A Mortar Domain Decomposition Method for Quasilinear Problems

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1 Introduction

As model problem for a quasilinear partial differential equation we consider the Richards equation, see, e.g., [2],

$$n\,\frac{\partial\theta(p)}{\partial t} - \nabla\cdot\left(\frac{K}{\mu}\,k(\theta(p))\nabla(p-d)\right) = f$$

to find the unknown pressure p. This equation results from the principle of mass balance and by using several laws from hydrology. The quantity $n(\mathbf{x})$ prescribes the porosity of the soil, $K(\mathbf{x})$ is the permeability of the soil, μ is just the constant viscosity of water, and $d(\mathbf{x}) := d(x_1,\ldots,x_d) = \varrho\,g\,x_d$ with the constant water density ϱ and with the gravitational constant g. The nonlinear parameter function θ describes the saturation of the soil in dependency of the pressure p. k is the relative permeability of the soil which depends on the saturation. There are several models available which describe the shape of θ and k. In this work we use the model of Brooks and Corey [5] where the saturation is given as

$$\theta(p) := \begin{cases} \left(\dfrac{p}{p_b}\right)^{-\lambda}(\theta_{\max}-\theta_{\min}) + \theta_{\min} & \text{for } p \le p_b, \\ \theta_{\max} & \text{for } p > p_b. \end{cases}$$

Here, $\theta_{\min}$ and $\theta_{\max}$ are the minimal and maximal saturation level, $p_b < 0$ is the so called bubbling pressure, and $\lambda > 0$ is the pore size distribution factor. The relative permeability is given as

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$$k(\theta) := \left(\frac{\theta - \theta_{min}}{\theta_{max} - \theta_{min}} \right)^{3+\frac{2}{\lambda}} .$$

Hence we conclude

$$k(\theta(p)) = \begin{cases} \left(\dfrac{p}{p_b} \right)^{-3\lambda-2} & \text{for } p \leq p_b, \\ 1 & \text{for } p > p_b. \end{cases}$$

The considerations made so far are valid for a single soil type only, see Fig. 1. In the case of several layers of different soil types we have to consider parameter functions θ and k which depend explicitely on $\mathbf{x}$, see Fig. 2 where we have a decomposition of Ω into N non–overlapping subdomains Ω_i representing a soil layer each with local parameter functions θ_i and k_i. Hence we define

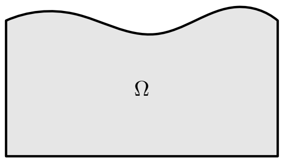


Fig. 1 Single soil type

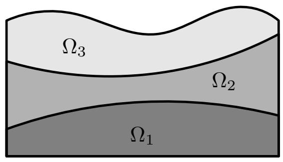


Fig. 2 Several soil layers

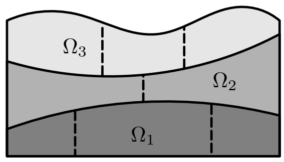


Fig. 3 Decomposition

global parameter functions as

$$\theta(\mathbf{x}, p(\mathbf{x},t)) = \theta_i(p(\mathbf{x},t)), \quad k(\mathbf{x}, \theta(\mathbf{x}, p(\mathbf{x},t))) = k_i(\theta_i(p(\mathbf{x},t))), \quad \mathbf{x} \in \Omega_i.$$

In what follows we will apply an implicit–explicit time discretization scheme and local Kirchhoff transformations to end up with a domain decomposition variational formulation of local linear elliptic partial differential equations, but with nonlinear transmission conditions. For the discretization we then use a mortar finite element approach.

2 Variational formulation

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain with boundary $\partial\Omega$ which is decomposed into two mutually disjoint parts Γ_D and Γ_N where boundary conditions of Dirichlet and Neumann type are given, respectively. We assume meas $\Gamma_D > 0$, and let $\mathbf{n}$ be the outer unit normal. For $T > 0$ we consider the initial boundary value problem to find $p : \Omega \times (0,T) \to \mathbb{R}$ such that

$$
\begin{aligned}
n\,\frac{\partial\theta(p)}{\partial t} - \nabla\cdot\left(\frac{K}{\mu}\,k(\theta(p))\nabla(p-d)\right) &= f && \text{in } \Omega\times(0,T), && \text{(1a)}\\
p &= p_D && \text{on } \Gamma_D\times(0,T), && \text{(1b)}\\
\frac{K}{\mu}\,k(\theta(p))\nabla(p-d)\cdot\mathbf{n} &= p_N && \text{on } \Gamma_N\times(0,T), && \text{(1c)}\\
p &= p_0 && \text{at } \Omega\times\{0\} && \text{(1d)}
\end{aligned}
$$

is satisfied.

For $M\in\mathbb{N}$ let $0=t_0<t_1<\ldots<t_M=T$ be a decomposition of the time interval $(0,T)$. For an implicit time discretization we use a backward Euler method to approximate the time derivative,

$$\frac{\partial}{\partial t}\theta(\mathbf{x},p(\mathbf{x},t))\Big|_{t=t_m} \approx \frac{\theta(p_m)-\theta(p_{m-1})}{\tau_m},\ \tau_m := t_m - t_{m-1},\ p_m(\mathbf{x})\approx p(t_m,\mathbf{x}).$$

After time discretization, the variational formulation of (1a) is to find, for all time steps $1\le m\le M$, $p_m\in H^1(\Omega)$, $p_{m|\Gamma_D}=p_D(t_m)$, such that

$$\int_\Omega \frac{n}{\tau_m}\theta(p_m)v\,\mathrm{d}\mathbf{x} + \int_\Omega \frac{K}{\mu}k(\theta(p_m))\nabla(p_m-d)\cdot\nabla v\,\mathrm{d}\mathbf{x} = \langle\widehat{F},v\rangle_\Omega$$

is satisfied for all $v\in V:=H^1_{0,\Gamma_D}(\Omega)$, where

$$\langle\widehat{F},v\rangle_\Omega := \int_\Omega\left(f(t_m)+\frac{n}{\tau_m}\theta(p_{m-1})\right)v\,\mathrm{d}\mathbf{x} + \int_{\Gamma_N} p_N(t_m)\,v\,\mathrm{d}s_{\mathbf{x}}.$$

For the remaining nonlinear term we apply an explicit discretization step,

$$k(\theta(p_m))\nabla(p_m-d)\approx k(\theta(p_m))\nabla p_m - k(\theta(p_{m-1}))\nabla d$$

where we keep the nonlinearity within the first term. Hence we end up with a variational formulation to find $p_m\in H^1(\Omega)$, $p_{m|\Gamma_D}=p_D(t_m)$, such that

$$\int_\Omega \frac{n}{\tau}\theta(p_m)v\,\mathrm{d}\mathbf{x} + \int_\Omega \frac{K}{\mu}k(\theta(p_m))\nabla p_m\cdot\nabla v\,\mathrm{d}\mathbf{x} = \langle F,v\rangle_\Omega \tag{2}$$

is satisfied for all $v\in V$, where

$$\langle F,v\rangle_\Omega := \langle\widehat{F},v\rangle_\Omega + \int_\Omega \frac{K}{\mu}k(\theta(p_{m-1}))\nabla d\cdot\nabla v\,\mathrm{d}\mathbf{x}\,.$$

Theorem 1. *Assume $n,K\in L^+_\infty(\Omega)=\{u\in L_\infty(\Omega)\mid \operatorname{ess\,inf}_{x\in\Omega} u>0\}$, $\tau,\mu\in\mathbb{R}_+$. Let $\theta_i=\theta_{|\Omega_i}\in C^{0,1}(\mathbb{R})$ be monotonically increasing, and we assume $k_i=k_{|\Omega_i}\in C^{0,1}(\mathbb{R})\cap L_\infty(\mathbb{R})$ and $k(s)\ge c>0$ for all $s\in\mathbb{R}$. Then there exists a unique solution of the variational problem (2).*

To handle the nonlinear term in the variational formulation (2) we will apply the Kirchhoff transformation [1, 3] locally within the subdomains Ω_i. Since this results in nonlinear Dirichlet transmission conditions, we will use a primal–hybrid formulation [4, 8] to split the global problem (2) into local ones with suitable transmission conditions.

In what follows we will skip the dependence on the time step, and we consider one time step only.

Let $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$ be a nonoverlapping domain decomposition which resolves the different soil layers, see Fig. 3. When defining the primal space

$$X := \left\{ p \in L^2(\Omega) \middle| p_{|\Omega_i} \in H^1(\Omega_i) \right\},$$

the Lagrange multiplier space

$$M := \left\{ \mu \in \prod_{i=1}^N H^{-1/2}(\partial\Omega_i) \middle| \, \exists\, \mathbf{q} \in H_{0,\Gamma_N}(\mathrm{div}, \Omega) : \mathbf{q} \cdot \mathbf{n}_i = \mu \text{ on } \partial\Omega_i \right\},$$

and the bilinear form

$$b(p, \nu) := -\sum_{i=1}^N \langle p_{|\Omega_i}, \nu \rangle_{\partial\Omega_i},$$

we obtain a variational problem to find $(p, \lambda) \in X \times M$ such that

$$\sum_{i=1}^N \left(\int_{\Omega_i} \frac{n}{\tau} \theta(p) v \,\mathrm{d}\mathbf{x} + \int_{\Omega_i} \frac{K}{\mu} k(\theta(p)) \nabla p \cdot \nabla v \,\mathrm{d}\mathbf{x} \right) + b(v, \lambda) = \langle F, v \rangle_\Omega,$$
$$b(p, \nu) = -\langle p_D, \nu \rangle_{\partial\Omega}$$

is satisfied for all $(v, \nu) \in X \times M$. Now we are in the position to apply local Kirchhoff transformations to shift the remaining nonlinearities from the subdomains Ω_i to the local boundaries $\partial\Omega_i$. We therefore introduce the generalized pressure $u \in X$ as $u_{|\Omega_i} := \kappa_i(p_{|\Omega_i})$ which satisfies, see [7],

$$\nabla u_{|\Omega_i} = k_i(\theta_i(p_{|\Omega_i})) \nabla p_{|\Omega_i}.$$

The mapping κ_i is a superposition operator induced by $\kappa_i : \mathbb{R} \to \mathbb{R}$ which is defined as

$$\kappa_i(r) = \int_0^r k_i(\theta_i(s)) \,\mathrm{d}s.$$

It can be shown that the nonlinear operators $\kappa_i : H^1(\Omega_i) \to H^1(\Omega_i)$ are continuous and bounded. If there exist positive constants $c_i > 0$ such that $k_i(s) \geq c_i$ for all $s \in \mathbb{R}$, i.e. κ_i being monotone, then the inverse operators κ_i^{-1} exist and are again continuous and bounded. Using these local nonlinear operators, we can define

$$\iota_i := \theta_i \circ \kappa_i^{-1}, \quad c(u,\nu) := -\sum_{i=1}^{N} \langle \kappa_i^{-1}(u_{|\Omega_i}), \nu \rangle_{\partial\Omega_i},$$

and we finally obtain a variational problem to find $(u, \lambda) \in X \times M$, such that

$$\begin{aligned} \sum_{i=1}^{N} \left(\int_{\Omega_i} \frac{n}{\tau} \iota(u) v \,\mathrm{d}\mathbf{x} + \int_{\Omega_i} \frac{K}{\mu} \nabla u \cdot \nabla v \,\mathrm{d}\mathbf{x} \right) + b(v,\lambda) &= \langle F, v \rangle_{\Omega}, \\ c(u,\nu) &= -\langle p_D, \nu \rangle_{\partial\Omega} \end{aligned} \tag{3}$$

is satisfied for all $(v, \nu) \in X \times M$. The variational problem (3) is by construction equivalent to (2), and hence we conclude unique solvability of (3).

3 Mortar finite element discretization

For the discretization of the variational problem (3) we use the mortar finite element method, see [9]. Let $\mathcal{T}_{h,i}$ be a local triangulation of the subdomain Ω_i, $i = 1, \ldots, N$, see Fig. 4. Note that the local triangulations do not have to coincide at neighbouring interfaces. With $\Gamma_{D,i} := \Gamma_D \cap \partial\Omega_i$ we define for each subdomain Ω_i the space

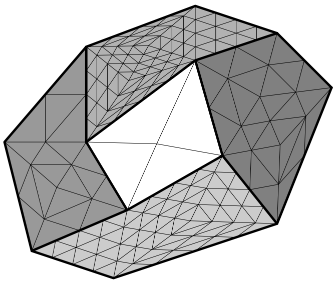


Fig. 4 Triangulation

$$H^1_\star(\Omega_i) := \begin{cases} H^1(\Omega_i) & \text{if meas}\, \Gamma_{D,i} = 0, \\ H^1_{0,\Gamma_{D,i}}(\Omega_i) & \text{else.} \end{cases}$$

We define the local finite element ansatz spaces $X_{h,i} := \mathcal{S}^1(\mathcal{T}_{h,i}) \cap H^1_\star(\Omega_i)$ as the space of all piecewise linear and continuous functions in Ω_i. The global ansatz space is then defined as $X_h := \prod_{i=1}^{N} X_{h,i}$. To define a discrete ansatz space for the Lagrange multiplier $\lambda \in M$ we consider each interface Γ_{ij} with $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $i \neq j$, separately. For a nonempty interface Γ_{ij} we have two neighbouring subdomains and their triangulations $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$. In view of a better approximation property, we choose the finer triangulation and denote its index by m_{ij}. The mesh $\mathcal{I}_{h,ij}$ of the interface Γ_{ij} is induced by $\mathcal{T}_{h,m_{ij}}$, that is $\mathcal{I}_{h,ij} = \mathcal{T}_{h,m_{ij}}|_{\Gamma_{ij}}$. By $\mathcal{I}'_{h,ij}$ we denote a modified dual mesh, i.e. we define $M_{h,ij} := \mathcal{S}^0(\mathcal{I}'_{h,ij})$ to be the space of all piecewise constant functions on the dual mesh, see Fig. 5. The global ansatz space is then defined as the product space $M_h := \prod_{\Gamma_{ij}} M_{h,ij}$. By construction, $u_h \in X_h$ satisfies $u_h = 0$ on Γ_D, and the discrete Lagrange multiplier $\lambda_h \in M_h$ are just defined on the interfaces within Ω. If we assume, that there exists a discrete extension $u_{h,D}$, satisfying the inhomogeneous Dirichlet boundary conditions, we obtain the following discrete nonlinear variational problem to find $(u_h, \lambda_h) \in X_h \times M_h$ such that $\widetilde{u}_h := u_h + u_{h,D}$ satisfies

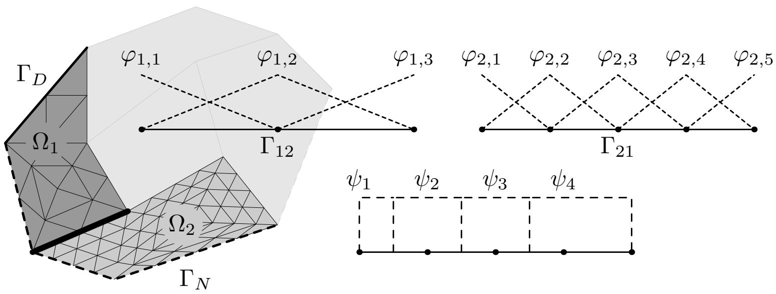


Fig. 5 Construction of ansatz space for Lagrange multiplier in $\mathbb{R}^2$

$$\sum_{i=1}^{N}\left(\int_{\Omega_i}\frac{n}{\tau}\iota(\widetilde{u}_h)v_h\,\mathrm{d}\mathbf{x}+\int_{\Omega_i}\frac{K}{\mu}\nabla\widetilde{u}_h\cdot\nabla v_h\,\mathrm{d}\mathbf{x}\right)+b(v_h,\lambda_h)=\langle F,v_h\rangle_\Omega,$$
$$c(u_h,\nu_h)=0$$

for all $(v_h,\nu_h)\in X_h\times M_h$. Since $M_{h,ij}\subset L_2(\Gamma_{ij})$, we can rewrite

$$b(v_h,\lambda_h):=-\sum_{\Gamma_{ij}}(v_{h|\Omega_i}-v_{h|\Omega_j},\lambda_h)_{\Gamma_{ij}}$$

as well as

$$c(v_h,\lambda_h):=b(\kappa^{-1}(v_h),\lambda_h)=-\sum_{\Gamma_{ij}}(\kappa_i^{-1}(v_{h|\Omega_i})-\kappa_j^{-1}(v_{h|\Omega_j}),\lambda_h)_{\Gamma_{ij}}.$$

Since the discrete variational problem is still nonlinear, we apply Newton's method and obtain the linearized problem: For $\widetilde{w}_h:=w_h+u_{h,D}$, $w_h\in X_h$, find $(u_h,\lambda_h)\in X_h\times M_h$, such that

$$\sum_{i=1}^{N}\left(\int_{\Omega_i}\frac{n}{\tau}\iota'(\widetilde{w}_h)u_hv_h\,\mathrm{d}\mathbf{x}+\int_{\Omega_i}\frac{K}{\mu}\nabla u_h\cdot\nabla v_h\,\mathrm{d}\mathbf{x}\right)+b(v_h,\lambda_h)=\langle\widetilde{F},v_h\rangle_\Omega,$$
$$c'(\widetilde{w}_h,u_h,\nu_h)=\langle\widetilde{G},\nu_h\rangle_S \tag{4}$$

is satisfied for all $(v_h,\nu_h)\in X_h\times M_h$. The linear forms of the discrete and linearized variational problem (4) are

$$\langle\widetilde{F},v_h\rangle_\Omega=\langle F,v_h\rangle_\Omega+\langle\overline{F},v_h\rangle_\Omega,\quad\langle\widetilde{G},\nu_h\rangle_S:=c'(\widetilde{w}_h,w_h,\nu_h)-c(\widetilde{w}_h,\nu_h)$$

with $c'(\widetilde{w}_h,u_h,\nu_h):=b\big((\kappa^{-1})'(\widetilde{w}_h)u_h,\nu_h\big)$ and

$$\langle\overline{F},v_h\rangle_\Omega:=\sum_{i=1}^{N}\left(\int_{\Omega_i}\frac{n}{\tau}\big(\iota'(\widetilde{w}_h)\widetilde{w}_h-\iota(\widetilde{w}_h)\big)v_h\,\mathrm{d}\mathbf{x}-\int_{\Omega_i}\frac{K}{\mu}\nabla u_{h,D}\cdot\nabla v_h\,\mathrm{d}\mathbf{x}\right).$$

The stability and error analysis of the mixed formulation (4) follows from related stability conditions of the underlying bilinear forms and appropriate finite element methods, see [6].

4 Numerical example

As an example we consider the domain $\Omega = (0,1) \times (0,2) \subset \mathbb{R}^2$, see Fig. 6, with Dirichlet conditions on $\Gamma_D := (0,1)\times\{2\}$, while on the remaining boundary Γ_N we have Neumann boundary conditions. The four layers behave like sand, sandy loam, loam and sand, see [6]. We assume that there are no sources or sinks within Ω, i.e. $f \equiv 0$. On Γ_D we prescribe a pressure which increases in time, that is

$$p_D(\mathbf{x}, t) := \begin{cases} -0.5\,(10 - t) & t < 10, \\ 0.0 & t \geq 10. \end{cases}$$

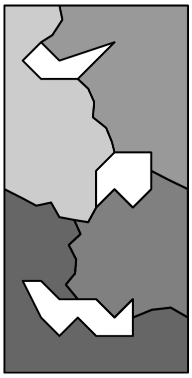
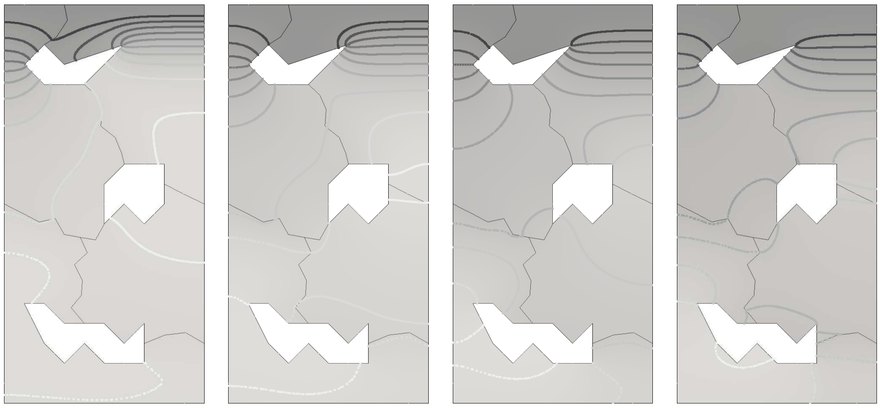


Fig. 6 Triangulation

On Γ_N we prescribe the no–outflow–condition $p_N(\mathbf{x}, t) \equiv 0$. Since we approximate the solution of the transformed variational problem (3), we have to consider the Dirichlet datum u_D for the generalized pressure which is given as $u_D(\mathbf{x}, t) = \kappa_i(p_D(\mathbf{x}, t))$ for $\mathbf{x} \in \Gamma_{D,i}$. The Neumann datum remains unchanged. The following snapshots show contour lines of the pressure p, which can be computed by the application of the inverse transformation, that is $p_{|\Omega_i} = \kappa_i^{-1}(u_{|\Omega_i})$. Due to the choice of the data, the problem evolutes to a pure diffusion equation. That is why the snapshots were taken at $t = 0, 250, 500, 1000, 2000, 4000, 8000, 10000$.



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