \in \underline{H}^2(\Omega) \cap \underline{H}_D^1(\Omega)$ *satisfy*

$$\operatorname{div} \underline{u} = 0. \tag{17}$$

For each $\lambda > 0$, *let* $\underline{\boldsymbol{u}}_\lambda^h \in \underline{\boldsymbol{V}}_c^h$ *satisfy*

$$a_\eta^h(\underline{\boldsymbol{u}}_\lambda^h, \underline{\boldsymbol{v}}^h) = a_\eta^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}^h) \quad \forall \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}_c^h, \tag{18}$$

where $\underline{\boldsymbol{u}} := \{\underline{u}, \underline{u}|_{\Gamma^h}\}$. *Assume that (L) holds. Then we have*

$$\left| \mathcal{J} \hat{\underline{u}}_\lambda^h \right|_{1,\Omega} \longrightarrow 0 \quad (\lambda \longrightarrow \infty). \tag{19}$$

Proof. We first introduce the following trace inequality: for all $h \in (0, \bar{h}]$, for all $K \in \mathcal{T}^h$, for all $e \in \mathcal{E}^K$, and for all $v \in H^1(K)$,

$$|v|_e^2 \le C \left(|e|^{-1} \|v\|_K^2 + |e| |v|_{1,K}^2 \right), \tag{20}$$

where C is a positive constant independent of h, K, e, and v.

It follows from (18), (17), and (20) that we have

$$\begin{aligned} a_\eta^h(\underline{\boldsymbol{u}}_\lambda^h, \underline{\boldsymbol{u}}_\lambda^h) &= a_\eta^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{u}}_\lambda^h) \\ &= 2\mu \sum_{K \in \mathcal{T}^h} \left[\left(\underline{\underline{\varepsilon}}(\underline{u}), \underline{\underline{\varepsilon}}(\underline{u}_\lambda^h) \right)_K + \left\langle \underline{\underline{\varepsilon}}(\underline{u}) \underline{n}, \hat{\underline{u}}_\lambda^h - \underline{u}_\lambda^h \right\rangle_{\partial K} \right] \\ &\le C \|\underline{u}\|_{2,\Omega} \left|\left|\left| \underline{\boldsymbol{u}}_\lambda^h \right|\right|\right|_h, \end{aligned} \tag{21}$$

where C is a positive constant independent of h, λ, and $\underline{u}$. Using (8), we obtain

$$\left|\left|\left| \underline{\boldsymbol{u}}_\lambda^h \right|\right|\right|_h \le C \|\underline{u}\|_{2,\Omega}. \tag{22}$$

Combining (6), (7), (2), (21), and (22) leads us to

$$\| \mathbf{div}^h \, \underline{\boldsymbol{u}}_\lambda^h \|_\Omega^2 \le \lambda^{-1} C \|\underline{u}\|_{2,\Omega}^2 \longrightarrow 0 \quad (\lambda \longrightarrow \infty),$$

and thus, by (16), we get (19). □

Theorem 1. *Assume that (L) holds for every $h \in (0, \bar{h}]$. There exists a positive constant C independent of h such that*

$$\liminf_{\lambda \longrightarrow \infty} \sup_{\underline{w} \in B_\lambda} \left|\left|\left| \underline{\boldsymbol{w}} - \underline{\boldsymbol{w}}_\lambda^h \right|\right|\right|_h \geq C \quad \forall h \in (0, \bar{h}], \tag{23}$$

where $\underline{\boldsymbol{w}} := \{\underline{w}, \underline{w}|_{\Gamma^h}\}$ and $\underline{\boldsymbol{w}}_\lambda^h \in \underline{\boldsymbol{V}}_c^h$ is the solution of (18) after replacing $\underline{\boldsymbol{u}}$ by $\underline{\boldsymbol{w}}$.

Proof. There exists a $\underline{u} \in \underline{H}^2(\Omega) \cap \underline{H}_D^1(\Omega)$ such that $\|\underline{u}\|_{2,\Omega} = 1$ and (17) holds Brenner and Scott [2008]. Then $\underline{u} \in B_\lambda$ for all $\lambda > 0$. For every $h \in (0, \bar{h}]$ and for every $\lambda > 0$,

$$\begin{aligned} \sup_{\underline{w} \in B_\lambda} \left|\left|\left| \underline{\boldsymbol{w}} - \underline{\boldsymbol{w}}_\lambda^h \right|\right|\right|_h \geq \left|\left|\left| \underline{\boldsymbol{u}} - \underline{\boldsymbol{u}}_\lambda^h \right|\right|\right|_h &\geq C \left| \underline{u} - \mathcal{J} \underline{\hat{u}}_\lambda^h \right|_{1,\Omega} \quad \text{(by (9))} \\ &\geq C \left(|\underline{u}|_{1,\Omega} - \left| \mathcal{J} \underline{\hat{u}}_\lambda^h \right|_{1,\Omega} \right), \end{aligned} \tag{24}$$

where C is independent of h and λ.

We can conclude from (19) and (24) that (23) holds. □

Remark 1. For a meaning of (23), see Brenner and Scott [2008]. Using (23), we can also prove that the HSIP-C method with $k = 1$ shows locking of order h^{-1} with respect to the solution set B_λ and the norm $|||\cdot|||_h$ in the sense of Babuška and Suri [1992] (see Koyama and Kikuchi [2016]).

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