Dual-Primal Domain Decomposition Methods for the Total Variation Minimization

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1 Introduction

Image denoising problem is one of classical problems in imaging science. In 1992, Rudin *et al.* [9] proposed the following denoising model,

$$\min_{u \in BV(\Omega)} \left\{ \frac{\lambda}{2} \int_{\Omega} (u-f)^2 \, dx + \int_{\Omega} |\nabla u| \, dx \right\}, \tag{1}$$

where Ω is the domain of image and f is an observed image corrupted by noise. Here, the space of functions of bounded variation is defined as

$$BV(\Omega) = \left\{ u \in L^1(\Omega) : \sup_{\phi \in C_c^1(\Omega, R^2), \|\phi\|_\infty \le 1} \int_{\Omega} u(x) \, \mathrm{div}\phi(x) \, dx < \infty \right\}.$$

This model has an anisotropic diffusion property so that the edge of the image is preserved.

Recently, as the number of CPUs and cores in a computer are increased, there have been attempts to solve this problem parallely using the domain decomposition technique. For example, see [3, 4, 5, 6, 7, 8, 11]. Since the problem is nonsmooth and not separable, it is not easy to show the convergence of the domain decomposition algorithm. Tseng [10] showed that if the function is separable, block Gauss-Seidel algorithm converges to the minimizer, but (1) is not of this case. Fornasier *et al.*[6] and Xu *et al.*[11] used overlapping domain decomposition methods to overcome this difficulty. Also, Fornasier and Schönlieb [5] proved the convergence of nonoverlaping domain decomposition method under certain assumptions.

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The main point of the domain decomposition approach is that instead of solving one large problem, several small problems are solved in parallel to reduce the computing time. In [4], Fornasier pointed out that the subproblems should reproduce the original problem at smaller dimensions, but it is difficult to satisfy this requirement since the boundary conditions of local subdomain problems should be considered.

In this paper, we propose new domain decomposition techniques considering this requirement. First we decompose the domain of the dual form of (1), discovered by Chambolle [1], into nonoverlapping rectangular subdomains. Then we change the local dual problems into the equivalent primal forms so that our methods use same algorithms to solve the original problem and local problems which can be solved in parallel.

2 Preliminaries

We assume that the image domain Ω consists of $N \times N$ discrete points, i.e.,

$$\Omega = [1, 2, ..., N] \times [1, 2, ..., N].$$

We define the function space V as a set of functions from Ω into $\mathbb{R}$ and V^* as a set of functions from Ω into $\mathbb{R}^2$ with the usual Euclidean inner product.

The operator ∇: $V \to V^*$ is defined by

$$(\nabla u)^1_{ij} = \begin{cases} u_{i+1,j} - u_{ij} & \text{for} \quad i = 1, ..., N-1, \\ 0 & \text{for} \quad i = N, \end{cases}$$
$$(\nabla u)^2_{ij} = \begin{cases} u_{i,j+1} - u_{ij} & \text{for} \quad j = 1, ..., N-1, \\ 0 & \text{for} \quad j = N. \end{cases}$$

We define an operator div: $V^* \to V$ by $-\nabla^*$ (the adjoint of ∇).

For simplicity, we decompose the image domain Ω into two subsets Ω_1 and Ω_2 such that

$$\Omega_1 = [1, ..., N] \times [1, ..., N_1],$$
$$\Omega_2 = [1, ..., N] \times [N_1, ..., N].$$

Then the interface Γ is

$$\Gamma = [1, ..., N] \times [N_1].$$

For each subdomain, we define the local function spaces

$$
\begin{aligned}
V_1 &= \{u \in V \,|\, \mathrm{supp}(u) \subset \Omega_1\}, \\
V_2 &= \{u \in V \,|\, \mathrm{supp}(u) \subset \Omega_2\}, \\
V_1^* &= \{\mathbf{p} \in V^* \,|\, \mathrm{supp}(\mathbf{p}) \subset \Omega_1 \backslash \Gamma\}, \\
V_2^* &= \{\mathbf{p} \in V^* \,|\, \mathrm{supp}(\mathbf{p}) \subset \Omega_2\}.
\end{aligned}
$$

Note that $V = V_1 + V_2$, and $V^* = V_1^* \oplus V_2^*$.

We also define the local operators as the restriction of global operators ∇ and div to these spaces. More precisely, the operator $\nabla_{\Omega_1}\colon V_1 \to V_1^*$ is defined as

$$
\begin{aligned}
(\nabla_{\Omega_1} u)^1_{ij} &= \begin{cases} u_{i+1,j} - u_{ij} & \text{for} \quad i = 1, ..., N-1, \\ 0 & \text{for} \quad i = N, \end{cases} \\
(\nabla_{\Omega_1} u)^2_{ij} &= \begin{cases} u_{i,j+1} - u_{ij} & \text{for} \quad j = 1, ..., N_1 - 1, \\ 0 & \text{for} \quad j = N_1, ..., N. \end{cases}
\end{aligned}
$$

We define $\nabla_{\Omega_2}\colon V_2 \to V_2^*$ with similar manner. We define $\mathrm{div}_{\Omega_1}\colon V_1^* \to V_1$ by $-\nabla^*_{\Omega_1}$ and $\mathrm{div}_{\Omega_2}\colon V_2^* \to V_2$ by $-\nabla^*_{\Omega_2}$.

3 Proposed Algorithms

We consider the following discrete version of (1),

$$
\min_{u \in V} \left\{ \frac{\lambda}{2} \|u - f\|_V^2 + \sum_{\Omega} |\nabla u| \right\} \quad \text{for } f \in V. \tag{2}
$$

Our result is based on the following two propositions which are summarized in Section 2 of [1].

Proposition 1. *The following two statements are equivalent.*

$$
(i) \quad \bar{u} = \arg\min_{u \in V} \left\{ \frac{\lambda}{2} \|u - f\|_V^2 + \sum_{\Omega} |\nabla u| \right\}
$$

$$
(ii) \quad \text{There exists } \mathbf{p} \in V^* \text{ such that } \begin{cases} f - \frac{1}{\lambda}\mathrm{div}\mathbf{p} = \bar{u} \\ \mathbf{p} = \arg\min_{|\mathbf{p}| \le \mathbf{1}} \left\| \frac{1}{\lambda}\mathrm{div}\mathbf{p} - f \right\|_V^2 \end{cases}
$$

Proposition 2 (Optimality Condition). *The following two statements are equivalent.*

$$(i)\ \ \mathbf{p} = \arg\min_{|\mathbf{p}|\leq \mathbf{1}} \left\| \frac{1}{\lambda}\mathrm{div}\mathbf{p} - f \right\|_V^2$$

$$(ii)\ \ \begin{cases} -\nabla(\frac{1}{\lambda}\mathrm{div}\mathbf{p} - f) + |\nabla(\frac{1}{\lambda}\mathrm{div}\mathbf{p} - f)|\mathbf{p} = 0 \quad in \quad \Omega \\ |\mathbf{p}| \leq \mathbf{1} \end{cases}$$

Now, we propose the block Gauss-Seidel algorithm for the primal problem (2).

Algorithm: Block Gauss-Seidel

Initialize $u_2^{(0)} := 0,\ f_2^{(0)} := 0$
For $n = 0, 1, ...$

$$\begin{aligned}
(f_1^{(n+1)})_{ij} &= (u_2^{(n)} - f_2^{(n)} + f)_{ij} \quad \text{for} \quad (i,j) \in \Omega_1 \\
u_1^{(n+1)} &= \arg\min_{u_1 \in V_1} \left\{ \frac{\lambda}{2}\|u_1 - f_1^{(n+1)}\|_{V_1}^2 + \sum_{\Omega_1 \backslash \Gamma} |\nabla_{\Omega_1} u_1| \right\} \\
(f_2^{(n+1)})_{ij} &= (u_1^{(n+1)} - f_1^{(n+1)} + f)_{ij} \quad \text{for} \quad (i,j) \in \Omega_2 \\
u_2^{(n+1)} &= \arg\min_{u_2 \in V_2} \left\{ \frac{\lambda}{2}\|u_2 - f_2^{(n+1)}\|_{V_2}^2 + \sum_{\Omega_2} |\nabla_{\Omega_2} u_2| \right\} \\
u^{(n+1)} &= f - f_1^{(n+1)} - f_2^{(n+1)} + u_1^{(n+1)} + u_2^{(n+1)}
\end{aligned}$$

end

Theorem 1. *The sequence $u^{(n)}$ of the block Gauss-Seidel algorithm converges to the minimizer of the problem* (2).

Proof. By the proposition 1, $u_1^{(n)}$, $u_2^{(n)}$, $f_1^{(n)}$, $f_2^{(n)}$, and $u^{(n)}$ are bounded sequences. Suppose that $u^{(\infty)}$ is the limit point of the sequence $u^{(n)}$. Then there exists a subsequence $u^{(n_k)}$ which converges to $u^{(\infty)}$. Now we claim that $u^{(\infty)}$ is the solution of (2).

By the propositions 1 and 2, there exists $\mathbf{p}_1^{(n)} \in V_1^*$, $\mathbf{p}_2^{(n)} \in V_2^*$ for all $n \geq 1$ such that in $\Omega_1 \backslash \Gamma$,

$$\begin{cases} f_1^{(n)} - \frac{1}{\lambda}\mathrm{div}_{\Omega_1}\mathbf{p}_1^{(n)} = u_1^{(n)}, \\ -\nabla_{\Omega_1}(\frac{1}{\lambda}\mathrm{div}_{\Omega_1}\mathbf{p}_1^{(n)} - f_1^{(n)}) + |\nabla_{\Omega_1}(\frac{1}{\lambda}\mathrm{div}_{\Omega_1}\mathbf{p}_1^{(n)} - f_1^{(n)})|\mathbf{p}_1^{(n)} = 0\,, \\ |\mathbf{p}_1^{(n)}| \leq \mathbf{1}, \end{cases}$$

and in Ω_2,

$$\begin{cases} f_2^{(n)} - \frac{1}{\lambda}\mathrm{div}_{\Omega_2}\mathbf{p}_2^{(n)} = u_2^{(n)}, \\ -\nabla_{\Omega_2}(\frac{1}{\lambda}\mathrm{div}_{\Omega_2}\mathbf{p}_2^{(n)} - f_2^{(n)}) + |\nabla_{\Omega_2}(\frac{1}{\lambda}\mathrm{div}_{\Omega_2}\mathbf{p}_2^{(n)} - f_2^{(n)})|\mathbf{p}_2^{(n)} = 0\,, \\ |\mathbf{p}_2^{(n)}| \leq \mathbf{1}. \end{cases}$$

By refining the subsequences, we can assume that $f_1^{(n_{k_j})} \to f_1^{(\infty)}$, $f_2^{(n_{k_j})} \to f_2^{(\infty)}$, $p_1^{(n_{k_j})} \to p_1^{(\infty)}$, $p_2^{(n_{k_j})} \to p_2^{(\infty)}$, $p_2^{(n_{k_j}-1)} \to \tilde{p}_2^{(\infty)}$, $u_1^{(n_{k_j})} \to u_1^{(\infty)}$, and $u_2^{(n_{k_j})} \to u_2^{(\infty)}$. By the proposition 2, the following monotone property holds for all $n \geq 1$;

$$\begin{aligned}\left\|\frac{1}{\lambda}\mathrm{div}(\mathbf{p}_1^{(n)} + \mathbf{p}_2^{(n)}) - f\right\| &\geq \left\|\frac{1}{\lambda}\mathrm{div}(\mathbf{p}_1^{(n+1)} + \mathbf{p}_2^{(n)}) - f\right\| \\ &\geq \left\|\frac{1}{\lambda}\mathrm{div}(\mathbf{p}_1^{(n+1)} + \mathbf{p}_2^{(n+1)}) - f\right\|\end{aligned}$$

so that $\mathrm{div}(\mathbf{p}_1^{(\infty)} + \mathbf{p}_2^{(\infty)}) = \mathrm{div}(\mathbf{p}_1^{(\infty)} + \tilde{\mathbf{p}}_2^{(\infty)})$. As $j \to \infty$, in $\Omega_1 \backslash \Gamma$,

$$\begin{cases} f_1^{\infty} - \frac{1}{\lambda}\mathrm{div}_{\Omega_1}\mathbf{p}_1^{(\infty)} = u_1^{(\infty)}, \\ -\nabla_{\Omega_1}(\frac{1}{\lambda}\mathrm{div}_{\Omega_1}\mathbf{p}_1^{(\infty)} - f_1^{(\infty)})) + |\nabla_{\Omega_1}(\frac{1}{\lambda}\mathrm{div}_{\Omega_1}\mathbf{p}_1^{(\infty)} - f_1^{(\infty)})|\mathbf{p}_1^{(\infty)} = 0\,, \\ |\mathbf{p}_1^{(\infty)}| \leq \mathbf{1}, \end{cases} \tag{3a}$$

and in Ω_2,

$$\begin{cases} f_2^{(\infty)} - \frac{1}{\lambda}\mathrm{div}_{\Omega_2}\mathbf{p}_2^{(\infty)} = u_2^{(\infty)}, \\ -\nabla_{\Omega_2}(\frac{1}{\lambda}\mathrm{div}_{\Omega_2}\mathbf{p}_2^{(\infty)} - f_2^{(\infty)}) + |\nabla_{\Omega_2}(\frac{1}{\lambda}\mathrm{div}_{\Omega_2}\mathbf{p}_2^{(\infty)} - f_2^{(\infty)})|\mathbf{p}_2^{(\infty)} = 0\,, \\ |\mathbf{p}_2^{(\infty)}| \leq \mathbf{1}. \end{cases} \tag{3b}$$

Let $\mathbf{p}^{(\infty)} = \mathbf{p}_1^{(\infty)} + \mathbf{p}_2^{(\infty)}$. We claim that

(i) $f - \dfrac{1}{\lambda}\mathrm{div}\mathbf{p}^{(\infty)} = f - f_1^{(\infty)} - f_2^{(\infty)} + u_1^{(\infty)} + u_2^{(\infty)}$.

(ii) $-\nabla\Big(\dfrac{1}{\lambda}\mathrm{div}\mathbf{p}^{(\infty)} - f\Big) + \Big|\nabla\Big(\dfrac{1}{\lambda}\mathrm{div}\mathbf{p}^{(\infty)} - f\Big)\Big|\mathbf{p}^{(\infty)} = 0$.

(iii) $|\mathbf{p}^{(\infty)}| \leq \mathbf{1}$.

The statement (i) is established by adding (3a) and (3b) and the statement (iii) is trivial. We have

$$\begin{aligned}\nabla_{\Omega_1}\Big(\frac{1}{\lambda}\mathrm{div}_{\Omega_1}\mathbf{p}_1^{(\infty)} - f_1^{(\infty)}\Big) &= \nabla\Big(\frac{1}{\lambda}\mathrm{div}_{\Omega_1}\mathbf{p}_1^{(\infty)} + \frac{1}{\lambda}\mathrm{div}_{\Omega_2}\tilde{\mathbf{p}}_2^{(\infty)} - f\Big) \\ &= \nabla\Big(\frac{1}{\lambda}\mathrm{div}\mathbf{p}^{(\infty)} - f\Big) \quad \text{in} \quad \Omega_1 \backslash \Gamma\,, \\ \nabla_{\Omega_2}\Big(\frac{1}{\lambda}\mathrm{div}_{\Omega_2}\mathbf{p}_2^{(\infty)} - f_2^{(\infty)}\Big) &= \nabla\Big(\frac{1}{\lambda}\mathrm{div}_{\Omega_1}\mathbf{p}_1^{(\infty)} + \frac{1}{\lambda}\mathrm{div}_{\Omega_2}\mathbf{p}_2^{(\infty)} - f\Big) \\ &= \nabla\Big(\frac{1}{\lambda}\mathrm{div}\mathbf{p}^{(\infty)} - f\Big) \quad \text{in} \quad \Omega_2,\end{aligned}$$

which proves the statement (ii) and $u^{(\infty)}$ is the solution of (2). Since the solution of (2) is unique, the result follows. $\square$

Next, we propose the relaxed block Jacobi algorithm as a parallel algorithm.

Algorithm: Relaxed Block Jacobi

Initialize $v_1^{(0)} := 0$, $v_2^{(0)} := 0$.
For $n = 0, 1, \ldots$

$$
\begin{aligned}
(f_1^{(n+1)})_{ij} &= (-v_2^{(n)} + f)_{ij} \quad \text{for} \quad (i,j) \in \Omega_1 \\
(f_2^{(n+1)})_{ij} &= (-v_1^{(n)} + f)_{ij} \quad \text{for} \quad (i,j) \in \Omega_2 \\
\tilde{u}_1^{(n+1)} &= \arg \min_{u_1 \in V_1} \left\{ \frac{\lambda}{2} \|u_1 - f_1^{(n+1)}\|^2 + \sum_{\Omega_1 \setminus \Gamma} |\nabla_{\Omega_1} u_1| \right\} \\
\tilde{u}_2^{(n+1)} &= \arg \min_{u_2 \in V_2} \left\{ \frac{\lambda}{2} \|u_2 - f_2^{(n+1)}\|^2 + \sum_{\Omega_2} |\nabla_{\Omega_2} u_2| \right\} \\
v_1^{(n+1)} &= \frac{v_1^{(n)} + f_1^{(n+1)} - \tilde{u}_1^{(n+1)}}{2} \\
v_2^{(n+1)} &= \frac{v_2^{(n)} + f_2^{(n+1)} - \tilde{u}_2^{(n+1)}}{2} \\
u^{(n+1)} &= f - v_1^{(n+1)} - v_2^{(n+1)}
\end{aligned}
$$

end

Lemma 1. *In the relaxed block Jacobi algorithm, we have* $\|v_1^{(n+1)} - v_1^{(n)}\|_{V_1} \to 0$ *and* $\|v_2^{(n+1)} - v_2^{(n)}\|_{V_2} \to 0$ *as* $n \to \infty$.

Sketch of Proof. By the proposition 1, there exist $\tilde{\mathbf{p}}_1^{(n+1)} \in V_1^*$ and $\tilde{\mathbf{p}}_2^{(n+1)} \in V_2^*$ such that

$$
\begin{aligned}
\tilde{\mathbf{p}}_1^{(n+1)} &= \arg \min_{\mathbf{p}_1 \in V_1^*} \left\| \frac{1}{\lambda} \mathrm{div}_{\Omega_1} \mathbf{p}_1 + v_2^{(n)} - f \right\|_{V_1}, \\
\tilde{\mathbf{p}}_2^{(n+1)} &= \arg \min_{\mathbf{p}_2 \in V_2^*} \left\| \frac{1}{\lambda} \mathrm{div}_{\Omega_2} \mathbf{p}_2 + v_1^{(n)} - f \right\|_{V_2}.
\end{aligned}
$$

By the triangle inequality and minimization property, the result follows. □

With this lemma, one can easily prove the following theorem.

Theorem 2. *The sequence* $u^{(n)}$ *of the relaxed block Jacobi algorithm converges to the minimizer of the problem* (2).

4 Numerical Results

In this section, we compare our domain decomposition algorithms with the first order primal dual algorithm in [2]. We used the following stop criterion to the relaxed block Jacobi algorithm and Algorithm 2 in [2] solving the full dimension problem (2):

$$\frac{\|u^{(n+1)} - u^{(n)}\|_V}{\|u^{(n+1)}\|_V} < 10^{-5}$$

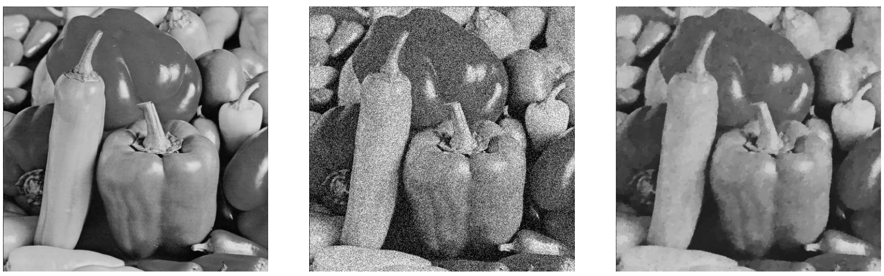
with the parameters $\tau = 1/\sqrt{8}$, $\sigma = 1/\sqrt{8}$, $\gamma = 0.7\lambda$, which are used to run Algorithm 2 in [2]. We choose the weight parameter λ in (1) as 7 empirically. For the local problems, we also used Algorithm 2 in [2] with the following stop criterion

$$\frac{\|u_i^{(n+1)} - u_i^{(n)}\|_V}{\|u_i^{(n+1)}\|_V} < 10^{-6}.$$

We tested two images of size 512×512 and 2048×3072, corrupted by additive zero mean Gaussian noise with variance 0.03. Table 1 shows the performance of the algorithm with the varying number of subdomains.

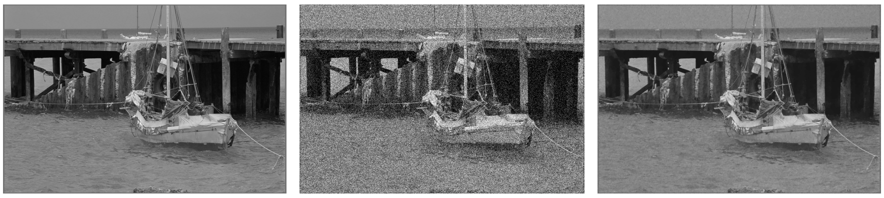
	Peppers 512×512			Boat 2048×3072		
domain	**iter**	**virtual wall-clock time (sec)**	**PSNR**	**iter**	**virtual wall-clock time (sec)**	**PSNR**
1x1	1	3.59	27.39	1	115.48	28.79
2x2	54	6.69	27.39	39	324.12	28.79
4x4	66	2.26	27.39	52	153.13	28.79
8x8	81	1.44	27.39	63	24.83	28.79
16x16	96	1.12	27.39	75	10.28	28.79

Table 1 Results of the proposed algorithm. The results for 1×1 domain are from Algorithm 2 in [2].



(a) (b) (c)

Fig. 1 **(a)** Original clean image of size 512×512, **(b)** Noisy image with Gaussian noise with zero mean and 0.03 variance (PSNR=15.66), **(c)** Denoised image with weight $\lambda = 7$ in (2).



(a) (b) (c)

Fig. 2 **(a)** Original clean image of size 2048×3072, **(b)** Noisy image with Gaussian noise with zero mean and 0.03 variance (PSNR=15.66), **(c)** Denoised image with weight $\lambda = 7$ in (2).

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