A Study of the Effects of Irregular Subdomain Boundaries on Some Domain Decomposition Algorithms

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1 Introduction.

In the standard domain decomposition theory the resulting subdomains are often assumed to have a certain regularity, as in [Toselli and Widlund, 2005, Assumption 4.3], where each subdomain is a finite union of coarse scale elements and the number of coarse elements forming the subdomain are uniformly bounded. This assumption does not always hold. Subdomains might be generated from a mesh partitioner, or be the result of a decomposition scheme with slight or systematic alterations of the subdomain following refinement, e.g. see the type 3 domain in [Dohrmann et al., 2008a, figure 5.1] and the snowflake domain in figure 1. In this paper we will assume that each subdomain is a connected union of fine scale elements.

Several papers, Dohrmann et al. [2008b,a], Klawonn et al. [2008], Widlund [2009], have developed theory for such less regular or irregular subdomains. In these studies the subdomains are assumed to be uniform or John domains; see Dohrmann et al. [2008a], Klawonn et al. [2008] for definitions of these families of domains. While these domains are not necessarily Lipschitz, a number of the tools important to the development of theory of domain decomposition algorithms have been developed for such domains in the plane. We note that the Poincaré inequality is particularly important; see Dohrmann et al. [2008a].

In this paper we primarily consider the Additive Average method, introduced in Bjørstad et al. [1997]. We note that [Toselli and Widlund, 2005, Assumption 4.3]

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was not needed in the original proof. The original proof uses the trace theorem, and to our knowledge this theorem is not available if the subdomains are only John domains. In Dryja and Sarkis [2010], the authors proved a condition number estimate of the Additive Average method for the scalar elliptic equation in $\mathbb{R}^2$ without the use of a trace theorem. Following the setup of Dryja and Sarkis [2010], we have extended the result to $\mathbb{R}^3$ and can show that this convergence estimate, with some modification, holds also when subdomain are John domains. To our knowledge convergence estimates for methods where the subdomains are John or uniform domains have previously only been available for methods in $\mathbb{R}^2$. We have obtained an estimate valid for both $\mathbb{R}^2$ and $\mathbb{R}^3$. In addition, when restricted to $\mathbb{R}^2$ our result may be improved so that it is comparable with the results of Dohrmann and Widlund [2012a]. In this paper we must leave out the proof due to page restrictions.

In certain cases of domain decomposition, the length of the subdomain boundaries can grow with refinement. One example is the snowflake domain shown in figure 1. In Dohrmann and Widlund [2012b,a] it was pointed out that such domains introduce a factor into the condition number bound which depends on the Hausdorff dimension of the resulting boundary as h goes to zero. For the snowflake domain in figure 1, we have a bound of this factor. Numerical results in section 4 are presented to indicate that this factor need to be present in the condition number bound.

This paper has the following layout. In section 2 we present the test problem, assumptions and definitions. In section 3 we introduce the additive average Schwarz preconditioner with convergence estimate as our main result. Finally we present some numerical results in section 4, mainly to illustrate effects of various subdomains on the condition number.

2 The Differential Problem

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v), \quad v \in H_0^1(\Omega), \tag{1}$$

where

$$a(u,v) := (\alpha(\cdot)\nabla u, \nabla v)_{L_2(\Omega)}, \quad f(v) := \int_\Omega f v dx \tag{2}$$

We assume that $\alpha \in L_\infty(\Omega)$, with $\alpha(x) \geq \alpha_0 > 0$ and that $f \in L_2(\Omega)$. Here Ω is a polygonal or polyhedral region in $\mathbb{R}^n$ where $n \in \{2,3\}$. Let $\mathscr{T}^h(\Omega)$ be the shape regular triangulation of Ω into triangular or tetrahedral elements. Let V_h be a space of piecewise linear continuous functions.

$$V_h(\Omega) := \left\{ v \in C_0(\Omega); v_{|e_k} \in P_1(x) \right\},$$

where e_k are elements of $\mathscr{T}^h(\Omega)$ and $P_1(x)$ is the set of linear polynomials.

The finite element problem is then defined as: Find $u_h \in V_h(\Omega)$ such that

$$a(u_h, v) = f(v), \quad v \in V_h(\Omega). \tag{3}$$

2.1 Assumptions

Let Ω be divided into disjoint subdomains Ω_i, $\overline{\Omega} = \cup_i \overline{\Omega}_i$, $i \in \{1, \cdots, N\}$, where each Ω_i is a John domain, as defined in Dohrmann et al. [2008a], with a uniformly bounded John constant. Let the boundary $\partial\Omega_i$ be aligned with the triangulation of $\mathscr{T}^h(\Omega)$ such that the inherited triangulation of Ω_i is shape regular with a mesh parameter h_i and $H_i := \mathrm{diam}(\Omega_i)$. According to Dohrmann et al. [2008a], $\mathrm{diam}(\Omega_i)$ can be estimated above and below by $|\Omega_i|^{\frac{1}{n}}$ with one of the constants depending on the John constant C_J. Denote by Ω_i^h the layer around $\partial\Omega_i$ which is a union the of $e_k^{(i)}$ the element of $\mathscr{T}^h(\Omega_i)$ which touch $\partial\Omega_i$, the boundary of Ω_i. We assume that all elements in Ω_i^h are quasi uniform. We also, as in Dryja and Sarkis [2010], introduce

$$\overline{\alpha}_i := \sup_{x \in \overline{\Omega}_i^h} \alpha(x), \quad \underline{\alpha}_i := \inf_{x \in \overline{\Omega}_i^h} \alpha(x). \tag{4}$$

2.2 The Snowflake Domain.

When proving the condition number estimate in Theorem 1, we needed to estimate the number of elements in the internal boundary layer given by $\Omega_i^h \cap \Omega_i$. Usually such an estimate is given by $c(H_i/h_i)^{n-1}$ where c is a constants not depending on the mesh parameter. This is not correct for all types of subdomains.

The snowflake domain follows a rule of refinement. It starts with a square with a boundary node in each corner. With each refinement all boundary edges are divided into three equal parts, and the middle part is replaced with an equilateral triangle. In figure 1, we see the first 3 refinements of the a snowflake domain. For the particular domain in the figure, we see that the triangles at the top and at the bottom always point into the domain, subtracting from its area, while the triangles at the left and the right side, always point outwards, adding to its area. The net change of the domains area is zero. With each refinement, the length of the boundary of the subdomain increases by a factor 4/3. It is possible to show that the asymptotic boundary of the snowflake domain is a von Koch curve with a Hausdorff dimension greater then 1. In Dohrmann and Widlund [2012b,a], it is pointed out that such a domain introduce a factor into the condition number which depends on the Hausdorff dimension, and particularly for the snowflake domain a bound for this factor is given by $c(4/3)^{\log(H_i/h_i)}$, with c independent of mesh parameters. This bound can be rewritten as $C(H_i/h_i)^{0.262}$ with C independent of mesh parameters.

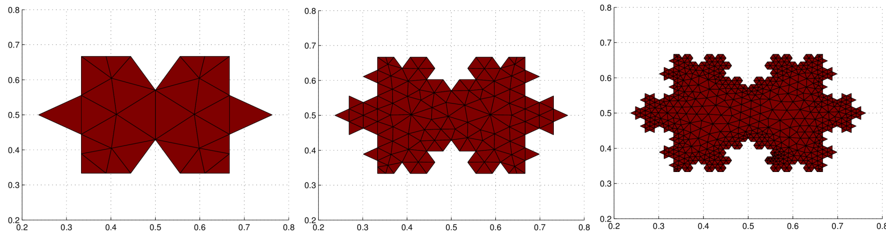

Fig. 1 Here we have 3 different levels of refinement of a snowflake domain. This domain has constant area but its boundary is growing by a factor $4/3$ with each refinement.

3 Additive Average Schwarz Method

Let us decompose $V_h(\Omega) = V_0(\Omega) + V_1(\Omega) + ... + V_N(\Omega)$, and define $V_i(\Omega) = V_h(\Omega) \cap H_0^1(\Omega_i)$ on Ω_i and extend by zero outside Ω_i for $i \in \{1, \cdots, N\}$. The coarse space $V_0(\Omega)$ is defined as the range of the following interpolation operator I_A. For $u \in V_h(\Omega)$, let $I_A u \in V_h(\Omega)$ be defined so that on Ω_i

$$I_A u = \begin{cases} u_j, & \text{if } x_j \in \partial\Omega_{ih} \\ \bar{u}_j, & \text{if } x_j \in \Omega_{ih} \setminus \partial\Omega_{ih} \end{cases} \tag{5}$$

where

$$\bar{u}_j := \frac{1}{n_i} \sum_{x_j \in \partial\Omega_{ih}} u_j. \tag{6}$$

Here Ω_{ih} and $\partial\Omega_{ih}$ are the sets of nodal points x_j on Ω_i and $\partial\Omega_i$, respectively, and n_i is the number of nodes on $\partial\Omega_{ih}$. u_j is the value of u at a nodal point.

For $i \in \{1, \cdots, N\}$, let us introduce

$$b_i(u, v) := a_i(u, v), \quad u, v \in V_i(\Omega), \tag{7}$$

where $a_i(\cdot, \cdot)$ is the restriction of $a(\cdot, \cdot)$ to Ω_i.

For $i = 0$ we introduce

$$b_0(u, v) := \sum_{i=1}^{N} \overline{\alpha} h_i^{n-2} \sum_{x_j \in \partial\Omega_{ih}} (u_j - \bar{u}_j) v_j. \tag{8}$$

3.1 The Preconditioner

For $i \in \{0, \cdots, N\}$, we define the operator $T_i^{(A)} : V_h(\Omega) \to V_i(\Omega)$ by $b_i(T_i^{(A)} u, v) = a(u, v)$, with $v \in V_i(\Omega)$. Of course, each of these problems have a unique solution. Let us introduce $T_A := T_0^{(A)} + T_1^{(A)} + \cdots + T_N^{(A)}$. We replace (3) by the operator equation

$$T_A u_h = g_h \tag{9}$$

where $g_h = \sum_{i=0}^{N} g_i$, and $g_i = T_i^{(A)} u_h$ and u_h is the solution of 3.

The main result

Theorem 1. *For any $u \in V_h(\Omega)$ the following holds:*

$$C_1 \beta_1^{-1} a(u,u) \leq a(Tu,u) \leq C_2 a(u,u), \tag{10}$$

where $\beta_1 = (\overline{\alpha}/\underline{\alpha}) \max_i \chi_i (H_i/h_i)^2$, and C_1 and C_2 depend on the parameter of an isoperimetric inequality, and the John constant, but not on the mesh parameter, and χ_i is a factor related to the Hausdorff dimension of the subdomain boundary. This factor χ_i might be mesh dependent, and can be estimated from the condition that $C\chi_i(H/h)^{n-1}$ are the number of patches needed to to cover Ω_i^h, where C is a mesh independent constant and n is the dimension of the problem.

Due to page restrictions, we leave out the proof. It is similar to that in Dryja and Sarkis [2010] but extended to $\mathbb{R}^3$, and valid for subdomains being John domains using some results from Dohrmann et al. [2008a].

Remark 1. When restricted to $\mathbb{R}^2$ with α constant in Ω, we can show that β_1 in Theorem 1 can be reduced to $\beta_1 = \max_i \chi_i((1 + log(H_i/h_i))(H_i/h_i))$.

4 Numerical Results

Here we present numerical results, for the simple Poisson equation in $\mathbb{R}^2$, for a variety of more or less irregular subdomains. The purpose of these results is to illustrate how the geometrical features of the subdomains impact the condition number. All tests have been done with the Additive Average method, and with the method in Dohrmann et al. [2008a]. In all the tests the two methods have shown similar performance. All methods are implemented in MATLAB using pcgeig with a default tolerance of 10^{-6}.

In table 1, we present results from solving the Poisson equation on the unit square with 16 subdomain of various shapes. We mainly look for effects on the condition number from boundary deformations, and from the use of subdomains with mesh dependent John constants. We use the results from the square subdomains with constant boundaries and a mesh independent John constant as a reference.

Based on the definition of a John domain in Dohrmann et al. [2008a], the subdomains with fingers, see figure 2, are designed to have a mesh dependent John constant that is doubling with each refinement of h. This does not cause an increase the condition number in the range of refinement tested as shown in table 1. Similar results where observed with the method in Dohrmann et al. [2008a]. Subdomains from the partitioner METIS result in an increase in the condition number, but it is hard to estimate what geometrical feature causes this increase. It is surprising that the type 2 subdomains of Dohrmann et al. [2008a] does not increase the condition number compared to the reference domain. The type 2 subdomain boundary is growing with refinement, however we see that the number of elements along the boundary is given by $C(H/h)$ with C independent of mesh parameters. This might explain why we do not see any increase in the condition number from this choice of subdomain geometry.

Table 1 This table shows iteration and condition numbers when solving the Poisson equation on different subdomains using additive average Schwarz method. The number of subdomains is fixed at $N = 16$ and $h = \left\{\frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}\right\}$.

		Square subdomains		Square subdomains with fingers		METIS subdomains		Type 2 subdomains	
N	h	itr	$cond$	itr	$cond$	itr	$cond$	itr	$cond$
16	1/16	17	6.22	13	4.20	21	9.18	13	4.20
16	1/32	26	16.61	28	18.91	38	25.26	22	14.00
16	1/64	46	38.34	43	44.47	62	66.85	35	36.41
16	1/128	68	82.32	65	97.42	91	126.29	53	82.86
16	1/256	84	170.58	94	205.13	135	282.69	81	171.98

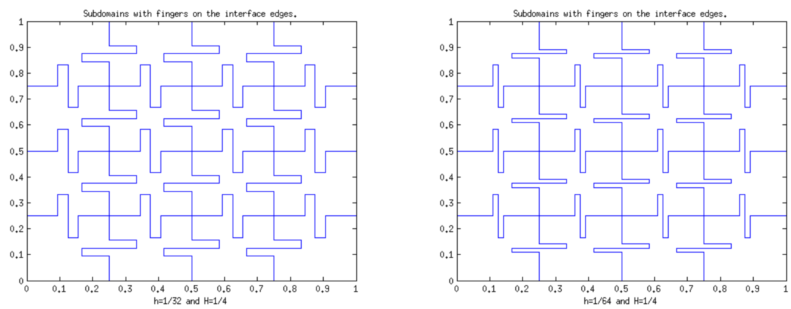

Fig. 2 Figures showing square subdomains with fingers on the edges. These fingers have length $1/3H$ and width h thus growing thiner with a refinement of h. This should give a growing John constant with refinement of h.

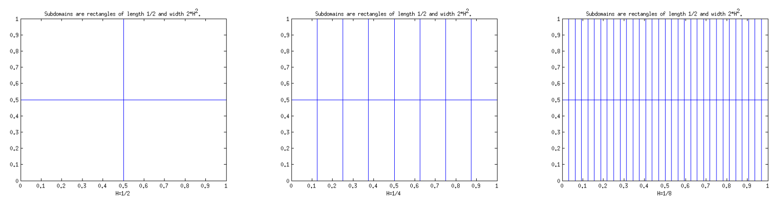

Fig. 3 Figures showing rectangular subdomains. Here theory for irregular subdomains estimates that $H_i = C_J|\Omega_i|^{\frac{1}{2}}$.

Table 2 This table shows iteration and condition numbers when solving the Poisson equation on both square and rectangular subdomains. The numerics is done with fixed $\frac{H}{h} = 16$ for $N = \{4, 16, 64\}$ subdomains. Using the method presented in Dohrmann et al. [2008a].

		Square subdomains		Rectangle subdomains	
N	H/h	itr	$cond$	itr	$cond$
4	16	13	20.57	13	20.57
16	16	27	20.66	36	55.94
64	16	32	20.69	84	350.61

Table 3 This table shows iteration and condition numbers when solving the Poisson equation on snowflake subdomains using additive average Schwarz method. Here $\beta = 1.262$.

Snowflake subdomains						
N	H/h	itr	$cond$	$\frac{cond}{(H/h)}$	$\frac{cond}{(H/h)^\beta}$	$\frac{cond}{\log(H/h)(H/h)^\beta}$
9	3	15	6.94	2.31	1.73	1.58
9	9	35	28.53	3.17	1.78	0.81
9	27	75	121.62	4.50	1.90	0.58
9	81	154	488.57	6.03	1.91	0.43

The deliberately poor choice of rectangular subdomains, as shown in figure 3, illustrate a type of domain where the John constant increases as the number of subdomains increases. Theory establishes that for the domains given in figure 3, we can estimate $H_i = C_J|\Omega_i|^{\frac{1}{2}}$ with a constant which depends on the John constant. In table 2, we observe an increase in the condition number even though the method in principle should be scalable and H/h is kept fixed.

Finally in table 3 the results for snowflake domains are listed. Looking at the ratio of the condition number with different proposed estimates it seems clear that the original estimate for the additive average Schwarz method given in Bjørstad et al. [1997] does not hold. If we take into account the Hausdorff dimension of the subdomain boundary, and adjust the classical convergence estimate by the bound of the factor χ, then this would result in an estimate $C(H/h)^\beta$ with $\beta = 1.262$. This estimate fits well with the numerical results. The condition number is well within the bounds established for irregular domains. Similar results were obtained when using the method of Dohrmann et al. [2008a] on snowflake subdomains.

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