Additive Schwarz Preconditioners for a State Constrained Elliptic Distributed Optimal Control Problem Discretized by a Partition of Unity Method

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1 Introduction

In this work, we are interested in solving a model elliptic optimal control problem of the following form: Find $(y,u) \in H_0^1(\Omega) \times L_2(\Omega)$ that minimize the functional

$$J(y,u) = \frac{1}{2}\int_\Omega (y-f)^2 dx + \frac{\beta}{2}\int_\Omega u^2 dx$$

subject to

$$-\Delta y = u \text{ in } \Omega, \quad y = 0 \text{ in } \partial\Omega, \tag{1}$$

and $y \le \psi$ in Ω, where Ω is a convex polygon in $\mathbb{R}^2$ and $f \in L_2(\Omega)$. We also assume $\psi \in C^2(\Omega) \cap H^3(\Omega)$ and $\psi > 0$ on $\partial\Omega$.

Using elliptic regularity (cf. [7]) for (1), we can reformulate the model problem as follows: Find $y \in K$ such that

$$y = \operatorname*{argmin}_{v\in K}\left[\frac{1}{2}a(v,v) - (f,v)\right], \tag{2}$$

where $K = \{v \in H^2(\Omega) \cap H_0^1(\Omega) : v \le \psi \text{ in } \Omega\}$,

$$a(w,v) = \beta\int_\Omega \Delta w \Delta v dx + \int_\Omega wv dx \quad \text{and} \quad (f,v) = \int_\Omega fv dx.$$

Once y is calculated, then u can be determined by $u = -\Delta y$.

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The minimization problem (2) is discretized in [4] by a partition of unity method (PUM). The goal of this paper is to use the ideas in [5] for an obstacle problem of clamped Kirchhoff plates to develop preconditioners for the discrete problems in [4]. We refer to these references for technical details and only present the important results here.

2 The Discrete Problem

We will use a variant of the PUM (cf. [11, 8, 1, 12]) to construct a conforming approximation space $V_h \subset H^2(\Omega) \cap H_0^1(\Omega)$. Below we present an overview of the construction of V_h.

Let $\{\Omega_i\}_{i=1}^n$ be an open cover of $\bar{\Omega}$ such that there exists a collection of nonnegative functions $\{\phi_i\}_{i=1}^n \in W_\infty^2(\mathbb{R}^2)$ with the following properties:

$$
\begin{aligned}
\operatorname{supp} \phi_i &\subset \Omega_i && \text{for } 1 \le i \le n,\\
\sum_{i=1}^n \phi_i &= 1 && \text{on } \Omega,\\
|\phi_i|_{W_\infty^m(\mathbb{R}^2)} &\le \frac{C}{(\operatorname{diam} \Omega_i)^m} && \text{for } 0 \le m \le 2,\ 1 \le i \le n.
\end{aligned}
$$

For $1 \le i \le n$, the local approximation space V_i consists of biquadratic polynomials satisfying the Dirichlet boundary conditions of (1), i.e. $v = 0$ on $\partial\Omega$ for all $v \in V_i$. Basis functions for V_i are tensor product Lagrange polynomials. Figure 1 (b) shows an illustration that depicts the interpolation nodes corresponding to the interior degrees of freedom for a given discretization.

In this work the patches $\{\Omega_i\}_{i=1}^n$ are open rectangles and $\{\phi_i\}_{i=1}^n$ are C^1 piecewise polynomial tensor product flat-top partition of unity functions. $\Omega_i^{\text{flat}} = \{x \in \Omega_i : \phi_i(x) = 1\}$. The interpolation nodes associated with V_i are distributed uniformly throughout Ω_i^{flat}, this is the reason the global basis functions have the Kronecker delta property. We will assume that the diameters of the patches are comparable to a mesh size h. We now define

$$V_h = \sum_{i=1}^n \phi_i V_i.$$

Let $\mathcal{N}_h$ be the set of all interior interpolation nodes used in the construction of V_h. The discrete problem is to find $y_h \in K_h$ such that

$$y_h = \operatorname*{argmin}_{v \in K_h} \left[\frac{1}{2} a(v, v) - (f, v) \right], \tag{3}$$

where $K_h = \{v \in V_h : v(p) \le \psi(p)\ \forall p \in \mathcal{N}_h\}$.

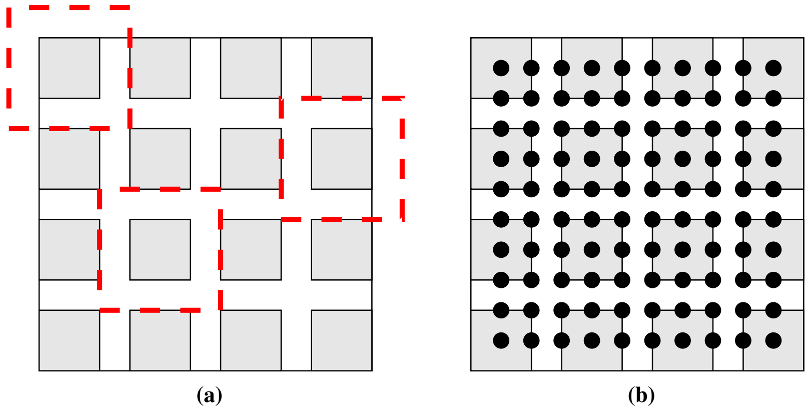

Fig. 1: (a) Ω_i (bounded by dotted lines) and Ω_i^{flat} (shaded in grey)
(b) nodes for the interior DOFs

By introducing a Lagrange multiplier $\lambda_h : \mathcal{N}_h \to \mathbb{R}$, the minimization problem (3) can be rewritten in the following form: Find $y_h \in K_h$ such that

$$\begin{aligned} a(y_h, v) - (f, v) &= -\sum_{p\in\mathcal{N}_h} \lambda_h(p)v(p) && \forall\, v \in V_h, \\ \lambda_h(p) &= \max(0, \lambda_h(p) + c(y_h(p) - \psi(p))) && \forall\, p \in \mathcal{N}_h, \end{aligned}$$

where c is a (large) positive number ($c = 10^8$ in our numerical experiments). This system can then be solved by a primal-dual active set (PDAS) algorithm (cf. [2, 3, 9, 10]). Given the k-th approximation (y_k, λ_k), the $(k+1)$-st iteration of the PDAS algorithm is to find (y_{k+1}, λ_{k+1}) such that

$$\begin{aligned} a(y_{k+1}, v) - (f, v) &= -\sum_{p\in\mathcal{N}_h} \lambda_{k+1}(p)v(p) && \forall v \in V_h, \\ y_{k+1}(p) &= \psi(p) && \forall p \in \mathfrak{A}_k, \\ \lambda_{k+1}(p) &= 0 && \forall p \in \mathcal{N}_h \backslash \mathfrak{A}_k, \end{aligned} \tag{4}$$

where $\mathfrak{A}_k = \{p \in \mathcal{N}_h : \lambda_k(p) + c(y_k(p) - \psi(p)) > 0\}$ is the set of active nodes determined from the approximations (y_k, λ_k). Below we present preconditioners for the linear systems encountered in (4).

3 The Preconditioners

The additive Schwarz preconditioners (cf. [6]) will be applied to a system associated with a subset $\tilde{\mathcal{N}}_h$ of $\mathcal{N}_h$. Let $\tilde{T}_h : V_h \to V_h$ be defined by

$$(\tilde{T}_h v)(p) = \begin{cases} v(p) & \text{if } p \in \tilde{\mathcal{N}}_h \\ 0 & \text{if } p \notin \tilde{\mathcal{N}}_h \end{cases}.$$

The approximation space for the subproblem is $\tilde{V}_h = \tilde{T}_h V_h$. The associated stiffness matrix is a symmetric positive definite operator $\tilde{A}_h : \tilde{V}_h \to \tilde{V}_h'$ defined by

$$\langle \tilde{A}_h v, w \rangle = a(v, w) \qquad \forall\, v, w \in \tilde{V}_h,$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $\tilde{V}_h' \times \tilde{V}_h$.

A One-Level Method Here we introduce a collection of shape regular subdomains $\{D_j\}_{j=1}^J$ with diam $D_j \approx H$ that overlap with each other by at most δ. Associated with each subdomain is a function space $V_j \subset \tilde{V}_h$ whose members vanish at the nodes outside D_j. Let $A_j : V_j \to V_j'$ be defined by

$$\langle A_j v, w \rangle = a(v, w) \qquad \forall v, w \in V_j.$$

The one-level additive Schwarz preconditioner $B_{\mathrm{OL}} : V_h' \to V_h$ is defined by

$$B_{\mathrm{OL}} = \sum_{j=1}^{J} I_j A_j^{-1} I_j^t,$$

where $I_j : V_j \to \tilde{V}_h$ is the natural injection.

Following the arguments in [5], we can obtain the following theorem.

Theorem 1 *There exists a positive constant C_{OL} independent of H, h, J, δ and $\tilde{N}_h$ such that*

$$\kappa(B_{OL}\tilde{A}_h) \leq C_{OL}\delta^{-3}H^{-1}.$$

Remark 1 The estimate given in Theorem 1 is identical to the one for the plate bending problem without an obstacle, i.e., the obstacle is invisible to the one-level additive Schwarz preconditioner.

A Two-Level Method Let $V_H \subset H^2(\Omega) \cap H_0^1(\Omega)$ be a coarse approximation space based on the construction in Section 2 where $H > h$. We assume the patches of V_H are of comparable size to the subdomains $\{D_j\}_{j=1}^J$. Let $\Pi_h : H^2(\Omega) \cap H_0^1(\Omega) \to V_h$ be the nodal interpolation operator. We define $V_0 \subset \tilde{V}_h$ by $V_0 = T_h \Pi_h V_H$, and $A_0 : V_0 \to V_0'$ by

$$\langle A_0 v, w \rangle = a(v, w) \qquad \forall\, v, w \in V_0.$$

The two-level additive Schwarz preconditioner $B_{\mathrm{TL}} : V_h' \to V_h$ is given by

$$B_{\mathrm{TL}} = \sum_{j=0}^{J} I_j A_j^{-1} I_j^t,$$

where $I_0 : V_0 \to \tilde{V}_h$ is the natural injection. Using the arguments in [5], we can obtain the following theorem.

Theorem 2 *There exists a positive constant C_{TL} independent of H, h, J, δ and $\tilde{N}_h$ such that*

$$\kappa(B_{TL}A_h) \le C_{TL} \min\left((H/h)^4, \delta^{-3}H^{-1}\right).$$

Remark 2 The two-level method is scalable as long as H/h remains bounded.

Remark 3 The estimate given in Theorem 2 is different from the estimate for the plate bending problem without obstacles that reads

$$\kappa\left(B_{TL}A_h\right) \le C\left(\frac{H}{\delta}\right)^3.$$

This difference is caused by the necessity of truncation in the construction of $\tilde{V}_0$ when the obstacle is present.

4 A Numerical Example

We consider Example 4.2 in [4], where $\Omega = (-0.5, 0.5)^2$, $\beta = 0.1$, $\psi = 0.01$, and $f = 10(\sin(2\pi(x_1 + 0.5)) + (x_2 + 0.5))$. We discretize (3) by the PUM with uniform rectangular patches so that $h \approx 2^{-\ell}$, where ℓ is the refinement level. As ℓ increases from 1 to 8, the number of degrees of freedom increases from 16 to 586756. The discrete variational inequalities are solved by the PDAS algorithm presented in Section 2, with $c = 10^8$.

For the purpose of comparison, we first solve the auxiliary systems in each iteration of the PDAS algorithm by the conjugate gradient (CG) method without a preconditioner. The average condition number during the PDAS iteration and the time to solve the variational inequality are presented in Table 1. The PDAS iterations fail to stop (DNC) within 48 hours beyond level 6.

Table 1: Average condition number (κ) and time to solve (t_{solve}) in seconds by the CG algorithm

ℓ	κ	t_{solve}
1	$3.1305\times10^{+2}$	2.6111×10^{-2}
2	$9.1118\times10^{+3}$	1.0793×10^{-1}
3	$2.0215\times10^{+5}$	9.7842×10^{-1}
4	$3.3705\times10^{+6}$	$3.3911\times10^{+1}$
5	$6.4346\times10^{+7}$	$6.2173\times10^{+2}$
6	$1.0537\times10^{+9}$	$8.8975\times10^{+3}$
7	DNC	DNC
8	DNC	DNC

We then solve the auxiliary systems by the preconditioned conjugate gradient (PCG) method, using the additive Schwarz preconditioners associated with J subdomains. The mesh size H for the coarse space V_H is $\approx 1/\sqrt{J}$. We say the PCG method has converged if $\|Br\|_2 \leq 10^{-15}\|b\|_2$, where B is the preconditioner, r is the residual, and b is the load vector. The initial guess for the PDAS algorithm is taken to be the solution at the previous level, or 0 if $2^{2\ell} = J$. To obtain a good initial guess for the two-level method, the one-level method is used when $2^{2\ell} = J$. The subdomain problems and the coarse problem are solved by a direct method based on the Cholesky factorization on independent processors.

Small Overlap Here we apply the preconditioners in such a way that $\delta \approx h$. The average condition numbers of the linear systems over the PDAS iterations are presented in Table 2. We can see that these condition numbers are significantly smaller than those for the unpreconditioned case and the condition numbers for the two-level method are smaller than those for the one-level method. For each ℓ, as J increases the condition numbers for the two-level method are decreasing, which demonstrates the scalability of the two-level method (cf. Remark 2).

Table 2: Average condition number for small overlap: one-level (left) and two-level (right)

ℓ	$J=4$	$J=16$	$J=64$	$J=256$	$J=4$	$J=16$	$J=64$	$J=256$
1	$1.00\times10^{+0}$	-	-	-	$1.00\times10^{+0}$	-	-	-
2	$4.94\times10^{+0}$	$7.40\times10^{+0}$	-	-	$5.46\times10^{+0}$	$7.40\times10^{+0}$	-	-
3	$1.51\times10^{+1}$	$4.41\times10^{+1}$	$6.61\times10^{+1}$	-	$1.22\times10^{+1}$	$1.14\times10^{+1}$	$6.61\times10^{+1}$	-
4	$7.82\times10^{+1}$	$1.90\times10^{+2}$	$5.35\times10^{+2}$	$8.19\times10^{+2}$	$2.85\times10^{+1}$	$2.79\times10^{+1}$	$1.26\times10^{+1}$	$8.19\times10^{+2}$
5	$6.47\times10^{+2}$	$1.64\times10^{+3}$	$3.17\times10^{+3}$	$9.50\times10^{+3}$	$6.29\times10^{+1}$	$9.19\times10^{+1}$	$4.61\times10^{+1}$	$1.98\times10^{+1}$
6	$5.07\times10^{+3}$	$1.31\times10^{+4}$	$2.58\times10^{+4}$	$5.04\times10^{+4}$	$3.67\times10^{+2}$	$3.48\times10^{+2}$	$1.31\times10^{+2}$	$5.77\times10^{+1}$
7	$4.07\times10^{+4}$	$1.06\times10^{+5}$	$2.10\times10^{+5}$	$4.15\times10^{+5}$	$2.74\times10^{+3}$	$2.11\times10^{+3}$	$1.03\times10^{+3}$	$2.86\times10^{+2}$
8	$3.26\times10^{+5}$	$8.55\times10^{+5}$	$1.70\times10^{+6}$	$3.38\times10^{+6}$	$2.16\times10^{+4}$	$1.48\times10^{+4}$	$9.19\times10^{+3}$	$1.87\times10^{+3}$

The times to solve the problem for each method are presented in Table 3. By comparing them with the results in Table 1, we can see that both of the two methods represents progress. For comparison purposes, the faster time of the two methods is highlighted in red for each ℓ and J. As h decreases and J increases, the two-level method performs better than the one-level method. These results are consistent with Theorems 1 and 2.

Generous Overlap Here we apply the preconditioners in such a way that $\delta \approx H$. When $J = 4$ and $J = 16$ both methods fail to converge at $\ell = 8$ within 48 hours due to the large size of the local problems. The average condition numbers of the linear systems over the PDAS iterations are presented in Table 4. They agree with Theorems 1 and 2. We can also see that these condition numbers are smaller than those in the case of small overlap.

The times to solve the problem for each method are presented in Table 5. Again both methods are superior to the unpreconditioned method and the scalability of the two-level method is observed.

Table 3: Time to solve in seconds for small overlap: one-level (left) and two-level (right). Times highlighted in red are the fastest between the two methods.

ℓ	$J=4$	$J=16$	$J=64$	$J=256$	$J=4$	$J=16$	$J=64$	$J=256$
1	$1.78\times10^{+0}$	-	-	-	$1.78\times10^{+0}$	-	-	-
2	3.04×10^{-1}	$1.55\times10^{+1}$	-	-	$1.06\times10^{+0}$	$1.55\times10^{+1}$	-	-
3	3.84×10^{-1}	$1.07\times10^{+1}$	$6.08\times10^{+1}$	-	$1.08\times10^{+0}$	$1.42\times10^{+1}$	$6.08\times10^{+1}$	-
4	$2.60\times10^{+0}$	$4.18\times10^{+1}$	$9.18\times10^{+1}$	$3.55\times10^{+2}$	$5.51\times10^{+0}$	$5.83\times10^{+1}$	$7.09\times10^{+1}$	$3.55\times10^{+2}$
5	$2.57\times10^{+1}$	$1.11\times10^{+2}$	$1.53\times10^{+2}$	$3.54\times10^{+2}$	$3.09\times10^{+1}$	$1.14\times10^{+2}$	$1.42\times10^{+2}$	$1.46\times10^{+2}$
6	$2.82\times10^{+2}$	$2.69\times10^{+2}$	$4.00\times10^{+2}$	$4.63\times10^{+2}$	$2.81\times10^{+2}$	$2.06\times10^{+2}$	$1.63\times10^{+2}$	$1.50\times10^{+2}$
7	$5.25\times10^{+3}$	$1.91\times10^{+3}$	$1.48\times10^{+3}$	$1.58\times10^{+3}$	$4.43\times10^{+3}$	$1.18\times10^{+3}$	$4.68\times10^{+2}$	$2.98\times10^{+2}$
8	$1.09\times10^{+5}$	$2.90\times10^{+4}$	$1.16\times10^{+4}$	$6.85\times10^{+3}$	$9.05\times10^{+4}$	$2.04\times10^{+4}$	$3.12\times10^{+3}$	$8.80\times10^{+2}$

Table 4: Average condition number for generous overlap: one-level (left) and two-level (right)

ℓ	$J=4$	$J=16$	$J=64$	$J=256$	$J=4$	$J=16$	$J=64$	$J=256$
1	$1.00\times10^{+0}$	-	-	-	$1.00\times10^{+0}$	-	-	-
2	$1.00\times10^{+0}$	$7.40\times10^{+0}$	-	-	$1.25\times10^{+0}$	$7.40\times10^{+0}$	-	-
3	$1.00\times10^{+0}$	$7.84\times10^{+0}$	$6.61\times10^{+1}$	-	$1.25\times10^{+0}$	$6.27\times10^{+0}$	$6.61\times10^{+1}$	-
4	$1.00\times10^{+0}$	$7.56\times10^{+0}$	$8.47\times10^{+1}$	$8.19\times10^{+2}$	$1.25\times10^{+0}$	$6.47\times10^{+0}$	$1.32\times10^{+1}$	$8.19\times10^{+2}$
5	$1.00\times10^{+0}$	$8.29\times10^{+0}$	$9.67\times10^{+1}$	$1.48\times10^{+3}$	$1.25\times10^{+0}$	$7.15\times10^{+0}$	$1.75\times10^{+1}$	$1.73\times10^{+1}$
6	$1.00\times10^{+0}$	$8.36\times10^{+0}$	$9.86\times10^{+1}$	$1.47\times10^{+3}$	$1.25\times10^{+0}$	$7.45\times10^{+0}$	$2.06\times10^{+1}$	$2.03\times10^{+1}$
7	$1.00\times10^{+0}$	$8.43\times10^{+0}$	$1.00\times10^{+2}$	$1.49\times10^{+3}$	$1.25\times10^{+0}$	$7.63\times10^{+0}$	$2.22\times10^{+1}$	$2.59\times10^{+1}$
8	DNC	DNC	$1.01\times10^{+2}$	$1.51\times10^{+3}$	DNC	DNC	$2.44\times10^{+1}$	$2.82\times10^{+1}$

We now compare the generous overlap methods with the small overlap methods. In Table 5, the times in red are the ones where the method with generous overlap outperforms the method with small overlap. It is evident from Table 5 that the performance of the two-level method with generous overlap suffers from a high communication cost for small h and large J.

Table 5: Time to solve in seconds for generous overlap: one-level (left) and two-level (right). Times highlighted in red are faster than the corresponding method with small overlap.

ℓ	$J=4$	$J=16$	$J=64$	$J=256$	$J=4$	$J=16$	$J=64$	$J=256$
1	1.33×10^{-1}	-	-	-	1.33×10^{-1}	-	-	-
2	1.90×10^{-1}	$1.66\times10^{+1}$	-	-	4.71×10^{-1}	$1.66\times10^{+1}$	-	-
3	2.88×10^{-1}	$7.17\times10^{+0}$	$6.14\times10^{+1}$	-	6.47×10^{-1}	$1.03\times10^{+1}$	$6.14\times10^{+1}$	-
4	$5.86\times10^{+0}$	$2.54\times10^{+1}$	$4.57\times10^{+1}$	$3.55\times10^{+2}$	$6.73\times10^{+0}$	$3.45\times10^{+1}$	$6.33\times10^{+1}$	$3.55\times10^{+2}$
5	$1.02\times10^{+2}$	$7.34\times10^{+1}$	$6.88\times10^{+1}$	$1.57\times10^{+2}$	$1.06\times10^{+2}$	$8.17\times10^{+1}$	$8.70\times10^{+1}$	$1.48\times10^{+2}$
6	$1.32\times10^{+3}$	$5.21\times10^{+2}$	$1.09\times10^{+2}$	$1.50\times10^{+2}$	$1.32\times10^{+3}$	$5.46\times10^{+2}$	$1.15\times10^{+2}$	$1.12\times10^{+2}$
7	$2.41\times10^{+4}$	$8.12\times10^{+3}$	$7.74\times10^{+2}$	$3.00\times10^{+2}$	$2.31\times10^{+4}$	$8.41\times10^{+3}$	$7.51\times10^{+2}$	$1.97\times10^{+2}$
8	DNC	DNC	$1.16\times10^{+4}$	$1.64\times10^{+3}$	DNC	DNC	$1.19\times10^{+4}$	$1.13\times10^{+3}$

5 Conclusion

In this paper we present additive Schwarz preconditioners for the linear systems that arise from the PDAS algorithm applied to an elliptic distributed optimal control problem with pointwise state constraints discretized by a PUM. Based on the condition number estimates and the numerical results, the two-level method with small overlap appears to be the best choice for small h and large J.

Acknowledgements The work of the first and third authors was supported in part by the National Science Foundation under Grant No. DMS-16-20273. Portions of this research were conducted with high performance computing resources provided by Louisiana State University (http://www.hpc.lsu.edu).

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