A BDDC algorithm for weak Galerkin discretizations

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1 Introduction

The weak Galerkin (WG) methods are a class of nonconforming finite element methods, which were first introduced for a second order elliptic problem in Wang and Ye [2014]. The idea of the WG is to introduce weak functions and their weak derivatives as distributions, which can be approximated by polynomials of different degrees. For second elliptic problems, weak functions have the form of $v = \{v_0, v_b\}$, where v_0 is defined inside each element and v_b is defined on the boundary of the element. v_0 and v_b can both be approximated by polynomials. The gradient operator is approximated by a *weak gradient* operator, which is further approximated by polynomials. These weakly defined functions and derivatives make the WG methods highly flexible and these WG methods have been extended to different applications such as Darcy in Lin et al. [2014], Stokes in Wang and Ye [2016], bi-harmonic in Mu et al. [2014], Maxwell in Mu et al. [2015c], Helmholtz in Mu et al. [2015b], and Brinkman equations in Mu et al. [2014]. In Mu et al. [2015a], the optimal order of polynomial spaces is studied to minimize the number of degrees of freedom in the computation.

The WG methods are closely related to the hybridizable discontinuous Galerkin (HDG) methods, which were introduced by Cockburn and his collaborators in Cockburn et al. [2009]. As most DG methods, the WG methods result in a large number of degrees of freedom and therefore require solving large linear systems with condition number deteriorating with the refinement of the mesh. Efficient fast solvers for the resulting linear system are necessary. However, so far there are relatively few fast solvers for the WG methods. Some multigrid methods, based on conforming finite element discretization, are studied in Chen et al. [2015].

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The BDDC algorithms, introduced by Dohrmann for second order elliptic problem in Dohrmann [2003], see also Mandel and Dohrmann [2003], Mandel et al. [2005], are non-overlapping domain decomposition methods, which are similar to the balancing Neumann-Neumann (BNN) algorithms. In the BDDC algorithm, the coarse problems are given in terms of a set of primal constraints. An important advantage with such a coarse problem is that the Schur complements that arise in the computation will all be invertible. The BDDC algorithms have been extended to the second order elliptic problem with mixed and hybrid formulations in Tu [2005, 2007] and the Stokes problem in Li and Widlund [2006b].

In this paper, we apply the BDDC preconditioner directly to the system arising from the WG discretization and estimate the condition number of the resulting preconditioned operator using its spectral equivalence with that of a hybridized RT method, which have been studied in Tu [2007].

The rest of the paper is organized as follows. An elliptic problem and its WG discretization are described in Section 2. We introduce the BDDC algorithms in Section 3 and analyze the condition number of the resulting preconditioned operator in Section 4. Finally, some computational results are given in Section 5.

2 An elliptic problem and its WG discretization

We consider the following elliptic problem on a bounded polygonal domain Ω, in two dimensions, with a Dirichlet boundary condition:

$$\begin{cases} -\nabla \cdot (\rho \nabla u) = f \text{ in } & \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where ρ is a positive definite matrix function with entries in $L^\infty(\Omega)$ satisfying

$$\xi^T \rho(\mathbf{x}) \xi \geq \alpha \left\|\xi\right\|^2, \quad \text{for a.e.} \quad \mathbf{x} \in \Omega,$$

for some positive constant α, $f \in L^2(\Omega)$, and $g \in H^{1/2}(\partial\Omega)$. Without loss of generality, we assume that $g = 0$. If Ω is convex or has a C^2 boundary, the equation (1), with sufficiently smooth coefficient ρ, has a unique solution $u \in H^2(\Omega)$.

We will approximate u by introducing discontinuous finite element spaces. Let $\mathcal{T}_h$ be a shape-regular and quasi-uniform triangulation of Ω and denote an the element in $\mathcal{T}_h$ by κ. Let h_κ be the diameter of κ and the mesh size be $h = \max_{\kappa \in \mathcal{T}_h} h_T$. Define $\mathcal{E}$ to be the union of edges of elements κ. $\mathcal{E}_i$ and $\mathcal{E}_\partial$ are the sets of the edges which are in interior of the domain and on its boundary, respectively.

Let $P_k(D)$ be the space of polynomials of order at most k on D and $\mathbf{P}_k(D) = [P_k(D)]^2$. Define the weak Galerkin finite element spaces associated

with $\mathcal{T}_h$ as:

$$\begin{aligned} V_k &= \{v = \{v_0, v_b\} : v_0|_\kappa \in P_k(\kappa),\ v_b|_e \in P_{k-1}(e), \quad \forall \kappa \in \mathcal{T}_h, e \in \partial\kappa\} \\ &= \{v = \{v_0, v_b\} : v_0 \in W_k,\ v_b \in M_{k-1}\}, \end{aligned}$$

where

$$\begin{aligned} W_k &= \{w_h \in L^2(\Omega) : w_h|_\kappa \in P_k(\kappa), \quad \forall \kappa \in \mathcal{T}_h\}, \\ M_k &= \{\mu_h \in L^2(\mathcal{E}) : \mu_h|_e \in P_k(e), \quad \forall e \in \mathcal{E}\}. \end{aligned}$$

A function $v \in V_k$ has a single value v_b on each $e \in \mathcal{E}$.

Let

$$V_k^0 = \{v \in V_k\ v_b = 0 \text{ on } \partial\Omega\}.$$

Denoted by $\nabla_{w,k-1}$, the discrete weak gradient operator on the finite element space V_k is defined as follows: for $v = \{v_0, v_b\} \in V_k$, on each element $\kappa \in \mathcal{T}_h$, $\nabla_{w,k-1} v|_\kappa \in \mathbf{P}_{k-1}(\kappa)$ is the unique solution of the following equation

$$(\nabla_{w,k-1} v|_\kappa, \mathbf{q})_\kappa = -(v_{0,\kappa}, \nabla \cdot \mathbf{q}) + < v_{b,\kappa}, \mathbf{q} \cdot \mathbf{n} >_{\partial\kappa}, \quad \forall \mathbf{q} \in \mathbf{P}_{k-1}(\kappa),$$

where $v_{0,\kappa}$ and $v_{b,\kappa}$ are the restrictions of v_0 and v_b to κ, respectively, $(u, w)_\kappa = \int_\kappa uwdx$, and $< u, w >_{\partial\kappa} = \int_{\partial\kappa} uwds$. To simplify the notation, we will drop the subscript $k-1$ in the discrete weak gradient operator $\nabla_{w,k-1}$.

The discrete problem resulting from the WG discretization of (1) can be written as: find $u_h = \{u_0, u_b\} \in V_k$ such that

$$a_s(u_h, v_h) = a(u_h, v_h) + s(u_h, v_h) = (f, v_h), \quad \forall v_h = \{v_0, v_b\} \in V_k, \tag{2}$$

where

$$\begin{aligned} a(u_h, v_h) &= \sum_{\kappa \in \mathcal{T}_h} (\rho \nabla_w u_h, \nabla_w v_h)_\kappa, \\ s(u_h, v_h) &= \sum_{\kappa \in \mathcal{T}_h} h_\kappa^{-1} < Q_b u_0 - v_b, Q_b v_0 - v_b >_{\partial\kappa}, \end{aligned}$$

and where Q_b is the L^2-projection from $L^2(e)$ to $P_{k-1}(e)$, for $e \in \partial\kappa$. In Mu et al. [2015a], (2) is proved to have a unique solution and the approximation properties of the WG methods are also studied.

Given a $u_h \in V_k$, let $\mathbf{q}|_\kappa = \nabla_w u_h|_\kappa$ and write (2) as a system of $\mathbf{q}$, u_0, u_b, which is similar to the linear system resulting from the HDG discretization with the local stabilization parameter h_κ^{-1}. Given the value of u_b on $\partial\kappa$, $\mathbf{q}_\kappa$ and u_0 can be uniquely determined, see Cockburn et al. [2009]. Therefore, by eliminating $\nabla_w u|_\kappa$ and u_0 locally in each element, (2) can be reduced to a system in u_b only

$$Au_b = b, \tag{3}$$

where b is the corresponding right-hand-side function.

In next section, we will develop a BDDC algorithm to solve the system in (3) for the u_b. To make the notation simple, we will denote u_b by λ and the finite element space for u_b by $\Lambda = \{\mu \in M_{k-1} : \mu|_e = 0 \ \forall e \in \partial\Omega\}$.

3 The BDDC algorithms and condition number bound

We decompose Ω into N non-overlapping subdomains Ω_i with diameters H_i, $i = 1, \cdots, N$, and set $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse triangles and that the number of such triangles forming an individual subdomain is uniformly bounded. We also assume $\rho(\mathbf{x})$, the coefficient of (1), is constant in each subdomain. We reduce the global problem (3) to a subdomain interface problem. Let Γ be the interface between subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h = (\cup_{i \neq j} \partial\Omega_{i,h} \cap \partial\Omega_{j,h}) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is the set of nodes on $\partial\Omega$.

We can decompose Λ into the subdomain interior and interface parts as

$$\Lambda = \bigoplus_{i=1}^{N} \Lambda_I^{(i)} \bigoplus \widehat{\Lambda}_\Gamma.$$

We denote the subdomain interface space of Ω_i by $\Lambda_\Gamma^{(i)}$, and the associate product space by $\Lambda_\Gamma = \prod_{i=1}^N \Lambda_\Gamma^{(i)}$. $R_\Gamma^{(i)}$ is the operator which maps functions in the continuous interface numerical trace space $\widehat{\Lambda}_\Gamma$ to their subdomain components in the space $\Lambda_\Gamma^{(i)}$. The direct sum of the $R_\Gamma^{(i)}$ is denoted by R_Γ. We can eliminate the subdomain interior variables $\lambda_I^{(i)}$ in each subdomain independently and define the subdomain Schur complement $S_\Gamma^{(i)}$ by: given $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$, $S_\Gamma^{(i)} \lambda_\Gamma^{(i)}$ is determined by such that

$$\begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \lambda_I^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ S_\Gamma^{(i)} \lambda_\Gamma \end{bmatrix}. \tag{4}$$

The global interface problem is assembled from the subdomain interface problems, and can be written as: find $\lambda_\Gamma \in \widehat{\Lambda}_\Gamma$, such that

$$\widehat{S}_\Gamma \lambda_\Gamma = b_\Gamma, \tag{5}$$

where $b_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} b_\Gamma^{(i)}$, and $\widehat{S}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} S_\Gamma^{(i)} R_\Gamma^{(i)}$. Thus, $\widehat{S}_\Gamma$ is a symmetric, positive definite operator defined on the interface space $\widehat{\Lambda}_\Gamma$. We will propose a BDDC preconditioner for solving (5) with a preconditioned conjugate gradient method.

In order to introduce the BDDC precondition, we first introduce a partially assembled interface space $\widetilde{\Lambda}_\Gamma$ by

$$\widetilde{\Lambda}_\Gamma = \widehat{\Lambda}_\Pi \bigoplus \Lambda_\Delta = \widehat{\Lambda}_\Pi \bigoplus \left(\prod_{i=1}^{N} \Lambda_\Delta^{(i)} \right).$$

Here, $\widehat{\Lambda}_\Pi$ is the coarse level, primal interface space which is spanned by subdomain interface edge basis functions with constant values at the nodes of the edge for two dimensions. We change the variables so that the degree of freedom of each primal constraint is explicit, see Li and Widlund [2006a] and Klawonn and Widlund [2006]. The new variables are called the primal unknowns. The space Λ_Δ is the direct sum of the $\Lambda_\Delta^{(i)}$, which are spanned by the remaining interface degrees of freedom with a zero average over each edge/face. In the space $\widetilde{\Lambda}_\Gamma$, we relax most continuity constraints across the interface but retain the continuity at the primal unknowns, which makes all the linear systems nonsingular.

We need to introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)}$ restricts functions in the space $\widetilde{\Lambda}_\Gamma$ to the components $\Lambda_\Gamma^{(i)}$ of the subdomain Ω_i. $R_\Delta^{(i)}$ maps the functions from $\widehat{\Lambda}_\Gamma$ to $\Lambda_\Delta^{(i)}$, its dual subdomain components. $R_{\Gamma\Pi}$ is a restriction operator from $\widehat{\Lambda}_\Gamma$ to its subspace $\widehat{\Lambda}_\Pi$. $\overline{R}_\Gamma : \widetilde{\Lambda}_\Gamma \to \Lambda_\Gamma$ is the direct sum of the $\overline{R}_\Gamma^{(i)}$ and $\widetilde{R}_\Gamma : \widehat{\Lambda}_\Gamma \to \widetilde{\Lambda}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_\Delta^{(i)}$. We define a positive scaling factor $\delta_i^\dagger(x)$ as follows: for $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(x) = \frac{\rho_i^\gamma(x)}{\sum_{j \in \mathcal{N}_x} \rho_j^\gamma(x)}, \qquad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$. We note that $\delta_i^\dagger(x)$ is constant on each edge/face, since we assume that the $\rho_i(x)$ is constant in each subdomain. Multiplying each row of $R_\Delta^{(i)}$, with the scaling factor $\delta_i^\dagger(x)$, gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_{D,\Delta}^{(i)}$.

The partially assembled interface Schur complement is defined by $\widetilde{S}_\Gamma = \overline{R}_\Gamma^T \mathrm{diag}(S_\Gamma^{(i)}) \overline{R}_\Gamma$ and the preconditioned BDDC operator is then of the form: find $\lambda_\Gamma \in \widehat{\Lambda}_\Gamma$, such that

$$\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} \widehat{S}_\Gamma \lambda_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} b_\Gamma. \tag{6}$$

This preconditioned problem is the product of two symmetric, positive definite operators and we can use the preconditioned conjugate gradient method to solve it.

4 Condition number bound

We first introduce one useful norm, which is defined in Gopalakrishnan [2003] and Cockburn et al. [2014]. For any domain D, we denote the L^2 norm by $\|\cdot\|_D$. For any $\lambda \in \Lambda(D)$, define

$$|||\lambda|||_D^2 = \left(\frac{1}{h} \sum_{\kappa \in \mathcal{T}_h, \kappa \subseteq \bar{D}} \|\lambda - m_\kappa(\lambda)\|^2_{L^2(\partial\kappa)} \right)^{1/2}, \tag{7}$$

where $m_\kappa = \frac{1}{|\partial\kappa|} \int_{\partial\kappa} \lambda ds$, and $|\partial\kappa|$ is the length of the boundary of κ.

We define the interface averaging operator E_D, by

$$E_D = \widetilde{R}_\Gamma \widetilde{R}^T_{D,\Gamma}, \tag{8}$$

which computes a weighted average across the subdomain interface Γ and then distributes the averages to the degrees of freedom on the boundary of the subdomains.

Similarly to the proof of [Tu and Wang, 2016, Lemma 5], using the spectral equivalence of A, defined in (3), the linear system from the hybridized RT method, and the norm defined in (7), we obtain that the interface averaging operator E_D satisfies the following bound:

Lemma 1. *For any $\lambda_\Gamma \in \widetilde{\Lambda}_\Gamma$,*

$$|E_D \lambda_\Gamma|^2_{\widetilde{S}_\Gamma} \leq C \left(1 + \log \frac{H}{h}\right)^2 |\lambda_\Gamma|^2_{\widetilde{S}_\Gamma},$$

where C is a positive constant independent of H, h, and the coefficient of (1).

As in the proof of [Li and Widlund, 2006b, Theorem 1] and [Tu and Wang, 2016, Theorem 1], using Lemma 1, we can obtain

Theorem 1. *The condition number of the preconditioned operator $M^{-1}\widehat{S}_\Gamma$ is bounded by $C(1 + \log \frac{H}{h})^2$, where C is a constant which is independent of h, H, and the coefficients ρ of (1).*

5 Numerical Experiments

We have applied our BDDC algorithms to the model problem (1), where $\Omega = [0,1]^2$. We decompose the unit square into $N \times N$ subdomains with the side-length $H = 1/N$. Equation (1) is discretized, in each subdomain, by the k*th*-order WG method with a element diameter h. The preconditioned conjugate

Table 1 Performance with $H/h = 8/\#$ sub=64

		$\rho = 1$				ρ checkboard pattern			
		$k = 1$		$k = 1$		$k = 1$		$k = 2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	2.22	6	3.50	7	1.80	5	2.37	5
	8×8	2.45	13	3.85	16	2.08	9	2.76	10
	16×16	2.45	14	3.86	17	2.16	14	2.87	15
	24×24	2.46	14	3.87	17	2.17	15	2.89	15
	32×32	2.46	14	3.87	17	2.18	15	2.90	16
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	1.78	11	2.90	14	1.67	9	2.33	10
	8	2.45	13	3.86	16	2.08	9	2.76	10
	16	3.29	15	4.95	18	2.49	10	3.18	10
	24	3.85	17	5.67	18	2.74	10	3.43	11
	32	4.28	17	6.21	19	2.91	10	3.60	11

gradient iteration is stopped when the relative l_2-norm of the residual has been reduced by a factor of 10^6.

We have carried out two different sets of experiments to obtain iteration counts and condition number estimates. In the first set of experiments, we take the coefficient $\rho \equiv 1$. In the second set of experiments, we take the coefficient $\rho = 1$ in half the subdomains and $\rho = 1000$ in the neighboring subdomains, in a checkerboard pattern. All the experimental results are fully consistent with our theory.

Acknowledgements This work was supported in part by National Science Foundation Contract No. DMS-1419069.

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