Optimized Schwarz Methods for Heterogeneous Helmholtz and Maxwell's Equations

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1 Introduction

The Helmholtz equation is very difficult to solve by iterative methods [15], and the time harmonic Maxwell's equations inherit these difficulties. Optimized Schwarz methods are among the most promising iterative techniques. For the Helmholtz equation, they have their roots in the seminal work of Deprés [5, 6], which led to the development of optimized transmission conditions [4, 17, 19, 16, 2], and these techniques were independently rediscovered for the sweeping preconditioner [14] and the source transfer domain decomposition method [3]. For the time harmonic Maxwell's equations, optimized transmission conditions were developed and tested for problems without conductivity in [1, 9, 20, 21, 13], and with conductivity in [7, 8]. Particular Galerkin discretizations of transmission conditions were studied in [11, 10], and for scattering applications, see [20, 21].

In [12, 18], it was discovered that heterogeneous media can actually improve the convergence of optimized Schwarz methods, provided that the coefficient jumps are aligned with the interfaces, and the jumps are taken into account in an appropriate way in the transmission conditions. Similar results were found for Maxwell's equations in [22] and [23]; it is even possible to obtain convergence independently of the mesh size in certain situations. We present and study here transmission conditions for the Helmholtz equation with heterogeneous media, and establish a relation to the results of [22, 23] written for Maxwell's equations. We then study improved convergence behavior for specific choices of the discretization parameters related to the pollution effect.

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2 Optimized Schwarz Methods for Helmholtz and Maxwell's Equations

We consider the two dimensional Helmholtz equation in discontinuous media with piece-wise constant density ρ and wave-speed c. The Helmholtz equation in $\Omega = \mathbb{R}^2$ is defined by

$$\nabla(\frac{1}{\rho}\nabla \cdot u) + \frac{\omega^2}{c^2\rho}u = f, \text{ in } \Omega, \tag{1}$$

with

$$\rho =: \begin{cases} \rho_1 & \text{in } \Omega_1, \\ \rho_2 & \text{in } \Omega_2, \end{cases} \qquad c := \begin{cases} c_1 & \text{in } \Omega_1, \\ c_2 & \text{in } \Omega_2, \end{cases}$$

where $\Omega_1 = \mathbb{R}^- \times \mathbb{R}$, $\Omega_2 = \mathbb{R}^+ \times \mathbb{R}$ and the Sommerfeld radiation condition is imposed at infinity,

$$\lim_{|x|\to\infty} \sqrt{|x|}\left(\partial_{|x|}u + i\omega u\right) = 0, \tag{2}$$

for every possible direction $\frac{x}{|x|}$.

We can naturally define a Schwarz algorithm for equation (1) with Robin transmission conditions at the interface aligned with the discontinuity between the coefficients, and parameters $s_1, s_2 \in \mathbb{C}$,

$$\begin{array}{rll} \nabla(\frac{1}{\rho_1}\nabla \cdot u_1^n) + \frac{\omega^2}{c_1^2\rho_1}u_1^n = f, & & \text{in } \Omega_1, \\ (\frac{1}{\rho_1}\partial_{n_1} + \frac{1}{\rho_2}s_2)u_1^n = (\frac{1}{\rho_2}\partial_{n_1} + \frac{1}{\rho_2}s_2)u_2^{n-1}, & & \text{on } \Gamma, \\ \nabla(\frac{1}{\rho_2}\nabla \cdot u_2^n) + \frac{\omega^2}{c_2^2\rho_1}u_2^n = f, & & \text{in } \Omega_2, \\ (\frac{1}{\rho_2}\partial_{n_2} + \frac{1}{\rho_1}s_1)u_2^n = (\frac{1}{\rho_1}\partial_{n_2} + \frac{1}{\rho_1}s_1)u_1^{n-1}, & & \text{on } \Gamma. \end{array} \tag{3}$$

Proposition 1. *The convergence factor of algorithm (3) is given by*

$$\rho_{opt}(k, \rho_1, \rho_2, \omega, c_1, c_2, s_1, s_2) = \left|\frac{(\lambda_1 - s_1)(\lambda_2 - s_2)}{(\lambda_1 + s_2\frac{\rho_1}{\rho_2})(\lambda_2 + s_1\frac{\rho_2}{\rho_1})}\right|^{1/2}, \tag{4}$$

with $\lambda_j = \sqrt{k^2 - \omega_j^2}$, $\omega_j = \frac{\omega}{c_j}$ *for* $j = 1, 2$.

The proof of Proposition 1 is based in Fourier analysis, see [24] for details.

In order to obtain an efficient algorithm, we have to choose s_1 and s_2 such that ρ_{opt} becomes as small as possible for all relevant numerical frequencies $k \in K := [k_{\min}, k_{\max}]$, where $k_{\min}$ is the lowest relevant frequency ($k_{\min}$ depends on the geometry of the media) and $k_{\max} = \frac{c_{\max}}{h}$ is the highest numerical frequency supported by the numerical grid with mesh size h.

In what follows, we only consider $s_1 = P_1(1 + i)$ and $s_2 = P_2(1 + i)$, a choice that has been justified in [19], and thus study the min-max problem

$$\rho_{\text{opt}}^{*} = \min_{P_1,P_2>0} \max_{k\in K} |\rho_{\text{opt}}(k,\rho_1,\rho_2,\omega,c_1,c_2,P_1(1+i),P_2(1+i))|. \quad (5)$$

Similarly we can define a Schwarz algorithm for the time-harmonic Maxwell equations in a given domain $\Omega = \mathbb{R}^3$

$$-i\omega\varepsilon\mathbf{E} + \nabla\times\mathbf{H} = \mathbf{J}, \quad i\omega\mu\mathbf{H} + \nabla\times\mathbf{E} = \mathbf{0}, \quad (6)$$

with the Silver Müller radiation condition

$$\lim_{r\to\infty} r(\mathbf{H}\times e_{\mathbf{r}} + \frac{1}{Z_j}\mathbf{E}) = 0, \quad (7)$$

where $r := |\mathbf{x}|$ and $e_{\mathbf{r}} = \mathbf{x}/r$ for any vector $\mathbf{x}\in\mathbb{R}^3$.

We also consider the heterogeneous case where the domain Ω consists of two non-overlapping subdomains $\Omega_1 := \mathbb{R}^- \times \mathbb{R}^2$ and $\Omega_2 := \mathbb{R}^+ \times \mathbb{R}^2$ with interface Γ, with piece-wise constant parameters ε_j and μ_j in Ω_j, $j = 1,2$. A general Schwarz algorithm for this configuration is

$$\begin{array}{ll}
-i\omega\varepsilon_1\mathbf{E}^{1,n} + \nabla\times\mathbf{H}^{1,n} = \mathbf{J}, \quad i\omega\mu_1\mathbf{H}^{1,n} + \nabla\times\mathbf{E}^{1,n} = \mathbf{0} & \text{in } \Omega_1,\\
(\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(\mathbf{E}^{1,n},\mathbf{H}^{1,n}) = (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(\mathbf{E}^{2,n-1},\mathbf{H}^{2,n-1}) & \text{on } \Gamma,\\
-i\omega\varepsilon_2\mathbf{E}^{2,n} + \nabla\times\mathbf{H}^{2,n} = \mathbf{J}, \quad i\omega\mu_2\mathbf{H}^{2,n} + \nabla\times\mathbf{E}^{2,n} = \mathbf{0} & \text{in } \Omega_2,\\
(\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(\mathbf{E}^{2,n},\mathbf{H}^{2,n}) = (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(\mathbf{E}^{1,n-1},\mathbf{H}^{1,n-1}) & \text{on } \Gamma,
\end{array} \quad (8)$$

where $\mathcal{S}_j$, $j = 1,2$ are tangential, possibly pseudo-differential operators, and

$$\mathcal{B}_{\mathbf{n}_j}(\mathbf{E}^{j,n},\mathbf{H}^{j,n}) = \frac{\mathbf{E}^{j,n}}{Z_j}\times\mathbf{n}_j + \mathbf{n}_j\times(\mathbf{H}^{j,n}\times\mathbf{n}_j)$$

are the characteristic conditions, with $Z_j = \sqrt{\mu_j/\epsilon_j}$, $j = 1,2$. Different choices of $\mathcal{S}_j$, $j = 1,2$ lead to different Schwarz methods, see [9].

Remark 1. A direct computation shows that algorithms (3) and (8) have the same convergence factor, when setting $\rho_j := \mu_j$ and $c_j := \frac{1}{\sqrt{\varepsilon_j\mu_j}}$ for $j = 1,2$. Hence we can use all the results presented in [22] for Maxwell's equations for the case of the Helmholtz equation (3). We thus focus in the remainder on the Helmholtz case, but keep in mind that all results we will obtain hold mutatis mutandis also for the Maxwell case.

Using Remark 1, we obtain from [22] and [23]

Corollary 1. *The solution of (5) for $c_1 \neq c_2$ is asymptotically*

$$\rho_{opt}^{*} = \begin{cases} 1 - \mathcal{O}(h^{1/4}) & \text{if } \rho_1 = \rho_2,\\ \sqrt{\frac{\rho_{min}}{\rho_{max}}} + \mathcal{O}(h^{1/2}) & \text{if } \frac{1}{\sqrt{2}} \le \frac{\rho_1}{\rho_2} \le \sqrt{2},\\ \sqrt[4]{\frac{1}{2}} + \mathcal{O}(h^{1/2}) & \text{if } \frac{\rho_1}{\rho_2} < \frac{1}{\sqrt{2}} \text{ or } \frac{\rho_1}{\rho_2} > \sqrt{2}. \end{cases} \quad (9)$$

If $\rho_1 \neq \rho_2$ and $c_1 = c_2$, we obtain after excluding the resonance frequency [9]

$$\rho_{opt}^* = \sqrt{\frac{\rho_{min}}{\rho_{max}}} + \mathcal{O}(h^{1/2}), \tag{10}$$

with $\rho_{min} = \min\{\rho_1, \rho_2\}$ *and* $\rho_{max} = \max\{\rho_1, \rho_2\}$.

The detailed proof of Corollary 1 and the values of P_j can be found in [24]. We see from Corollary 1 that in most of the cases the optimized convergence factor ρ_{opt}^* has an asymptotic behavior independent of the mesh size h.

3 Scaling Results when Controlling the Pollution Effect

The core of our study is the asymptotic analysis of algorithms (3) and (8) when the mesh size h is related to the wave number ω to control the pollution effect. We will focus on the first case of Corollary 1, because this is the only case where the convergence can deteriorate in the mesh size h, see the first line in (9). We will consider three particular relationships between ω and h: $\omega h = C_\omega$, C_ω a constant, where the pollution effect is not controlled, $\omega^2 h = C_\omega$ where the pollution effect is provably controlled, and finally $\omega^{3/2} h = C_\omega$ which is widely believed to suffice to control the pollution effect.

Theorem 1. *Let* $\rho_1 = \rho_2$, $c_1 \neq c_2$ *and* $\omega h = C_\omega$. *If* $|\rho_{opt}|$ *defined in (4) is maximal for the frequencies* $k = \omega_1$, $k = \omega_2$ *and* $k = k_{\max}$, *and* $s_j = (1+i)P_j$, *then the solution of the min-max problem (5) is*

$$P_1^* = \frac{\overline{p}_1}{h}, \quad P_2^* = \frac{\overline{p}_2}{h}, \quad \rho_{opt}^* = \left(\frac{\overline{p}_1^2(2\overline{p}_2^2 - 2\overline{p}_2 c_r + c_r^2)}{\overline{p}_2^2(2\overline{p}_1^2 + 2\overline{p}_1 c_r + c_r^2)} \right)^{\frac{1}{4}}, \tag{11}$$

where $\{\overline{p}_1, \overline{p}_2\}$ *is solution of the system of equations*

$$\frac{p_1^2(2p_2^2-2p_2c_r+c_r^2)}{p_2^2(2p_1^2+2p_1c_r+c_r^2)} = \frac{\rho^2 p_2^2(2p_1^2-2p_1c_r+c_r^2)}{p_1^2(2p_2^2+2p_2c_r+c_r^2)},$$
$$\frac{p_1^2(2p_2^2-2p_2c_r+c_r^2)}{p_2^2(2p_1^2+2p_1c_r+c_r^2)} = \frac{\rho^2(2p_2^2-2p_2c_{\max_2}+c_{\max_2}^2)(2p_1^2-2p_1c_{\max_1}+c_{\max_1}^2)}{(2p_2^2+2p_2c_{\max_2}+c_{\max_2}^2)(2p_1^2+2p_1c_{\max_1}+c_{\max_1}^2)},$$

$c_r := rh := \sqrt{|\omega_1^2 - \omega_2^2|}h$, $c_{\max_1} := \sqrt{c_{\max}^2 - C_\omega^2/c_1^2}$, $c_{\max_2} := \sqrt{c_{\max}^2 - C_\omega^2/c_2^2}$.

Proof. Evaluating $|\rho_{\text{opt}}|^4$ from (4) at $s_j := \frac{p_j}{h}(1+i)$ for $k = \omega_1$, $k = \omega_2$ and $k = k_{\max}$ yields

$$R_1 = \frac{(h^2r^2-2p_2hr+2p_2^2)p_1^2}{p_2^2(h^2r^2+2p_1hr+2p_1^2)}, \quad R_2 = \frac{\rho^2 p_2^2(h^2r^2-2p_1hr+2p_1^2)}{(2p_2^2+2p_2hr+h^2r^2)p_1^2},$$
$$R_3 = \frac{\left(h^2(\frac{c_{\max}^2}{h^2}-\frac{C_\omega^2}{c_2^2h^2})-2p_2h\sqrt{\frac{c_{\max}^2}{h^2}-\frac{C_\omega^2}{c_2^2h^2}}+2p_2^2\right)}{\left(h^2(\frac{c_{\max}^2}{h^2}-\frac{C_\omega^2}{c_2^2h^2})-2p_1h\sqrt{\frac{c_{\max}^2}{h^2}-\frac{C_\omega^2}{c_2^2h^2}}+2p_1^2\right)} \frac{\left(h^2(\frac{c_{\max}^2}{h^2}-\frac{C_\omega^2}{c_1^2h^2})-2p_1h\sqrt{\frac{c_{\max}^2}{h^2}-\frac{C_\omega^2}{c_1^2h^2}}+2p_1^2\right)}{\left(h^2(\frac{c_{\max}^2}{h^2}-\frac{C_\omega^2}{c_1^2h^2})-2p_2h\sqrt{\frac{c_{\max}^2}{h^2}-\frac{C_\omega^2}{c_1^2h^2}}+2p_2^2\right)}.$$

Replacing rh by c_r, $c_{\max_1} = \sqrt{c_{\max}^2 - C_\omega^2/c_1^2}$ and $c_{\max_2} = \sqrt{c_{\max}^2 - C_\omega^2/c_2^2}$, the expressions can be simplified to

$$R_1 = \frac{p_1^2(2p_2^2 - 2p_2c_r + c_r^2)}{p_2^2(2p_1^2 + 2p_1c_r + c_r^2)}, \quad R_2 = \frac{\rho^2 p_2^2(2p_1^2 - 2p_1c_r + c_r^2)}{p_1^2(2p_2^2 + 2p_2c_r + c_r^2)},$$
$$R_3 = \frac{(2p_2^2 - 2p_2c_{\max_2} + c_{\max_2}^2)(2p_1^2 - 2p_1c_{\max_1} + c_{\max_1}^2)}{(2p_2^2 + 2p_2c_{\max_2} + c_{\max_2}^2)(2p_1^2 + 2p_1c_{\max_1} + c_{\max_1}^2)}.$$

Equioscillation between R_1, R_2 and R_3 then gives the result.

Remark 2. Note that Theorem 1 gives a closed form solution of the min-max problem (5), not just an asymptotic one.

For the special case of equal transmission conditions, we have

Corollary 2. *Under the same assumptions as in Theorem 1, if $s_j = (1+i)P_j$ with $P_1 = P_2$, then the solution of the min-max problem (5) is given by*

$$P_1^* = P_2^* = \frac{\overline{p}}{h}, \quad \rho_{opt}^* = \left(\frac{(2\overline{p}^2 - 2\overline{p}c_r + c_r^2)}{(2\overline{p}^2 + 2\overline{p}c_r + c_r^2)}\right)^{\frac{1}{4}},$$

with $\overline{p}$ the solution of the equation

$$\frac{(2p^2 - 2pc_r + c_r^2)}{(2p^2 + 2pc_r + c_r^2)} = \frac{(2p^2 - 2pc_{\max_2} + c_{\max_2}^2)(2p^2 - 2pc_{\max_1} + c_{\max_1}^2)}{(2p^2 + 2pc_{\max_2} + c_{\max_2}^2)(2p^2 + 2pc_{\max_1} + c_{\max_1}^2)}.$$

Proof. The proof follows along the same lines as the proof of Theorem 1.

Theorem 2. *Let $\rho_1 = \rho_2$, $c_1 \neq c_2$ and $\omega^2 h = C_\omega$. If $|\rho_{opt}|$ defined in (4) is maximal for the frequencies $k = \omega_1$, $k = \omega_2$, $k = k_m := \frac{c_m}{h^{3/4}}$ and $k = k_{\max}$, and $s_j = (1+i)P_j$, $P_1 = \frac{p_1}{h}$ and $P_2 = \frac{p_2}{\sqrt{h}}$, then the asymptotic solution of the min-max problem (5) for h small is given by*

$$P_1^* = \frac{c_{\max}^{3/4} c_r^{1/4}}{2^{1/4} h^{7/8}}, \quad P_2^* = \frac{1}{2}\frac{c_{\max}^{1/4} c_r^{3/4}}{2^{3/4} h^{5/8}}, \quad \rho_{opt}^* = 1 - \frac{r^{1/4}}{2^{1/4} c_{\max}^{1/4}} h^{1/8} + \mathcal{O}(h^{1/4}).$$

Interchanging the role of P_1 and P_2 leads to the same result.

Proof. The proof is based again on equioscillation.

Theorem 3. *Let $\rho_1 = \rho_2$, $c_1 \neq c_2$ and $\omega^{3/2} h = C_\omega$. If the frequencies $k = \omega_1$, $k = \omega_2$, $k = k_m := \frac{c_m}{h^{5/6}}$ and $k = k_{\max}$ are the local maxima of the convergence factor ρ_{opt} from (4), and if $s_1 = (1+i)P_1$, $s_2 = (1+i)P_2$, with $P_1 = \frac{p_1}{h^{11/12}}$ and $P_2 = \frac{p_2}{h^{3/4}}$, then the asymptotic solution of the min-max problem (5) for h small is given by*

$$P_1^* = \frac{c_{\max}^{3/4} c_r^{1/4}}{2^{1/4} h^{11/12}}, \quad P_2^* = \frac{1}{2}\frac{c_{\max}^{1/4} c_r^{3/4}}{2^{3/4} h^{3/4}}, \quad \rho_{opt}^* = 1 - \frac{r^{1/4}}{2^{1/4} c_{\max}^{1/4}} h^{1/12} + \mathcal{O}(h^{1/6}).$$

Interchanging the role of P_1 and P_2 leads to the same result.

	$\omega = C_\omega$	$\omega^2 h = C_\omega$	$\omega^{3/2} h = C_\omega$	$\omega h = C_\omega$
$\rho_1 = \rho_2,\ c_1 \neq c_2$	$1 - \mathcal{O}(h^{1/4})$ (Corollary 1)	$1 - \mathcal{O}(h^{1/8})$ (Theorem 2)	$1 - \mathcal{O}(h^{1/12})$ (Theorem 3)	< 1 (Theorem 1)
$\rho_1 \neq \rho_2,\ c_1 \neq c_2$	$\max\{\sqrt[4]{\frac{1}{2}}, \sqrt{\frac{\rho_{\min}}{\rho_{\max}}}\}$ (Corollary 1)	$\max\{\sqrt[4]{\frac{1}{2}}, \sqrt{\frac{\rho_{\min}}{\rho_{\max}}}\}$ (Remark 3)	$\max\{\sqrt[4]{\frac{1}{2}}, \sqrt{\frac{\rho_{\min}}{\rho_{\max}}}\}$ (Remark 3)	< 1 (Remark 3)
$\rho_1 \neq \rho_2,\ c_1 = c_2$	$\sqrt{\frac{\rho_{\min}}{\rho_{\max}}}$ (Corollary 1)	$\sqrt{\frac{\rho_{\min}}{\rho_{\max}}}$ (Remark 3)	$\sqrt{\frac{\rho_{\min}}{\rho_{\max}}}$ (Remark 3)	< 1 (Remark 3)

Table 1 Comparison of the convergence factors with different relationships between ω and h.

Proof. The proof is similar to the proof of Theorem 2.

One can justify the choice of the frequencies $k = \omega_1$, $k = \omega_2$, $k = k_m$ and $k = k_{\max}$ as the correct candidates for the $|\rho_{\text{opt}}|$ using asymptotic analysis, but this exceeds the space available, see [24] for more details.

Remark 3. One can obtain similar results also for the cases $\rho_1 \neq \rho_2$ but this will only reduce the order of the second asymptotic term, as in Theorems 2 and 3. For the relationship $\omega h = C_\omega$ one can also obtain a similar result to Theorem 1.

We give a summary of all these results in Table 1.

4 Conclusions

We studied the performance of optimized Schwarz methods for Helmholtz and Maxwell's equations for heterogeneous media. Using Fourier analysis, we showed that the convergence factor of the optimized Schwarz methods for the Helmholtz equation and the Maxwell's equations are the same, and it suffices therefore to study the algorithms only for the Helmholtz equation. We then studied in detail the performance for three different choices of the relationship between the wave number and the mesh size to control the pollution effect, and showed that increasing the resolution improves the performance of the optimized Schwarz methods. It was not possible to show all the proofs in detail in this short manuscript, but more information can be found in the PhD thesis [24].

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